Proximal Point Algorithms for Fixed Point Problem and Convex Minimization Problem

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Abstract
In this paper, a new algorithm for finding common elements of the set of minimizers of a convex function and the set of fixed points of a multivalued quasi-nonexpansive mapping is constructed. Convergence theorems are also proved in Hilbert spaces without any compactness assumption. The results here are significant improvements on several important recent results in this direction for this class of mappings.

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1 Background

Let $H$ be a real Hilbert space with inner product $\langle ., . \rangle$ and induced norm $\| \cdot \|$. A map $A : H \to H$ is said to be Lipschitz if there exists a positive constant $L \geq 0$ such that
\[ \| Ax - Ay \| \leq L \| x - y \| \quad \text{for all } x, y \in H. \] (1)

If $L < 1$, $A$ is called contraction and if $L = 1$, $A$ is called nonexpansive. We denote by $\text{Fix}(A)$ the set of fixed points of the mapping $A$, that is $\text{Fix}(A) := \{ x \in D(A) : x = Ax \}$. An operator $A : H \to H$ is said to be strongly positive bounded linear if there exists a constant $k > 0$ such that
\[ \langle Ax, x \rangle_H \geq k \| x \|^2, \quad \forall x \in H. \]

An operator $A : H \to H$ is called monotone if
\[ \langle Ax - Ay, x - y \rangle_H \geq 0, \quad \forall x, y \in H, \]
and it is called $k$-strongly monotone if there exists $k \in (0, 1)$ such that for each $x, y \in H$
\[ \langle Ax - Ay, x - y \rangle_H \geq k \| x - y \|^2. \]

Remark 1.1. From the definition of $A$, we note that strongly positive bounded linear operator $A$ is a $\| A \|$-Lipschitzian and $k$-strongly monotone operator.

Recently, iterative methods for nonexpansive mappings have been applied to solve convex minimization problems; see, e.g., [35, 21] and the references therein. A typical problem is to minimize a quadratic function over the set of fixed points of nonexpansive mappings in real Hilbert spaces:
\[ \min_{x \in \text{Fix}(T)} \frac{1}{2} \langle Ax, x \rangle - \langle b, x \rangle. \] (2)

In [35], Xu proved that the sequence $\{ x_n \}$ defined iteratively from arbitrary initial guess $x_0 \in H$ by:
\[ x_{n+1} = \alpha_n b + (I - \alpha_n A)Tx_n, \quad n \geq 0, \] (3)
converges strongly to the unique solution of the minimization problem (2), where $T$ is a nonexpansive mapping in $H$ and $A$ is a strongly positive bounded linear operator. Moudafi, in [23] introduced the viscosity approximation method. Marino and Xu [21] improved the result of Moudafi by considering a general iterative method for nonexpansive mappings : let $f$ be a contraction map on
H and $A : H \to H$ be a strongly positive bounded linear operator. Let $\{x_n\}$ be the sequence defined iteratively from arbitrary initial guess $x_0 \in H$ by:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad n \geq 0. \quad (4)$$

He then proved that the sequence $\{x_n\}$ converges strongly to the fixed point of $T$, which is the unique solution of the following variational inequality

$$\langle Ax^* - \gamma f(x^*), x^* - p \rangle \leq 0, \quad \forall p \in \text{Fix}(T).$$

However, it is noted that few results of iterative solutions for multivalued mappings are established.

Let $(X, d)$ be a metric space, $K$ be a nonempty subset of $X$ and $T : K \to 2^K$ be a multivalued mapping. An element $x \in K$ is called a fixed point of $T$ if $x \in Tx$. The fixed point set of $T$ is denoted by $\text{Fix}(T) := \{x \in D(T) : x \in Tx\}$. It is easy to see that single-valued mapping is a particular case of multivalued mapping.

The theory of set-valued mappings has applications in control theory, convex optimization, differential equations and economics. Fixed point theory for set-valued mappings has been studied by many authors, (see, for example, Brouwer [2], Kakutani [18], Nash [27, 28], Geanakoplos [15], Nadla [26], Downing and Kirk [12], Chidume et al. [8, 11, 9, 10], Djitté et al. [14], Diop et al. [13], Sene et al. [32], Sow et al. [34]).

Interest in the study of fixed point theory for multi-valued nonlinear mappings stems, perhaps, mainly from its usefulness in real-world applications such as Game Theory and Non-Smooth Differential Equations.

Let $D$ be a nonempty subset of a normed linear space $E$. The set $D$ is called proximinal (see, e.g., [29]) if for each $x \in E$, there exists $u \in D$ such that

$$d(x, u) = \inf \{\|x - y\| : y \in D\} = d(x, D),$$

where $d(x, y) = \|x - y\|$ for all $x, y \in E$. Every nonempty, closed and convex subset of a real Hilbert space is proximinal. Let $CB(D)$, $K(D)$ and $P(D)$ denote the family of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximinal bounded subsets of $D$ respectively. The Hausdorff metric on $CB(D)$ is defined by:

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}$$
for all $A, B \in CB(D)$. A multi-valued mapping $T : D(T) \subseteq E \to CB(E)$ is
called $L$-Lipschitzian if there exists $L > 0$ such that
\[ H(Tx, Ty) \leq L\|x - y\| \quad \forall x, y \in D(T). \] (5)
When $L \in (0, 1)$, we say that $T$ is a contraction, and $T$ is called nonexpansive
if $L = 1$.
A multivalued map $T$ is called quasi-nonexpansive if
\[ H(Tx, Tp) \leq \|x - p\| \]
holds for all $x \in D(T)$ and $p \in Fix(T)$. A mapping $T : D \to CB(D)$ is said
to be nonsparing-type multi-valued mapping [3] if
\[ 2H(Tx, Ty)^2 \leq d(x, Ty)^2 + d(y, Tx)^2, \quad x, y \in D. \]
It is easy to see that, if $T$ is a nonsparing-type multivalued mapping, and
$Fix(T) \neq \emptyset$, then $T$ is a quasi-nonexpansive multivalued mapping.

The minimization problem (MP) is one of the most important problems in
nonlinear analysis and optimization theory. The MP is defined as follows: find
\( x \in H, \) such that
\[ g(x) = \min_{y \in H} g(y), \]
where $g : H \to (-\infty, +\infty)$ is a proper convex and lower semi-continuous.
The set of all minimizers of $g$ on $H$ is denoted by $\text{argmin}_{y \in H} g(y)$. A successful
and powerful tool for solving this problem is the well-known proximal point
algorithm (shortly, the PPA) which was initiated by Martinet [22] in 1970 and
later studied by Rockafellar [31] in 1976. Let $H$ be a real Hilbert space and
$g : H \to R$ be a proper lower semi-continuous and convex function. The PPA
is defined as follows:
\[
\begin{aligned}
& x_1 \in H, \\
& x_{n+1} = \text{argmin}_{y \in H} \left[ g(y) + \frac{1}{2\lambda_n} \|x_n - y\|^2 \right],
\end{aligned}
\] (6)
where $\lambda_n > 0$ for all $n \geq 1$. It was proved that the sequence $\{x_n\}$ converges
weakly to a minimizer of $g$ provided $\sum_{n=0}^{\infty} \lambda_n = \infty$. In [31] Rockafellar proved
that the sequence $\{x_n\}$ given by (6) converges weakly to a minimizer of $g$. He
then posed the following question: does the sequence $\{x_n\}$ converges strongly?
This question was resolved in the negative by Guler [17] (1991). He produced
a proper lower semi continuous and convex function $g$ in $l_2$ for which the PPA
converges weakly but not strongly. This leads naturally to the following ques-
tion: Can the PPA be modified to guarantee strong convergence? In response
to Q2, several works have been done (see, e.g., Güler [17], Solodov and Svaiter [33], Kamimura and Takahashi [19], Lehdili and Moudafi [20], Reich, [30], Chidume and Djitte [6, 7] and the references therein).

Over years, researcher have been able to further extend the convex minimization problems by investigating on problems consisting of finding common elements of sets of solutions of various convex minimization problems and sets of fixed points of single and multivalued mappings in Banach spaces.

Recently, Chang et al. [4] motivated by the fact that the proximal point algorithm is remarkably useful for solving fixed point problems and convex minimization problems, proved the following theorem.

**Theorem 1.2** (Chang et al. [3]). Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$ and $g : C \rightarrow (-\infty, +\infty)$ be a proper convex and lower semi-continuous function. Let $T : K \rightarrow K(C)$ be a multivalued nonspradling-type multivalued mapping such that $\Omega := \text{Fix}(T) \cap \text{argmin}_{u \in C} g(u) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated as follows:

\[
\begin{align*}
x_1 & \in C, \\
n_n & = \text{argmin}_{y \in H} \left[ g(y) + \frac{1}{2\lambda_n} \| x_n - y \|^2 \right], \\
zw_n & = (1 - \beta_n) x_n + \beta_n w_n, \\
x_{n+1} & = (1 - \alpha_n) x_n + \alpha_n v_n,
\end{align*}
\]

where $\{\alpha_n\} \subset (0, 1)$ and $\{\beta_n\}$ are real sequences in $[0, 1]$ such that $0 < a \leq \alpha_n$, $\beta_n \leq b < 1$ for all $n \geq 1$, and $\{\lambda_n\}$ is a real sequence such that $\lambda_n \geq \lambda > 0$ for all $n \geq 1$ and some $\lambda$. Then the sequence $\{x_n\}$ converges weakly to an element of $\Omega$.

It is our purpose in this paper to construct a new algorithm for approximating common elements to the set of minimizers of proper lower semicontinuous convex function and the set fixed points multivalued quasi-nonexpansive mappings in Hilbert spaces. Then, under appropriate conditions, we establish some strong convergence theorems. The class of multivalued quasi-nonexpansive mappings contains those of multivalued nonexpansive and mappings and multivalued nonspreading mappings as subclasses. The results obtained here extend and unify the result of Chang et al. [3] and most of the recent results in this direction. Our technique of proof is of independent interest.

### 2 Preliminaries

The following lemmas will be useful in the sequel. Let $H$ be a real Hilbert space. Let $\{x_n\}$ be a sequence in $H$, and $x \in H$. We denote the weak convergence of $x_n$ to $x$ by $x_n \rightharpoonup x$ and the strong convergence $x_n$ to $x$ by $x_n \rightarrow x$. 
Let $K$ be a nonempty, closed convex subset of $H$. The nearest point projection from $H$ to $K$ denoted by $P_K$, assigns to each $x \in H$ the unique point of $K$, $P_Kx$ such that
\[ \|x - P_Kx\| \leq \|x - y\|, \]
for all $y \in K$. It is well known that for every $x \in H$,
\[ \langle x - P_Kx, y - P_Kx \rangle \leq 0, \]
for all $y \in K$.

**Definition 2.1.** Let $H$ be a real Hilbert space and $T : D(T) \subseteq H \to 2^H$ be a multivalued mapping. The multivalued map \( I - T \) is said to be demiclosed at 0 if for any sequence \( \{x_n\} \subseteq D(T) \) such that \( \{x_n\} \) converges weakly to $p$ and \( d(x_n, Tx_n) \) converges to zero, then $p \in Tp$, where $I$ is the identity map of $H$.

**Lemma 2.2 (Cholamjiak, [4]).** Let $H$ be a real Hilbert space and $C$ be a nonempty closed and convex subset of $H$. Let $T : C \to K(C)$ be a multivalued nonspreading mapping. Then \( I - T \) is demi-closed at zero.

**Lemma 2.3 (Chidume, [5]).** Let $H$ be a real Hilbert space. Then, for every $x, y \in H$, and every $\lambda \in [0, 1]$, the following hold:
\[ \|x + y\|^2 \leq \|x\|^2 + 2\langle x, x + y \rangle. \]
\[ \|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - (1 - \lambda)\lambda\|x - y\|^2, \quad \lambda \in (0, 1). \]

**Lemma 2.4 (Xu, [36]).** Assume that \( \{a_n\} \) is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \alpha_n)a_n + \sigma_n$ for all $n \geq 0$, where \( \{\alpha_n\} \) is a sequence in $(0, 1)$ and \( \{\sigma_n\} \) is a sequence in $\mathbb{R}$ such that
\[ (a) \sum_{n=0}^{\infty} \alpha_n = \infty, \quad (b) \limsup_{n \to \infty} \frac{\sigma_n}{\alpha_n} \leq 0 \text{ or } \sum_{n=0}^{\infty} |\sigma_n| < \infty. \] Then $\lim_{n \to \infty} a_n = 0$.

**Lemma 2.5 (Wang, [37]).** Let $H$ be a real Hilbert space. Let $K$ be a nonempty, closed convex subset of $H$ and $A : K \to H$ be a $k$-strongly monotone and $L$-Lipschitzian operator with $k > 0$, $L > 0$. Assume that $0 < \eta < \frac{2k}{L^2}$ and \( \tau = \eta \left( k - \frac{L^2 \eta}{2} \right) \). Then for each $t \in \left( 0, \min\{1, \frac{1}{\tau}\} \right)$, we have
\[ \|(I - t\eta A)x - (I - t\eta A)y\| \leq (1 - t\tau)\|x - y\|, \quad x, y \in K. \]
Iterative algorithm

Let $H$ be a real Hilbert space and $K$ be a nonempty convex subset of $H$. Let $g : K \rightarrow (-\infty, +\infty)$ be a proper, lower semi-continuous and convex function. For every $\lambda > 0$, the Moreau-Yosida resolvent of $g$, $J^g_\lambda$ is defined by:

$$J^g_\lambda x = \arg\min_{u \in K} \left[ g(u) + \frac{1}{2\lambda} \|x - u\|^2 \right],$$

for all $x \in H$. It was shown in [17] that the set of fixed points of the resolvent associated to $g$ coincides with the set of minimizers of $g$. Also, the resolvent $J^g_\lambda$ of $g$ is nonexpansive for all $\lambda > 0$ (see, e.g., [21]).

**Lemma 2.6.** (Miyadera [25]) Let $H$ be a real Hilbert space and $K$ be a nonempty convex subset of $H$. Let $g : K \rightarrow (-\infty, +\infty)$ be a proper, lower semi-continuous and convex function. For every $r > 0$ and $\mu > 0$, the following holds:

$$J^g_r x = J^g_\mu x(\frac{\mu}{r} x + (1 - \frac{\mu}{r})J^g_r x).$$

**Lemma 2.7** (Sub-differential inequality, Ambrosio et al., [1]). Let $H$ be a real Hilbert space and and $g : H \rightarrow (-\infty, +\infty)$ be a proper, lower semicontinuous and convex function. Then, for every $x, y \in H$ and $\lambda > 0$, the following sub-differential inequality holds:

$$\frac{1}{\lambda} \|J^g_\lambda x - y\|^2 - \frac{1}{\lambda} \|x - y\|^2 + \frac{1}{\lambda} \|x - J^g_\lambda x\|^2 + g(J^g_\lambda x) \leq f(y). \quad (9)$$

**Lemma 2.8** (Mainge, [24]). Let $\{t_n\}$ be a sequence of real numbers that does not decreases at infinity in the sense that there exists a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that $t_{n_i} \leq t_{n_{i+1}}$, for all $i \geq 0$. For $n \in \mathbb{N}$, sufficiently large, let $\{\tau(n)\}$ be the sequence of integers defined as follows:

$$\tau(n) = \max\{k \leq n : t_k \leq t_{k+1}\}.$$

Then, $\tau(n) \rightarrow \infty$ as $n \rightarrow \infty$ and

$$\max\{t_{\tau(n)}, t_n\} \leq t_{\tau(n)+1}.$$

**3 Main Results**

The main objective of this section is to study a new iterative algorithm for approximating a common element of the set of minimizers of a proper, lower semicontinuous and convex function and the set of fixed points of a multivalued quasi-nonexpansive mappings.
Theorem 3.1. Let $K$ be a nonempty, closed convex subset of a real Hilbert space $H$ and $g : K \to (-\infty, +\infty)$ be a proper, lower semi-continuous and convex function. Let $f : K \to H$ be an $b$-Lipschitzian mapping and $T : K \to \overline{CB}(K)$ be a multivalued quasi-nonexpansive mapping such that $F := \text{Fix}(T) \cap \text{argmin}_{u \in K} g(u) \neq \emptyset$ and $Tp = \{p\}$ $\forall p \in \text{Fix}(T)$. Let $A : K \to H$ be an $k$-strongly monotone and $L$-Lipschitzian operator. Assume that $0 < \eta < 2k/L^2$, $0 < \gamma b < \tau$, where $\tau = \eta\left(k - \frac{L^2\eta}{2}\right)$ and $I - T$ is demiclosed at the origin. Let $\{x_n\}$ be a sequence defined as follows:

\[
\begin{align*}
  x_0 &\in K, \\
  z_n &= \text{argmin}_{u \in K} \left[ g(u) + \frac{1}{2\lambda_n} \|u - x_n\|^2 \right], \\
  y_n &= \beta_n z_n + (1 - \beta_n)w_n, \quad w_n \in Tz_n, \\
  x_{n+1} &= P_K(\alpha_n \gamma f(x_n) + (I - \eta \alpha_n A)y_n),
\end{align*}
\]

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ such that $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim \inf \beta_n(1 - \beta_n) > 0$ and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \geq 1$ and some $\lambda$. Then, the sequence $\{x_n\}$ generated by (10) converges strongly to $x^* \in F$, the unique solution of the variational inequality problem:

\[
\langle \eta Ax^* - \gamma f(x^*), x^* - p \rangle \leq 0, \quad \forall p \in F.
\]

Proof. Without loss of generality, we can assume that $\alpha_n \in \left(0, \min\{1, \frac{1}{\tau}\}\right)$.

We prove that the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. For this, let $p \in F$. Then, $Tp = \{p\}$ and $g(p) \leq g(u)$ for all $u \in K$. It then follows that

\[
g(p) + \frac{1}{2\lambda_n} \|p - p\|^2 \leq g(u) + \frac{1}{2\lambda_n} \|u - p\|^2
\]

Hence, $J_{\lambda_n}^g p = p$ for all $n \geq 1$, where $J_{\lambda_n}^g$ is the Moreau-Yosida resolvent of $g$ in $K$. Since, $J_{\lambda_n}^g$ is nonexpansive, we have

\[
\|z_n - p\| = \|J_{\lambda_n}^g x_n - p\| \leq \|x_n - p\|, \quad \forall n \geq 0.
\]

From (10) and the fact that $Tp = \{p\}$, we have

\[
\|y_n - p\| = \|\beta_n z_n + (1 - \beta_n)w_n - p\|
\leq \beta_n \|z_n - p\| + (1 - \beta_n) \|w_n - p\|
\leq \beta_n \|z_n - p\| + (1 - \beta_n) H(Tz_n, Tp)
\leq \beta_n \|z_n - p\| + (1 - \beta_n) \|z_n - p\|.
\]
By induction, it is easy to see that
\[ \|y_n - p\| \leq \|z_n - p\| \leq \|x_n - p\|. \] (12)
Using (10), inequality (12) and Lemma 2.5, we have
\[
\|x_{n+1} - p\| \leq \|\alpha_n \gamma f(x_n) + (I - \eta \alpha_n A)y_n - p\|
\leq \alpha_n \gamma \|f(x_n) - f(p)\| + (1 - \tau \alpha_n)\|y_n - p\| + \alpha_n \|\gamma f(p) - \eta A p\|
\leq (1 - \alpha_n (\tau - b \gamma))\|x_n - p\| + \alpha_n \|\gamma f(p) - \eta A p\|
\leq \max \{\|x_n - p\|, \frac{\|\gamma f(p) - \eta A p\|}{\tau - b \gamma}\}.
\]
By induction, it is easy to see that
\[ \|x_n - p\| \leq \max \{\|x_0 - p\|, \frac{\|\gamma f(p) - \eta A p\|}{\tau - b \gamma}\}, \quad n \geq 1. \]
Hence \{x_n\} is bounded also are \{f(x_n)\}, and \{Ax_n\}.

Using Lemma 2.3 and (10), we have
\[
\|y_n - p\|^2 = \|\beta_n z_n + (1 - \beta_n)w_n - p\|^2
\leq (1 - \beta_n)\|w_n - p\|^2 + \beta_n \|z_n - p\|^2 - \beta_n (1 - \beta_n)\|w_n - z_n\|^2
\leq (1 - \beta_n)\|x_n - p\|^2 + \beta_n \|x_n - p\|^2 - \beta_n (1 - \beta_n)\|z_n - w_n\|^2.
\]
Hence,
\[ \|y_n - p\|^2 \leq \|x_n - p\|^2 - \beta_n (1 - \beta_n)\|z_n - w_n\|^2. \] (13)
Therefore, by Lemma 2.5, inequalities (13) and (12), we have
\[
\|x_{n+1} - p\|^2 \leq \|\alpha_n (\gamma f(x_n) - \eta A p) + (I - \eta \alpha_n A)(y_n - p)\|^2
\leq \alpha_n^2 \|\gamma f(x_n) - \eta A p\|^2 + (1 - \tau \alpha_n)^2\|y_n - p\|^2 + 2 \alpha_n (1 - \tau \alpha_n)\|\gamma f(x_n) - \eta A p\|\|y_n - p\|
\leq \alpha_n^2 \|\gamma f(x_n) - \eta A p\|^2 + (1 - \tau \alpha_n)^2\|x_n - p\|^2 - (1 - \tau \alpha_n)^2 \beta_n (1 - \beta_n)\|z_n - w_n\|^2
+ 2 \alpha_n (1 - \tau \alpha_n)\|\gamma f(x_n) - \eta A p\|\|x_n - p\|.
\]
Since \{x_n\} is bounded, then there exists a constant \(B > 0\) such that
\[ (1 - \tau \alpha_n)^2 \beta_n (1 - \beta_n)\|z_n - w_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2 \alpha_n B. \] (14)
We show the uniqueness of the solution of the variational inequality problem (11). Suppose both \(x^* \in F\) and \(x^{**} \in F\) are solutions to (11). Then,
\[ \langle \eta Ax^* - \gamma f(x^*), x^* - x^{**} \rangle \leq 0 \] (15)
and
\[ \langle \eta Ax^{**} - \gamma f(x^{**}), x^{**} - x^* \rangle \leq 0. \] (16)
Adding up (15) and (16) yields

\[ \langle \eta Ax^* - \eta Ax - \gamma f(x) - \gamma f(x^*), x^* - x \rangle \leq 0. \]  

(17)

Observing that,

\[ \frac{L^2}{2} \eta > 0 \iff k - \frac{L^2}{2} \eta < k \]
\[ \iff \eta(k - \frac{L^2}{2} \eta) < k \eta \]
\[ \iff \tau < k \eta, \]

it follows that

\[ 0 < b \gamma < \tau < k \eta. \]

Noticing that

\[ \langle \eta Ax^* - \eta Ax + \gamma f(x) - \gamma f(x^*), x^* - x \rangle \geq (k \eta - b \gamma) \| x^* - x^* \|^2, \]

we conclude that \( x^* = x^* \) which proves the uniqueness.

In what follows, we denote \( x^* \) to be the unique solution of (11). Let \( t_0 \) be a fixed real number in \( t_0 \in \left( 0, \min \{1, \frac{1}{\tau} \} \right) \). We observe that \( P_F(I + (t_0 \gamma f - t_0 \eta A)) \) is a contraction. Indeed, for all \( x, y \in H \), by Lemma 2.5, we have

\[
\| P_F(I + (t_0 \gamma f - t_0 \eta A))x - P_F(I + (t_0 \gamma f - t_0 \eta A))y \| \leq \|(I + (t_0 \gamma f - t_0 \eta A))x - (I + (t_0 \gamma f - t_0 \eta A))y\|
\]
\[
\leq t_0 \gamma \| f(x) - f(y) \|
\]
\[
+ \|(I - t_0 \eta A)x - (I - t_0 \eta A)y\|
\]
\[
\leq (1 - t_0(\tau - b \gamma)) \| x - y \|.
\]

Banach’s Contraction Mapping Principle guarantees that \( P_F(I + (t_0 \gamma f - t_0 \eta A)) \) has a unique fixed point, say \( x_1 \in H \). That is, \( x_1 = P_F(I + (t_0 \gamma f - t_0 \eta A))x_1 \).

Thus, in view of inequality (8), it is equivalent to the following variational inequality problem

\[ \langle \eta Ax_1 - \gamma f(x_1), x_1 - p \rangle \leq 0, \quad \forall \ p \in F. \]

By the uniqueness of the solution of (11), we have \( x_1 = x^* \). Now we prove that \( \{x_n\} \) converges strongly to \( x^* \). We divide the proof into two cases.

**Case 1.** Assume that the sequence \( \{\|x_n - p\|\} \) is monotonically decreasing sequence. Then \( \{\|x_n - p\|\} \) is convergent. Clearly, we have

\[
\lim_{n \to \infty} \left[ \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \right] = 0.
\]

(18)
Iterative algorithm

It then implies from (14) that

\[ \lim_{n \to \infty} (1 - \tau \alpha_n)^2 \beta_n (1 - \beta_n) \|z_n - w_n\|^2 = 0. \]  

(19)

Using the fact that \( \beta_n \in [a, b] \subset (0, 1) \), we have

\[ \lim_{n \to \infty} \|z_n - w_n\| = 0. \]  

(20)

Hence,

\[ \lim_{n \to \infty} d(z_n, T z_n) = 0. \]  

(21)

Let \( p \in F \). Using Lemma 2.7 and since \( g(p) \leq g(z_n) \), we get

\[ \|x_n - z_n\|^2 \leq \|x_n - p\|^2 - \|z_n - p\|^2. \]  

(22)

Therefore, from (10), inequality (22) Lemmas 2.3 and 2.5, we get that

\[
\|x_{n+1} - p\|^2 \leq \|a_n \gamma f(x_n) + (I - \eta \alpha_n A) y_n - p\|^2 \\
\leq (1 - \tau \alpha_n)^2 \|y_n - p\|^2 + 2a_n \langle \gamma f(x_n) - \eta Ap, x_{n+1} - p \rangle \\
\leq (1 - \tau \alpha_n)^2 \|z_n - p\|^2 + 2a_n \gamma \langle f(x_n) - f(p), x_{n+1} - p \rangle + 2a_n \langle \gamma f(p) - \eta Ap, x_{n+1} - p \rangle \\
\leq (1 - \tau \alpha_n)^2 \|x_n - p\|^2 - \|x_n - z_n\|^2 + 2a_n \gamma \|x_n - p\| \|x_{n+1} - p\| \\
+ 2a_n \| \gamma f(p) - \eta Ap \| \|x_{n+1} - p\| \\
= (1 - 2\tau \alpha_n + (\tau \alpha_n)^2)\|x_n - p\|^2 - (1 - \tau \alpha_n)^2 \|x_n - z_n\|^2 + 2a_n \gamma \|x_n - p\| \|x_{n+1} - p\| \\
+ 2\alpha_n \| \gamma f(p) - \eta Ap \| \|x_{n+1} - p\| \\
\leq \|x_n - p\|^2 + \tau^2 \alpha_n \|x_n - p\|^2 - (1 - \tau \alpha_n)^2 \|x_n - z_n\|^2 + 2a_n \gamma \|x_n - p\| \|x_{n+1} - p\| \\
+ 2\alpha_n \| \gamma f(p) - \eta Ap \| \|x_{n+1} - p\|,
\]

and hence

\[
(1 - \tau \alpha_n)^2 \|x_n - z_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \tau^2 \alpha_n \|x_n - p\|^2 + 2a_n \gamma \|x_n - p\| \|x_{n+1} - p\| \\
+ 2\alpha_n \| \gamma f(p) - \eta Ap \| \|x_{n+1} - p\|.
\]

Hence,

\[ \lim_{n \to \infty} \|x_n - z_n\| = 0. \]  

(23)

Next, we prove that \( \limsup_{n \to +\infty} \langle \eta Ax^* - \gamma f(x^*), x^* - x_n \rangle \leq 0 \). Since \( H \) is a Hilbert space and \( \{x_n\} \) is bounded, there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) which converges weakly to \( \omega \) in \( K \) and

\[
\limsup_{n \to +\infty} \langle \eta Ax^* - \gamma f(x^*), x^* - x_n \rangle = \lim_{k \to +\infty} \langle \eta Ax^* - \gamma f(x^*), x^* - x_{n_k} \rangle.
\]

From (21), inequality (23) and \( I - T \) is demiclosed, we obtain \( \omega \in Fix(T) \).
Using (10) and Lemma 2.6, we arrive at
\[
\|x_n - J_\lambda^g x_n\| \leq \|z_n - J_\lambda^g x_n\| + \|z_n - x_n\|
\]
\[
\leq \|J_\lambda^g x_n - J_\lambda^g x_n\| + \|z_n - x_n\|
\]
\[
\leq \|z_n - x_n\| + \|J_\lambda^g \left( \frac{\lambda_n - \lambda}{\lambda_n} J_\lambda^g x_n + \frac{\lambda}{\lambda_n} x_n \right) - J_\lambda^g x_n\|
\]
\[
\leq \|z_n - x_n\| + \|\frac{\lambda_n - \lambda}{\lambda_n} J_\lambda^g x_n + \frac{\lambda}{\lambda_n} x_n - x_n\|
\]
\[
\leq \|z_n - x_n\| + \left( 1 - \frac{\lambda}{\lambda_n} \right) \|z_n - x_n\|
\]
\[
\leq (2 - \frac{\lambda}{\lambda_n}) \|z_n - x_n\|.
\]

Hence,
\[
\lim_{n \to \infty} \|x_n - J_\lambda^g x_n\| = 0. \tag{24}
\]

Since $J_\lambda^g$ is single valued and nonexpansive, using (24) and Lemma 2.2, then $\omega \in Fix(J_\lambda^g) = \arg \min_{u \in K} g(u)$. Therefore, $\omega \in F$. Hence,
\[
\lim \sup_{n \to +\infty} \langle \eta Ax^* - \gamma f(x^*), x^* - x_n \rangle = \lim_{k \to +\infty} \langle \eta Ax^* - \gamma f(x^*), x^* - x_{n_k} \rangle
\]
\[
= \langle \eta Ax^* - \gamma f(x^*), x^* - \omega \rangle \leq 0.
\]

Finally, we show that $x_n \to x^*$. From (10), inequality (12) and Lemma 2.5, we get that
\[
\|x_{n+1} - x^*\|^2 \leq \|a_n \gamma f(x_n) + (I - a_n \eta A)y_n - x^*\|^2
\]
\[
\leq \|a_n (\gamma f(x_n) - \gamma f(x^*)) + (I - a_n \eta A)(y_n - x^*)\|^2 + 2a_n \langle \eta Ax^* - \gamma f(x^*), x^* - x_{n+1} \rangle
\]
\[
\leq (a_n \|f(x_n) - f(x^*)\| + \|I - a_n \eta A\| (y_n - x^*))^2 + 2a_n \langle \eta Ax^* - \gamma f(x^*), x^* - x_{n+1} \rangle
\]
\[
\leq \left[ 1 - (\tau - b\eta)a_n \right] \|x_n - x^*\|^2 + 2a_n \langle \eta Ax^* - \gamma f(x^*), x^* - x_{n+1} \rangle
\]
\[
\leq \left[ 1 - (\tau - b\eta)a_n \right] \|x_n - x^*\|^2 + 2a_n \langle \eta Ax^* - \gamma f(x^*), x^* - x_{n+1} \rangle.
\]

We can check that all the assumptions of Lemma 2.4 are satisfied. Therefore, we deduce $x_n \to x^*$.

**Case 2.** Assume that the sequence $\{\|x_n - x^*\|\}$ is not monotonically decreasing sequence. Set $B_n = \|x_n - x^*\|$ and $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some $n_0$ large enough) by $\tau(n) = \max\{k \in \mathbb{N} : k \leq n, B_k \leq B_{k+1}\}$. We have $\tau$ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $B_{\tau(n)} \leq B_{\tau(n)+1}$ for $n \geq n_0$. From (14), we have
\[
(1 - \alpha_{\tau(n)} \gamma^2 \beta_{\tau(n)} (1 - \beta_{\tau(n)})) \|z_{\tau(n)} - w_{\tau(n)}\|^2 \leq 2\alpha_{\tau(n)} B \to 0 \text{ as } n \to \infty.
\]

Furthermore, we have
\[
\lim_{n \to \infty} \|z_{\tau(n)} - w_{\tau(n)}\| = 0.
\]
Hence,

\[
\lim_{n \to \infty} d\left(z_{\tau(n)}, Tz_{\tau(n)}\right) = 0.
\] (25)

By similar argument as above in Case 1, we conclude immediately that

\[
\lim_{n \to +\infty} \langle \eta Ax^* - \gamma f(x^*), x^* - x_{\tau(n)} \rangle \leq 0
\]

for all \(n \geq n_0\), where

\[
0 \leq \|x_{\tau(n)+1} - x^*\|^2 - \|x_{\tau(n)} - x^*\|^2 \leq \alpha_{\tau(n)}[-(\tau-b)\gamma \|x_{\tau(n)} - x^*\|^2 + 2(\eta Ax^* - \gamma f(x^*), x^* - x_{\tau(n)+1})]
\]

which implies that

\[
\|x_{\tau(n)} - x^*\|^2 \leq \frac{2}{\tau - b\gamma} \langle \eta Ax^* - \gamma f(x^*), x^* - x_{\tau(n)+1} \rangle.
\]

Then, we have

\[
\lim_{n \to \infty} \|x_{\tau(n)} - x^*\|^2 = 0.
\]

Therefore,

\[
\lim_{n \to \infty} B_{\tau(n)} = \lim_{n \to \infty} B_{\tau(n)+1} = 0.
\]

Thus, by Lemma 2.8, we conclude that

\[
0 \leq B_n \leq \max\{B_{\tau(n)}, B_{\tau(n)+1}\} = B_{\tau(n)+1}
\]

Hence, \(\lim_{n \to \infty} B_n = 0\), that is \(\{x_n\}\) converges strongly to \(x^*\). This completes the proof.

We now apply Theorem 3.1 to approximate common element of the set of minimizers of a proper lower semi-continuous convex function and the set of fixed points of a multivalued nonspreading-type mappings without any compactness type assumption assumption.

**Theorem 3.2.** Let \(K\) be a nonempty, closed convex subset of a real Hilbert space \(H\) and \(g : K \to (-\infty, +\infty)\) be a proper lower semi-continuous convex function. Let \(f : K \to H\) be an \(L\)-Lipschitzian mapping and \(T : K \to CB(K)\) be a multivalued nonspreading mapping such that \(F := Fix(T) \cap \text{argmin}_{u \in K} g(u) \neq \emptyset\) and \(Tp = \{p\} \ \forall p \in Fix(T)\). Let \(A : K \to H\) be an \(k\)-strongly monotone and \(L\)-Lipschitzian operator. Assume that \(0 < \eta < \frac{2k}{L^2}, 0 < \gamma b < \tau\), where \(\tau = \eta \left(k - \frac{L^2\eta}{2}\right)\).

Let \(\{x_n\}\) be a sequence defined as follows:

\[
\begin{align*}
x_0 &\in K, \\
z_n &= \text{argmin}_{u \in K}\left[ g(u) + \frac{1}{2\lambda_n} \|x_n - u\|^2 \right], \\
y_n &= \beta_n z_n + (1 - \beta_n)w_n, \quad w_n \in Tz_n, \\
x_{n+1} &= P_K(\alpha_n \eta f(x_n) + (I - \eta\alpha_n A)y_n),
\end{align*}
\] (26)
where \( \{ \alpha_n \} \) and \( \{ \beta_n \} \) are sequences in \([0,1]\) such that \( \lim_{n \to \infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty, \) \( \lim \inf_{n \to \infty} \beta_n(1 - \beta_n) > 0 \) and \( \{ \lambda_n \} \) is a sequence such that \( \lambda_n \geq \lambda > 0 \) for all \( n \geq 1 \) and some \( \lambda. \) Then, the sequence \( \{ x_n \} \) generated by (26) converges strongly to \( x^* \in F, \) which is a together minimizer of \( g \) in \( K \) and a fixed of point of \( T \) in \( K. \)

**Proof.** Since every nonspreading-type multivalued is multivalued quasi-nonexpansive, then the proof follows Lemma 2.2 and Theorem 3.1.

If \( T : K \to K \) is a single-valued quasi-nonexpansive mapping and \( A \) be a strongly positive bounded linear operator, then the following theorem can be obtained from Theorem 3.1 and Remark 1.1 immediately.

**Theorem 3.3.** Let \( K \) be a nonempty, closed convex subset of a real Hilbert space \( H \) and \( g : K \to (\infty, +\infty) \) be a proper, lower semi-continuous and convex function. Let \( f : K \to H \) be an \( h \)-Lipschitzian mapping and \( T : K \to K \) be a multivalued quasi-nonexpansive mapping such that \( F := \text{Fix}(T) \cap \text{argmin}_{u \in K} g(u) \neq \emptyset. \) Let \( A : K \to H \) be strongly bounded linear operator with coefficient \( k > 0. \) Assume that \( 0 < \eta < \frac{2k}{\|A\|^2}, 0 < \gamma b < \tau, \) where \( \tau = \eta \left( k - \frac{\|A\|^2 \eta}{2} \right) \) and \( I - T \) is demiclosed at the origin.

Let \( \{ x_n \} \) be a sequence defined as follows:

\[
\begin{align*}
  x_0 & \in K, \\
  z_n & = J_{\lambda_n}^g x_n, \\
  y_n & = \beta_n z_n + (1 - \beta_n) T z_n, \\
  x_{n+1} & = P_K(\alpha_n \gamma f(x_n) + (I - \eta \alpha_n A)y_n),
\end{align*}
\]

(27)

where \( \{ \alpha_n \} \) and \( \{ \beta_n \} \) are sequences in \([0,1]\) such that \( \lim_{n \to \infty} \alpha_n = 0, \sum_{n=0}^{\infty} \alpha_n = \infty, \) \( \lim \inf_{n \to \infty} \beta_n(1 - \beta_n) > 0 \) and \( \{ \lambda_n \} \) is a sequence such that \( \lambda_n \geq \lambda > 0 \) for all \( n \geq 1 \) and some \( \lambda. \) Then, the sequence \( \{ x_n \} \) generated by (27) converges strongly to \( x^* \in F, \) which satisfies the optimality condition of the minimization problem:

\[
\min_{x \in F} \frac{\eta}{2} \langle Ax, x \rangle - h(x),
\]

(28)

where \( h \) is a potential function for \( \gamma f \) (i.e. \( h'(x) = \gamma f(x) \) on \( H \)).
Remark 3.4. It is known that the class of multivalued quasi-nonexpansive mappings contains the classes of multivalued nonexpansive mappings and that of multivalued nonspreading mappings. The method of proof used in our work is different from the one of Chang et al. [3] for multivalued nonspreading-type mappings. Our results improve many recent results involving the proximal point algorithm for solving the problem of finding common elements of the set of minimizers of a convex proper lower semi-continuous function and the set of fixed points of certain multivalued mappings.

References


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