An Alternative Proof of an Inequality by Zhu

Kwara Nantomah

Department of Mathematics, Faculty of Mathematical Sciences
University for Development Studies, Navrongo Campus
P. O. Box 24, Navrongo, UE/R, Ghana

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Abstract

In this short note, we provide a new and relatively simple proof of an inequality established by Zhu in 2009. The main tools employed are Lazarevic inequality and the arithmetic-geometric mean inequality.

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1 Introduction

In 1989, Wilker[5, p. 55] put forward the following two problems.

(a) Prove that if $0 < z < \pi/2$, then

$$\left(\frac{\sin z}{z}\right)^2 + \frac{\tan z}{z} > 2.$$  \hfill (1)

(b) For $0 < z < \pi/2$, determine the largest number $\lambda$ such that

$$\left(\frac{\sin z}{z}\right)^2 + \frac{\tan z}{z} > 2 + \lambda z^3 \tan z.$$  \hfill (2)
In 1991, Sumner et al. [4] provided solutions to the above problems and further proved that

\[ \frac{16}{\pi^3} z^3 \tan z < \left( \frac{\sin z}{z} \right)^2 + \frac{\tan z}{z} - 2 < \frac{8}{45} z^3 \tan z, \quad (3) \]

for \( 0 < z < \pi/2 \), where the constants \( 16/\pi \) and \( 8/45 \) are the best possible.

In 2003, Guo et al. [1] gave new proofs of (1) and (2). In 2005, Zhu [9] also gave another proof of (1). Also in 2007, Zhang and Zhu [8] provided a new and elementary proof of (3). Since then, these elegant inequalities continue to attract the attention of researchers. For instance, Wu and Srivastava [7] established the Wilker-type inequality

\[ \left( \frac{z}{\sin z} \right)^2 + \frac{z}{\tan z} > 2, \quad (4) \]

where \( 0 < z < \pi/2 \). The hyperbolic counterpart of (1) was established by Zu [10] as

\[ \left( \frac{\sinh z}{z} \right)^2 + \frac{\tanh z}{z} > 2, \quad (5) \]

where \( z \in \mathbb{R} \setminus \{0\} \). Also, the hyperbolic counterpart of (4) was established by Wu and Debnath [6] as

\[ \left( \frac{z}{\sinh z} \right)^2 + \frac{z}{\tanh z} > 2, \quad (6) \]

where \( z \in \mathbb{R} \setminus \{0\} \). Then in [11], Zhu generalized (5) and (6) by proving the following theorem among other things.

**Theorem 1.1.** Let \( z > 0 \) and \( a \geq 1 \). Then the inequality

\[ \left( \frac{\sinh z}{z} \right)^{2a} + \left( \frac{\tanh z}{z} \right)^{a} > \left( \frac{z}{\sinh z} \right)^{2a} + \left( \frac{z}{\tanh z} \right)^{a} > 2, \quad (7) \]

holds.

By using some classical inequalities, the motive of this note is to provide a new and relatively simple proof of this theorem. The results are given in the following section.

## 2 Results

To start with, we recall the following result which is well known in the literature as Lazarevic inequality (see [2] or [3, p. 270]).
Lemma 2.1. Let \( z \in \mathbb{R} \setminus \{0\} \). Then the inequality
\[
\left( \frac{\sinh z}{z} \right)^q > \cosh z,
\]
holds if and only if \( q \geq 3 \).

Proof of Theorem 1.1. Since \((\alpha^2 + \beta)/(1/\alpha^2 + 1/\beta) = \alpha^2 \beta\) for all \( \alpha, \beta \in \mathbb{R} \), then by applying Lemma 2.1, we obtain
\[
\frac{(\sinh z)^2}{z} + \left( \frac{\tan z}{\sinh z} \right)^2 = \left( \frac{\sinh z}{z} \right)^a \left( \frac{\tan z}{z} \right)^a
\]
\[
= \left\{ \left( \frac{\sinh z}{z} \right)^3 \frac{1}{\cosh z} \right\}^a
\]
\[
> 1,
\]
which gives the left-hand side of (7). Next, let \( H \) be defined for \( z > 0 \) and \( a \geq 1 \) as
\[
H(z) = \left( \frac{z}{\sinh z} \right)^{2a} + \left( \frac{z}{\tanh z} \right)^a.
\]
Then by differentiating and applying the arithmetic-geometric mean inequality, we obtain
\[
H'(z)
\]
\[
= 2a \left( \frac{z}{\sinh z} \right)^{2a-1} \left[ \frac{1}{\sinh z} - z \frac{\cosh z}{\sinh^2 z} \right] + a \left( \frac{z}{\sinh z} \right)^{a-1} \left[ \frac{\cosh z}{\sinh z} - \frac{z}{\sinh^2 z} \right]
\]
\[
= \frac{a}{\sinh z} \left( \frac{z}{\sinh z} \right)^{2a} \left[ \frac{\cosh^a z \sinh^{a+1} z}{z^{a+1}} + \left( \frac{2 \sinh z}{z^{a+1}} - \frac{\sinh^a z}{x^a} \right) - 2 \cosh z \right]
\]
\[
\geq \frac{a}{\sinh z} \left( \frac{z}{\sinh z} \right)^{2a} \left[ 2 \sqrt{\cosh^a z \sinh^{a+1} z} \left( \frac{2 \sinh z}{z^{a+1}} - \frac{\sinh^a z}{x^a} \right) - 2 \cosh z \right]
\]
\[
= \frac{2a}{\sinh z} \left( \frac{z}{\sinh z} \right)^{2a} \sqrt{\cosh^a z} \left[ \frac{\sinh^{2a+1} z}{z} \left( \frac{2 \sinh z}{z} - 1 \right) - \sqrt{\cosh z} \right]
\]
\[
> 0,
\]
which is as a result of Lemma 2.1 and the fact that \( \frac{\sinh z}{z} > 1 \). Thus \( H(z) \) is increasing and hence, for \( z > 0 \), we have
\[
H(z) > H(0+) = \lim_{z \to 0} H(z) = 2
\]
which gives the right-hand side of (7). These complete the proof.

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References


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