Gelin-Cesàro Identities for Generalized Fibonacci Sequences

Younseok Choo

Department of Electronic and Electrical Engineering
Hongik University
2639 Sejong-Ro, Sejong, 30016, Korea

In this paper we derive the Gelin-Cesàro identities for two types of generalized Fibonacci sequences.

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1 Introduction

The classical Fibonacci sequence \( \{F_n\} \) is a sequence of nonnegative integers defined by the recurrence relation

\[
F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad (n \geq 2).
\]

The Binet’s formula for \( \{F_n\} \) is given by

\[
F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},
\]

where \( \alpha \) and \( \beta \) are solutions of the equation \( x^2 - x - 1 = 0 \).

As is well known, the following identities hold for the Fibonacci sequence:

\[
F_n^2 - F_{n-r}F_{n+r} = (-1)^{n-r}F_r^2 \quad \text{(Catalan’s identity)},
\]

\[
F_{n-2}F_{n-1}F_{n+1}F_{n+2} = F_n^4 - 1 \quad \text{(Gelin-Cesàro identity)}.
\]
Morgado [6] showed that (3) can be easily obtained from (2).

Many authors generalized the Fibonacci sequence by changing initial conditions and/or recurrence relation. Horadam [4] considered the sequence \( \{W_n\} \) defined by

\[
W_0 = w_0, \quad W_1 = w_1, \quad W_n = aW_{n-1} + bW_{n-2} \quad (n \geq 2).
\]

(4)

The Binet’s formula for \( \{W_n\} \) is [4]

\[
W_n = \frac{(W_1 - W_0 \beta) \alpha^n - (W_1 - W_0 \alpha) \beta^n}{\alpha - \beta},
\]

where \( \alpha \) and \( \beta \) are solutions of the equation \( x^2 - ax - b = 0 \).

The Catalan’s identity for \( \{W_n\} \) is given by [4]

\[
W_n^2 - W_{n-r}W_{n+r} = (-b)^{n-r}(w_1^2 - aw_0w_1 - bw_0^2)U_r^2,
\]

(5)

where \( \{U_n\} \) is defined by

\[
U_0 = 0, \quad U_1 = 1, \quad U_n = aU_{n-1} + bU_{n-2} \quad (n \geq 2).
\]

Using (5), Horadam and Shannon [5] derived the Gelin-Cesàro identity for \( \{W_n\} \) as follows:

\[
W_{n-2}W_{n-1}W_{n+1}W_{n+2} = W_n^4 - (-b)^{n-2}(a^2 - b)(w_1^2 - aw_0w_1 - bw_0^2)W_n^2
\]

\[\quad + (-b)^{2n-3}a^2(w_1^2 - aw_0w_1 - bw_0^2)^2.\]

(6)

Edson and Yayenie [3] introduced a sequence \( \{f_n\} \) defined by

\[
f_0 = 0, \quad f_1 = 1, \quad f_n = \begin{cases} af_{n-1} + f_{n-2}, & \text{if } n \text{ is even} \\ bf_{n-1} + f_{n-2}, & \text{if } n \text{ is odd} \end{cases} \quad (n \geq 2).
\]

(7)

Edson and Yayenie [3] derived the Binet’s formula for \( \{f_n\} \) as

\[
f_n = \frac{\alpha^{\zeta(n+1)}}{(ab)^{\left\lfloor \frac{n}{2} \right\rfloor}} \left( \frac{\alpha^n - \beta^n}{\alpha - \beta} \right),
\]

where \( \alpha \) and \( \beta \) are solutions of the equation \( x^2 - abx - ab = 0 \), and \( \zeta(\cdot) \) is the parity function such that \( \zeta(n) = 0 \) if \( n \) is even and \( \zeta(n) = 1 \) if \( n \) is odd.

The Catalan’s identity for \( \{f_n\} \) is [3]

\[
a^{\zeta(n)}b^{1-\zeta(n)}f_n^2 - a^{\zeta(n-r)}b^{1-\zeta(n-r)}f_{n-r}f_{n+r} = (-1)^{n-r}a^{\zeta(r)}b^{1-\zeta(r)}f_r^2.
\]

(8)

Using the Catalan’s identity given in (8), Sahin [8] obtained the Gelin-Cesàro identity for \( \{f_n\} \) as follows:

\[
f_{n-2}f_{n-1}f_{n+1}f_{n+2} = \left( \frac{b}{a} \right)^{1-2\zeta(n)}f_n^4 - (-1)^n \left( \frac{a}{b} \right)^{\zeta(n)}(ab - 1)f_n^2 - a^2.
\]

(9)
Sahin [8] and Yayenie [9] considered a sequence \( \{q_n\} \) defined by
\[
q_0 = 0, \quad q_1 = 1, \quad q_n = \begin{cases} 
  aq_{n-1} + cq_{n-2}, & \text{if } n \text{ is even} \\
  bq_{n-1} + dq_{n-2}, & \text{if } n \text{ is odd}
\end{cases} (n \geq 2).
\] (10)

According to Yayenie [9], the Binet’s formula for \( \{q_n\} \) is
\[
q_n = a^\frac{\zeta(n+1)}{(ab)^{\frac{n}{2}}} \left( \frac{\alpha^{\frac{n}{2}}(\alpha + d - c)^{\frac{n}{2}} - \beta^{\frac{n}{2}}(\beta + d - c)^{\frac{n}{2}}}{\alpha - \beta} \right),
\]
where \( \alpha \) and \( \beta \) are solutions of the equation \( x^2 - (ab + c - d)x - abd = 0 \).

For two nonnegative even integers \( n \) and \( r \) with \( n \geq r \), Sahin [8] derived the Catalan’s identity for \( \{q_n\} \) as follows:
\[
q_{2n}^2 - q_{n-r}^2 r_{n+r} = (cd)^{\frac{n-r}{2}} q_r^2.
\] (11)

Using (11), Sahin [8] also obtained the Gelin-Cesàro identity for \( \{q_n\} \). However, there is a slight mistake in [8, Theorem 3.5], and we will give a correct identity.

In this paper we consider two sequences \( \{p_n\} \) and \( \{q_n\} \), where \( \{p_n\} \) is generated from the recurrence relation
\[
p_n = \begin{cases} 
  ap_{n-1} + cp_{n-2}, & \text{if } n \text{ is even} \\
  bp_{n-1} + cp_{n-2}, & \text{if } n \text{ is odd}
\end{cases} (n \geq 2),
\] (12)
with arbitrary initial conditions \( p_0 \) and \( p_1 \), and \( \{q_n\} \) is as defined in (10). For these sequences, we derive the Gelin-Cesàro identities.

## 2 Main results

Consider the sequence \( \{p_n\} \) given in (12). According to [1], the Catalan’s identity for \( \{p_n\} \) is given by
\[
\frac{p_n^2}{\gamma_n^2} - \frac{p_{n-r}p_{n+r}}{\gamma_{n-r}\gamma_{n+r}} = (-c)^{n-r} \left( ap_r \frac{p_r}{\gamma_r} - \sqrt{ab} p_0 \frac{p_{r+1}}{\gamma_{r+1}} \right) D_r,
\] (13)
where the sequence \( \{D_n\} \) is defined by
\[
D_0 = 0, \quad D_1 = 1, \quad D_n = \sqrt{ab} D_{n-1} + cD_{n-2} \quad (n \geq 2),
\]
and
\[
\gamma_l = \frac{1}{a^{\xi(l)}(\sqrt{ab})^{1-\xi(l)}}.
\]

**Theorem 1.** For any integer \( n \geq 2 \), we have
\[
p_{n-2}p_{n-1}p_{n+1}p_{n+2} = \left( \frac{b}{a} \right)^{1-2\zeta(n)} \left( \frac{b}{a} \right)^{\zeta(n+1)} (ab - c) \frac{\Delta}{b} + (-c)^{2n-3} \Delta^2,
\] (14)
where
\[ \Delta =: ap_1^2 - abp_0 p_1 - bcp_0^2. \]

**Proof.** It is easily seen that
\[ \frac{1}{\gamma_{n-r}\gamma_{n+r}} = a^{1+\zeta(n+r)}b^{1-\zeta(n+r)}. \]

For \( r = 1 \), we have
\[ ab \left( \frac{a}{b} \right)^{\zeta(n+1)} p_{n-1} p_{n+1} = ab \left( \frac{a}{b} \right)^{\zeta(n)} p_n^2 - (-c)^{n-1} (a^2 p_1^2 - abp_0 p_2) \]
\[ = ab \left( \frac{a}{b} \right)^{\zeta(n)} p_n^2 - (-c)^{n-1} a \Delta, \]

or
\[ p_{n-1} p_{n+1} = \left( \frac{b}{a} \right)^{1-2\zeta(n)} p_n^2 - (-c)^{n-1} \left( \frac{b}{a} \right)^{\zeta(n+1)} \Delta \frac{a}{b}. \]

For \( r = 2 \), we have
\[ ab \left( \frac{a}{b} \right)^{\zeta(n)} p_{n-2} p_{n+2} = ab \left( \frac{a}{b} \right)^{\zeta(n)} p_n^2 - (-c)^{n-2} (a^2 b p_1 p_2 - a^2 b p_0 p_3) \]
\[ = ab \left( \frac{a}{b} \right)^{\zeta(n)} p_n^2 - (-c)^{n-2} a^2 b \Delta, \]

or
\[ p_{n-2} p_{n+2} = p_n^2 - (-c)^{n-2} \left( \frac{b}{a} \right)^{\zeta(n)} a \Delta. \]

Then
\[ p_{n-2} p_{n-1} p_{n+1} p_{n+2} = \left( \left( \frac{b}{a} \right)^{1-2\zeta(n)} p_n^2 - (-c)^{n-1} \left( \frac{b}{a} \right)^{\zeta(n+1)} \Delta \frac{a}{b} \right) \left( p_n^2 - (-c)^{n-2} \left( \frac{b}{a} \right)^{\zeta(n)} a \Delta \right) \]
\[ = \left( \frac{b}{a} \right)^{1-2\zeta(n)} p_n^4 - (-c)^{n-2} \left( \frac{b}{a} \right)^{\zeta(n+1)} \Delta \frac{a}{b} + (-c)^{2n-3} \Delta^2, \]

and the proof is completed.

If we set \( b = a \) and \( c = b \), then (14) reduces to the identity given in (6).

If \( p_0 = 0, p_1 = 1 \) and \( c = 1 \), then (14) reduces to the identity given in (9).

Now consider the sequence \( \{ q_n \} \) defined in (10). For any nonnegative integer \( n \) and even integer \( r \) with \( n \geq r \), the Catalan’s identity for \( \{ q_n \} \) is given by [2]
\[ q_n^2 - q_{n-r} q_{n+r} = (-1)^n \left( \frac{b}{a} \right)^{\zeta(n)} c^{\frac{n-r+\zeta(n)}{2}} d^{\frac{n-r-\zeta(n)}{2}} q_r^2. \]

(15)
Theorem 2. For any integer $n \geq 4$, we have

$$q_{n-4q_n-2q_{n+2}q_{n+4}} = q_n^4 + (-1)^n \left( \frac{b}{a} \right)^{\zeta(n)} \frac{c^{n-4+\zeta(n)}}{2} \frac{d^{n-4-\zeta(n)}}{2} q_n^2 \left( cd + (ab + c + d)^2 \right) q_n^2$$

$$+ \left( \frac{b}{a} \right)^{2\zeta(n)} \frac{c^{n-3+\zeta(n)}}{2} \frac{d^{n-3-\zeta(n)}}{2} q_n^2 (ab + c + d)^2.$$  \hspace{1cm} (16)

Proof. For $r = 2$, we have

$$q_{n-2q_{n+2}} = q_n^2 + (-1)^n \left( \frac{b}{a} \right)^{\zeta(n)} \frac{c^{n-2+\zeta(n)}}{2} \frac{d^{n-2-\zeta(n)}}{2} q_2^2.$$  

For $r = 4$, we have

$$q_{n-4q_{n+4}} = q_n^2 + (-1)^n \left( \frac{b}{a} \right)^{\zeta(n)} \frac{c^{n-4+\zeta(n)}}{2} \frac{d^{n-4-\zeta(n)}}{2} q_4^2.$$  

Then

$$q_{n-4q_{n-2q_{n+2}q_{n+4}}} = \left( q_n^2 + (-1)^n \left( \frac{b}{a} \right)^{\zeta(n)} \frac{c^{n-2+\zeta(n)}}{2} \frac{d^{n-2-\zeta(n)}}{2} q_2^2 \right) \left( q_n^2 + (-1)^n \left( \frac{b}{a} \right)^{\zeta(n)} \frac{c^{n-4+\zeta(n)}}{2} \frac{d^{n-4-\zeta(n)}}{2} q_4^2 \right)$$

$$= q_n^4 + (-1)^n \left( \frac{b}{a} \right)^{\zeta(n)} \frac{c^{n-2+\zeta(n)}}{2} \frac{d^{n-2-\zeta(n)}}{2} q_2^2 q_4^2 + c^{n-4+\zeta(n)} \frac{d^{n-4-\zeta(n)}}{2} q_2^2 q_4^2$$

$$+ \left( \frac{b}{a} \right)^{2\zeta(n)} \frac{c^{n-3+\zeta(n)}}{2} \frac{d^{n-3-\zeta(n)}}{2} q_2^2 q_4^2.$$  

Since $q_2 = a$ and $q_4 = a(ab + c + d)$, the proof is completed.

If $n \geq 4$ is even, then (16) is given by

$$q_{n-4q_{n-2q_{n+2}q_{n+4}}} = q_n^4 - \left( \frac{b}{a} \right)^{\zeta(n)} \frac{c^{n-2+\zeta(n)}}{2} \frac{d^{n-2-\zeta(n)}}{2} \left( cd + (ab + c + d)^2 \right) q_n^2$$

$$+ (cd)^{n-3} a^4 (ab + c + d)^2.$$  \hspace{1cm} (17)

The Gelin-Cesàro identity given in [8, Theorem 3.5] should be corrected as above.

References


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