On One Disturbance of Laguerre-Hahn Forms

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Abstract
We show that if \( u \) is a regular Laguerre-Hahn form of class \( s \), then
the form \( v = u + \lambda \delta_c \) is also a regular and Laguerre-Hahn form of class
\( \tilde{s} \leq s + 2 \) for every complex \( \lambda \) except for a discrete set of numbers
depending on both \( u \) and \( c \).

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1 Introduction

To obtain orthogonal polynomials of type Legendre, Jacobi, Laguerre and
Bessel, several authors have used the method of adding one or more Dirac
measures to a classic forms. This point of view was considerably developed
in the works of Krall and L. L. Littlejohn [6, 7]. The general problem of the
addition of one or more Dirac measures has been studied in [5]. For reasons
of simplification, we treats here the case of a single Dirac measure.
Let \( u \) be a regular Laguerre-Hahn form of class \( s \), we will show that \( v = u + \lambda \delta_c \)
is also a regular Laguerre-Hahn form for

\[
\lambda \neq -\left( \sum_{\nu=0}^{n} \frac{B_{\nu}^2(c)}{\tau_{\nu}} \right)^{-1}, \quad n \geq 0,
\]
where \( \{B_n\}_{n \geq 0} \) denotes the orthogonal sequence polynomials with respect to the regular form \( u \). We prove that \( v \) is of class \( \tilde{s} \) less than or equal \( s + 2 \) and we discuss the cases when \( \tilde{s} = s + 1 \) and when \( \tilde{s} = s + 2 \).

Let \( P \) be the vector space of polynomials with coefficients in \( \mathbb{C} \) and let \( P' \) be its topological dual. We denote by \( \langle u, f \rangle \) the action of \( u \in P' \) on \( f \in P \). In particular, we denote by \( (u)_n := \langle u, x^n \rangle \), \( n \geq 0 \), the moments of \( u \). For any linear form \( u \), any polynomial \( h \) and any complex number \( c \), let \( Du = u', hu, \delta_c \) and \((x - c)^{-1}u \) be the linear forms defined by

\[
\begin{align*}
\langle u', f \rangle &= -\langle u, f' \rangle \\
\langle (x - c)^{-1}u, f \rangle &= \langle u, \theta_c f \rangle, \quad (\theta_c f)(x) := \frac{f(x) - f(c)}{x - c} \\
\langle hu, f \rangle &= \langle u, hf \rangle, \quad f \in P \\
\langle \delta_c, f \rangle &= f(c)
\end{align*}
\]

(1.1)

The form \( hu \) is the left-multiplication of a linear form by a polynomial. We also define the right-multiplication of a linear form by a polynomial as

\[
(uh)(x) := iu, \quad xh(x) - \xi_h(x) = \sum_{k=0}^{n} (\sum_{i=k}^{n} a_i(u)_{i-k}) x^k, \quad h(x) = \sum_{j=1}^{n} a_j x^j
\]

It is possible to define the product of two linear forms:

\[
\forall u, v \in P', \quad \forall f \in P, \quad \langle uv, f \rangle := \langle u, vf \rangle
\]

**Definition 1.1** A linear form \( u \) is said to be regular if we can associate with it a sequence of monic orthogonal polynomials (MOPS) \( \{B_n\}_{n \geq 0} \), \( \deg B_n = n \), i.e:

- \( \langle u, B_n B_m \rangle = r_n \delta_{n,m}, \quad n, m \geq 0, \quad r_n \neq 0, \quad n \geq 0, \)
- the leading coefficient of \( B_n \) is equal to 1.

The sequence \( \{B_n\}_{n \geq 0} \) is said to be orthogonal polynomials sequence with respect to the linear form \( u \).

It is well known that the sequence \( \{B_n\}_{n \geq 0} \) satisfies the following second-order recurrence relation (see [2, 4, 7]):

\[
\begin{align*}
B_0(x) &= 1 \\
B_1(x) &= x - \beta_0 \\
B_{n+2}(x) &= (x - \beta_{n+1})B_{n+1}(x) - \gamma_{n+1}B_n(x), \quad n \geq 0 \\
\gamma_0 &= 1, \quad \gamma_{n+1} \in \mathbb{C} - \{0\}, \quad \beta_n \in \mathbb{C}
\end{align*}
\]

(1.2)
2 Characterization of the orthogonal polynomial sequence with respect to the form \( v = u + \lambda \delta c \)

Let \( \lambda \) and \( c \) be two complex numbers, \( u \) a regular linear form on \( P \) and \( \{B_n\}_{n \geq 0} \) be the orthogonal polynomials sequence with respect to \( u \). We consider the form

\[
 v = u + \lambda \delta c.
\] (2.1)

In this section we answer the following question: for what value of \( \lambda \) \( v \) is regular? From (2.1), we have

\[
(x - c)v = (x - c)u = \eta.
\] (2.2)

Remark that if \( \{\tilde{B}_n\}_{n \geq 0} \) is the orthogonal polynomials sequence with respect to \( v \), then it is strict quasi-orthogonal of order 1 with respect to \( \eta \) (see [7]).

We will determine the sequence \( \{\tilde{B}_n\}_{n \geq 0} \) in the form

\[
(x - c)\tilde{B}_n(x) = \sum_{\nu=0}^{n+1} \lambda_{n,\nu} B_{\nu}(x) , \ n \geq 0.
\] (2.3)

**Remark 2.1** The form \( \eta \) is not necessarily regular.

From (2.2), we deduce that

\[
< \eta, B_m \tilde{B}_n > = \lambda_{n,m} < u, B_m^2 > , \ 0 \leq m \leq n + 1.
\] (2.4)

From the condition of the strict quasi-orthogonality, we have

\[
\begin{cases} 
< \eta, B_m \tilde{B}_n > = 0 , \ 0 \leq m \leq n - 2 , \ n \geq 2 \\
< \eta, B_{n-1} \tilde{B}_n > \neq 0 , \ n \geq 1.
\end{cases}
\] (2.5)

From (2.4) and (2.5), we have

\[
\lambda_{n,m} = 0, \ 0 \leq m \leq n - 2, \ n \geq 2; \ \lambda_{n,n-1} = a_{n-1} \neq 0, \ n \geq 1,
\] (2.6)

therefore

\[
\begin{cases} 
(x - c)\tilde{B}_n(x) = B_{n+1}(x) + b_n(x)B_n(x) + a_{n-1}B_{n-1}(x) , \ n \geq 0 \\
b_n = \lambda_{n,n}
\end{cases}
\] (2.7)

with the convention \( B_{-1}(x) = 0 \).

If \( B_1(x) = x - \beta_0 \), then

\[
(x - c)\tilde{B}_0(x) = B_1(x) + b_0B_0(x)
\]
thus
\[ b_0 = \lambda_{0,0} = \beta_0 - c. \] (2.8)

The orthogonality condition for the sequence \( \{ \tilde{B}_n \}_{n \geq 0} \) can be written in the two following relations
\[ <v, \tilde{B}_{n+1}> = 0 \quad ; \quad <v, \tilde{B}_n^2> \neq 0 \quad , \quad n \geq 0. \] (2.9)

But, from (2.7), we get
\[ B_{n+2}(c) + b_{n+1}B_{n+1}(c) + a_nB_n(c) = 0, \quad n \geq 0. \] (2.10)

Then
\[ \tilde{B}_{n+1}(x) = \frac{B_{n+2}(x) - B_{n+2}(c)}{x-c} + b_{n+1} \frac{B_{n+1}(x) - B_{n+1}(c)}{x-c} + a_n \frac{B_n(x) - B_n(c)}{x-c} \] (2.11)

and considering (2.1), we obtain
\[ <u, \tilde{B}_{n+1}> = B^{(1)}_{n+1}(c) + \lambda B'_{n+2}(c) + b_{n+1}(B^{(1)}_n(c) + \lambda B'_{n+1}(c)) + a_n(B^{(1)}_{n-1}(c) + \lambda B'_{n}(c)) = 0, \quad n \geq 0, \] (2.12)

where
\[ B^{(1)}_n(x) = iu, B_{n+1}(x) - B_{n+1}(\xi)x - \xi \frac{d}{dx} = (u \theta_0 B_{n+1})(x), \quad n \geq 0. \]

\( \{ B^{(1)}_n \}_{n \geq 0} \) is called the sequence associated to the sequence \( \{ B_n \}_{n \geq 0} \) with respect to the form \( u \) (see [2, 3, 7]).

The coefficients \( b_{n+1} \) and \( a_n \) are solutions of the following system
\[
\begin{cases}
    B_{n+1}(c)b_{n+1} + B_n(c)a_n = -B_{n+2}(c) \\
    (B^{(1)}_n(c) + \lambda B'_{n+1}(c))b_{n+1} + (B^{(1)}_{n-1}(c) + \lambda B'_{n}(c))a_n = -\left( B^{(1)}_{n+1}(c) + \lambda B'_{n+2}(c) \right).
\end{cases} \] (2.13)

Knowing that
\[ B_{n+1}(x)B^{(1)}_{n-1}(x) - B_n(x)B^{(1)}_n(x) = \prod_{\nu=0}^{n-1} \gamma_\nu = \tau_n \]

and using the Cristoffel-Darboux identity (see [4]), the determinant of the above system is
\[ d_n = \tau_n \left\{ 1 + \lambda \sum_{\nu=0}^{n} \frac{B^2_{\nu}(c)}{\tau_\nu} \right\}, \quad n \geq 0, \] (2.14)
which satisfies the following recurrence relation

\[ d_{n+1} = \gamma_n d_n + \lambda B^2_{n+1}(c). \]  

(2.15)

Taking into account the relations (1.2) and (2.14), when \( d_n \neq 0, \ n \geq 0 \), we have,

\[ a_n = \frac{d_{n+1}}{d_n} \neq 0, \ n \geq 0; \ (a_{-1} = d_0 = 1 + \lambda) \]  

(2.16)

\[ b_{n+1} = \beta_{n+1} - c - \frac{\lambda}{d_n} B_n(c) B_{n+1}(c), \ n \geq 0. \]  

(2.17)

Then, using the Christoffel-Darboux identity, we deduce the expression of \( \tilde{B}_n \):

\[ \tilde{B}_{n+1}(x) = -\frac{\lambda \tau_n}{d_n} B_{n+1}(c) \sum_{\nu=0}^n \frac{B_{\nu}(c) B_{\nu}(x)}{\tau_{\nu}} + B_{n+1}(x), \ n \geq 0. \]  

(2.18)

Then the regularity of \( v \) is equivalent to \( d_n \neq 0, \ n \geq 0 \).

So, if we put

\[ \lambda_n = -\left( \sum_{\nu=0}^n \frac{B^2_{\nu}(c)}{\tau_{\nu}} \right)^{-1}, \ n \geq 0, \]  

(2.19)

one gets the following result

**Theorem 2.1** Let \( u \) be a regular form. Then, the form \( v = u + \lambda \delta c, (c \in \mathbb{C}) \) is regular if and only if \( \lambda \neq \lambda_n, \ n \geq 0 \).

**Proposition 2.1** (See [8]) Let \( u \) be a regular form, \( v = u + \lambda \delta c \ (\lambda \neq \lambda_n) \) and put

\[ \rho_n = a_{n-1} - \gamma_n; \ \sigma_n(x) = x - \beta_n + b_n, \ n \geq 0. \]

Then

\[ (x - c)B_n(x) = \frac{\rho_n}{a_{n-1}} \tilde{B}_{n+1}(x) + \frac{\gamma_n}{a_{n-1}} \sigma_{n+1}(x) \tilde{B}_n(x), \ n \geq 0 \]  

(2.20)

\[ (x - c)B_{n+1}(x) = (\sigma_n(x) - \rho_n) \frac{b_n}{a_{n-1}} \tilde{B}_{n+1}(x) - \frac{\gamma_n}{a_{n-1}} \rho_{n+1} \tilde{B}_n(x), \ n \geq 0. \]  

(2.21)

**Remark 2.2** From Proposition 2.1, for \( x = c \), we deduce that

\[ \frac{\rho_n}{a_{n-1}} \tilde{B}_{n+1}(c) + \frac{\gamma_n}{a_{n-1}} \sigma_{n+1}(c) \tilde{B}_n(c) = 0 \]  

(2.22)

and

\[ \left\{ \sigma_n(c) - \rho_n \frac{b_n}{a_{n-1}} \right\} \tilde{B}_{n+1}(c) - \frac{\gamma_n}{a_{n-1}} \rho_{n+1} \tilde{B}_n(c) = 0 \]  

(2.23)
In the sequel, we suppose that the form $v$ is regular ($\lambda \neq \lambda_n$). The following result gives the coefficients $\tilde{\gamma}_{n+1}$ and $\tilde{\beta}_n$, $n \geq 0$, of the recurrence relation of the sequence $\{\tilde{B}_n\}_{n \geq 0}$.

**Proposition 2.2** The recurrence coefficients $\tilde{\gamma}_{n+1}$ and $\tilde{\beta}_n$ of the sequence $\{\tilde{B}_n\}_{n \geq 0}$, are given by:

$$
\tilde{\gamma}_0 = 1 + \lambda; \quad \tilde{\gamma}_{n+1} = \gamma_n \frac{d_{n+1}d_{n-1}}{d_n^2}, \quad n \geq 0, \quad d_{-1} = 1 \quad (2.24)
$$

$$
\tilde{\beta}_n = \beta_{n+1} + b_n - b_{n+1}, \quad n \geq 0. \quad (2.25)
$$

**Proof.**

The sequence $\{\tilde{B}_n\}_{n \geq 0}$ verifies

$$
\begin{cases}
\tilde{B}_{n+2}(x) = (x - \tilde{\beta}_{n+1})\tilde{B}_{n+1}(x) - \tilde{\gamma}_{n+1}\tilde{B}_n(x) & n \geq 0 \\
\tilde{B}_0(x) = 0 & \tilde{B}_1(x) = x - \tilde{\beta}_0.
\end{cases} \quad (2.26)
$$

and we have

$$
\tilde{\gamma}_0 = \langle v, 1 \rangle = \langle u, 1 \rangle + \lambda < \delta_c, 1 \rangle = \gamma_0 + \lambda.
$$

From (2.26), we have

$$
\langle v, x\tilde{B}_{n+1}\tilde{B}_n \rangle = \tilde{\gamma}_{n+1} \langle v, \tilde{B}^2_n \rangle = \langle u, \tilde{B}^2_{n+1} \rangle.
$$

$$
\langle v, (x - c)\tilde{B}_n\tilde{B}_{n+1} \rangle = \langle u + \lambda \delta_c, (x - c)\tilde{B}_n\tilde{B}_{n+1} \rangle = \langle u, (x - c)\tilde{B}_n\tilde{B}_{n+1} \rangle = \langle v, \tilde{B}^2_{n+1} \rangle. \quad (2.27)
$$

This relation together with (2.7) give

$$
\langle v, (x - c)\tilde{B}_n\tilde{B}_{n+1} \rangle = \langle u, \tilde{B}_n(B_{n+2} + b_{n+1}B_{n+1} + a_nB_n) \rangle = a_n < u, \tilde{B}_nB_n > = a_n < u, B^2_n >,
$$

therefore

$$
\prod_{\nu=0}^{n+1} \tilde{\gamma}_\nu = \langle v, \tilde{B}^2_{n+1} \rangle = a_n < u, B^2_n > \frac{d_{n+1}d_{n-1}}{d_n^2} \prod_{\nu=0}^{n} \gamma_\nu.
$$

From where $\tilde{\gamma}_{n+1} = \gamma_n \frac{d_{n+1}d_{n-1}}{d_n^2}$, $n \geq 0$, $d_{-1} = 1$.

On the other hand, we have

$$
\tilde{B}_{n+1}\tilde{B}_n = (x - \tilde{\beta}_n)\tilde{B}_{n+1}^2 - \tilde{\gamma}_n\tilde{B}_n\tilde{B}_{n-1}.
$$
Consequently, \( <v, (x - \tilde{\beta}_n)\tilde{B}^2_n> = 0 \), if and only if

\[
<v, x\tilde{B}^2_n> = \tilde{\beta}_n <v, \tilde{B}^2_n> - <(x - c)\tilde{B}^2_n> + c <v, \tilde{B}^2_n>.
\]

But from (2.7), we have

\[
(x - c)\tilde{B}^2_n = B_{n+1}\tilde{B}_n + b_n B_n \tilde{B}_n + a_{n-1} B_{n-1} \tilde{B}_n.
\]

So,

\[
\tilde{\beta}_n <v, \tilde{B}^2_n> = b_n <v, \tilde{B}^2_n> + <v, B_{n+1} \tilde{B}_n> + c <v, \tilde{B}^2_n> = (b_n + c) <v, \tilde{B}^2_n> + \lambda B_{n+1}(c) \tilde{B}_n(c).
\]

Furthermore, from (2.18) we have, for \( n \geq 1 \),

\[
\tilde{B}_n(c) = -\frac{\lambda \tau_{n-1}}{d_{n-1}} B_n(c) \sum_{\nu=0}^{n-1} \frac{B^2_n(\tau)}{\tau^\nu} + B_n(c)
\]

\[
= -\frac{B_n(c)}{d_{n-1}} \left( d_{n-1} - \tau_{n-1} \right) + B_n(c) = \frac{<u, \tilde{B}^2_n>}{d_n} B_n(c).
\]

Thus, \( \tilde{\beta}_n = b_n + c + \frac{\lambda}{d_n} B_n(c) B_{n+1}(c) \). Finally, considering (2.17), we get

\[
\tilde{\beta}_n = \beta_{n+1} + b_n - b_{n+1}, \quad n \geq 0.
\]

\[\blacksquare\]

**Exemple 2.1**

1. When \( u \) is defined positive and \( c \in \mathbb{R} \), the form \( v \) is regular for each value of \( \lambda \in \mathbb{C} - (-\infty, 0] \).

2. If \( u \) is symmetric and real, then for each \( c, \lambda \in \mathbb{C} \) such that \( \Re(c) = 0 \) and \( \Im(\lambda) \neq 0 \), the form \( v \) is regular (in particular, if \( u \) is definite positive).

## 3 Case of Laguerre-Hahn form

Henceforth \( u \) will denote a Laguerre-Hahn form of class \( s \) (see [1], [2]). It then verifies the functional equation

\[
(\Phi(x)u)' + \Psi(x)u + B(x^{-1}u^2) = 0
\]

and the following condition

\[
\prod_{\alpha} \{|\Psi(\alpha) + \Phi'(\alpha)| + |B(\alpha)| + |<u, \theta_{\alpha} \Psi + \theta_{\alpha}^2 \Phi + u(\theta_{\alpha} B)>|\} \neq 0,
\]

\[\text{(3.2)}\]
where \( \alpha \) goes through the zeros of \( \Phi \). We recall that the class of the form \( u \) is given by

\[
s = \max \left[ \deg \Psi - 1, \max (\deg \Psi, \deg B) - 2 \right].
\]

In the following theorem, we give the functional equation of the form \( v \) and specify its class.

**Theorem 3.1** Let \( u \) be a Laguerre-Hahn form of class \( s \) satisfying (3.1) such that \( v = u + \lambda \delta_c \) supposed regular. Then \( v \) is a Laguerre-Hahn form of class \( s' \leq s + 2 \) and verifies the functional equation:

\[
(\tilde{\Phi}v) + \tilde{\Psi}(x)v + \tilde{B}(x^{-1}v^2) = 0,
\]

with the following cases:

1. if \( \Phi(c) + \lambda B(c) \neq 0 \), then

\[
\begin{align*}
\tilde{\Phi}(x) &= (x - c)^2 \Phi(x) \\
\tilde{\Psi}(x) &= (x - c) \left\{ (x - c) \Psi(x) - 2 \Phi(x) - 2 \lambda B(x) \right\} \\
\tilde{B}(x) &= (x - c)^2 B(x),
\end{align*}
\]

and \( v \) is of class \( s + 2 \);

2. if \( \Phi(c) = B(c) = 0 \) and \( \Psi(c) - \lambda B'(c) \neq 0 \), then

\[
\begin{align*}
\tilde{\Phi}(x) &= (x - c) \Phi(x) \\
\tilde{\Psi}(x) &= (x - c) \Psi(x) - \Phi(x) - 2 \lambda B(x) \\
\tilde{B}(x) &= (x - c) B(x),
\end{align*}
\]

and \( v \) is of class \( s + 1 \);

3. if \( \Phi(c)B(c) \neq 0 \), \( \Psi(c) - \lambda B'(c) \neq 0 \) and \( \lambda = -\frac{\Phi(c)}{B(c)} \) is not a singular value, then \( v \) is of class \( s + 1 \) and verifies the functional equation given by (3.5);

4. if we have \( \Psi(c) - \lambda B'(c) = 0 \) together with \( \Phi(c) = B(c) = 0 \) or \( \Phi(c)B(c) \neq 0 \) then the form \( v \) is of class less than \( s \).

**Proof.**

Using (1.1), one can prove easily that

\[
(x - c) B(x^{-1}u \delta_c) = B(x)u.
\]

\((*)\)

Since the form \( u \) verifies the functional equation

\[
(\Phi u)' + \Psi u + B(x^{-1}u^2) = 0,
\]
so, the form \( v = u + \lambda \delta_c \) satisfies the following functional equation

\[
(\Phi(v - \lambda \delta_c))' + \Psi(v - \lambda \delta_c) + B(x^{-1}(v - \lambda \delta_c)^2) = 0.
\]

which is equivalent to

\[
(\Phi v)' + \Psi v + B(x^{-1}v^2) = \lambda(\Phi \delta_c)' + \lambda(\Psi \delta_c) + 2\lambda B(x^{-1}v \delta_c) - \lambda^2 B(x^{-1}\delta_c^2). \tag{3.6}
\]

From (*), this last expression multiplied by \( x - c \), becomes:

\[
(\Phi v)' + (x - c)\Phi v + (x - c)B(x^{-1}v^2) = \lambda(\Phi \delta_c)' - \Phi \delta_c + 2\lambda B(x)v - \lambda^2 B\delta_c.
\]

That is

\[
(\Phi v)' + (x - c)\Phi v - 2\lambda B(x)v + (x - c)B(x^{-1}v^2) = \lambda\left\{\Phi(c) + \lambda B(c)\right\}\delta_c. \tag{3.7}
\]

The equation (3.7) becomes after multiplication by \( x - c \):

\[
(\tilde{\Phi} v)' + \tilde{\Psi} v + \tilde{B}(x^{-1}v^2) = 0, \tag{3.8}
\]

where

\[
\tilde{\Phi} = (x - c)^2\Phi, \quad \tilde{\Psi} = (x - c)\left((x - c)\Psi - 2\Phi - 2\lambda B\right), \quad \tilde{B} = (x - c)^2B.
\]

Let us write

\[
\deg \Psi = p, \quad \deg \Phi = t, \quad \deg B = r, \quad d = \max(t, r),
\]

\[
\deg \tilde{\Psi} = \tilde{p}, \quad \deg \tilde{\Phi} = \tilde{t}, \quad \deg \tilde{B} = \tilde{r},
\]

\[
\tilde{d} = \max(\tilde{r}, \tilde{t}) \quad \text{and} \quad s_1 = \max(\tilde{p} - 1, \tilde{d} - 2).
\]

Then,

\[
\tilde{p} \leq \sup(p + 2, t + 1, r + 1), \quad \tilde{t} = t + 2 \quad \text{and} \quad \tilde{r} = r + 2.
\]

Furthermore, we have the following cases:

- if \( 0 \leq d \leq p \) then \( \tilde{p} = p + 2 \) and \( 0 \leq \tilde{d} \leq \tilde{p} \), so
  \[
  s_1 = \tilde{p} - 1 = p + 1 = s + 2.
  \]

- if \( d = p + 1 \) then \( \tilde{d} = p + 3 \), so \( \tilde{p} \leq p + 2 = \tilde{d} - 1 \) and we have the two following subcases:
\(\tilde{d} = \tilde{p} + 1\), then \(s_1 = \tilde{p} - 1 = d - 2 = p + 1 = s + 2\).

- if \(\tilde{d} \geq \tilde{p} + 2\) then \(s_1 = \tilde{d} - 2 = (p + 3) - 2 = p + 1 = s + 2\).

- if \(d \geq p + 2\), then \(\tilde{p} \leq d + 1 = \tilde{d} - 1\) so \(s_1 = \tilde{d} - 2 = d = s + 2\).

Does the triplet \((\tilde{\Phi}, \tilde{\Psi}, \tilde{B})\) given by (3.4) and (3.5) provide the class of \(v\)? For the answer, we shall use (3.2).

From (3.4), \(c\) is a root of \(\tilde{\Phi}\) and \(\tilde{\Psi}(c) + \tilde{\Phi}'(c) = \tilde{B}(c) = 0\). We should therefore calculate
\[
< v, \theta_c(\tilde{\Psi} + \theta_c\tilde{\Phi})u + v(\theta_0\theta_c\tilde{B}) > .
\]

We have
\[
< v, \theta_c\tilde{\Psi} + \theta_c^2\tilde{\Phi} + v(\theta_0\theta_c\tilde{B}) > = < v, (x - c)\Psi - 2\Phi - 2\lambda B(x) + \Phi + v(\theta_0(x - c)B > \\
= < u, (x - c)\Psi(x) > - < u, \Phi > - 2\lambda < u, B > + < u^2, \theta_0(x - c)B > \\
+ \lambda(< \delta_c, (x - c)\Psi(x) > - < \delta_c, \Phi > - 2\lambda < \delta_c, B >)
\]
\[
= < (\Phi u)' + \Psi u, (x - c) > + < u^2, \theta_0((x - c)B) > + 2\lambda < u, \delta_c\theta_0(x - c)B > \\
+ \lambda^2 < \delta_c, \delta_c\theta_0(x - c)B > - 2\lambda < u, B > - \lambda\left\{\Phi(c) + 2\lambda B(c)\right\}
\]
\[
= < (\Phi u)' + \Psi u + B(x^{-1}u^2), x - c > - \lambda\left\{\Phi(c) + \lambda B(c)\right\}
\]
\[
= -\lambda\left(\Phi(c) + \lambda B(c)\right) .
\]

For \(v\) to be of class \(s + 2\), it is necessary that \(\Phi(c) + \lambda B(c) \neq 0\).

**Case 1:** \(\Phi(c) + \lambda B(c) \neq 0\)

Let \(a \neq c\) be one root of \(\tilde{\Phi}\), then it is a root of \(\Phi\). That is
\[
\tilde{\Phi}(x) = (x - a)\tilde{\Phi}_a(x) ; \Phi(x) = (x - a)\Phi_a(x),
\]

From (3.4), we deduce that
\[
\tilde{\Phi}_a(x) = (x - c)^2\Phi_a(x) .
\]

By Euclidean division, we obtain:
\[
\Psi(x) + \Phi_a(x) = (x - a)q_a(x) + r_a ; \quad r_a \in \mathbb{C} \\
\tilde{\Psi}(x) + \tilde{\Phi}_a(x) = (x - a)\tilde{q}_a(x) + \tilde{r}_a ; \quad \tilde{r}_a \in \mathbb{C} .
\]

The two relations (3.4) and (3.10) imply that
\[
\tilde{r}_a = \tilde{\Psi}(a) + \tilde{\Phi}'(a) = (a - c)^2\left\{\Psi(a) + \Phi'(a)\right\} - 2\lambda(a - c)B(a) .
\]

- **a)** \(B(a) \neq 0 \iff \tilde{B}(a) \neq 0\), from (3.2), we deduce that the class of the form \(v\) is equal to \(s + 2\).
b) \( B(a) = 0 \iff \tilde{B}(a) = 0 \), from (3.11) we have:

\[
\tilde{q}_a = (a - c) \tilde{g}_a(c) = (a - c)^2 \left\{ \Psi(a) + \Phi'(a) \right\}.
\]

Two cases are envisaged:

i) \( \Psi(a) + \Phi'(a) \neq 0 \) then \( \tilde{\Psi}(a) + \tilde{\Phi}'(a) \neq 0 \), So, from (3.11) \( v \) is of class \( s + 2 \).

ii) \( \Psi(a) + \Phi'(a) = 0 \) then \( \Psi(x) + \Phi_a(x) = (x - a)q_a(x) \).

From (3.4) and (3.10), we have

\[
\tilde{q}_a(x) = (x - c)^2 q_a(x) - 2(x - c) \left\{ \Phi_a(x) + \lambda B_a(x) \right\}.
\]

Then,

\[
< v, \theta_a \tilde{\Psi} + \theta_a^2 \tilde{\Phi} + v(\theta_0 \theta_a \tilde{B}) > = < v, \theta_a (\tilde{\Psi} + \tilde{\Phi}_a) + v(\theta_0 \tilde{B}_a) > = < v, \tilde{q}_a + v(\theta_0 \tilde{B}_a) >.
\]

But,

\[
< v, \tilde{q}_a > = < v, (x - c)^2 q_a(x) - 2(x - c) (\Phi_a(x) + \lambda B_a(x)) >
\]

\[
= < u, (x - c)^2 q_a(x) - 2(x - c) (\Phi_a(x) + \lambda B_a(x)) >
\]

\[
= < u, (x - c)^2 \theta_a \Psi > + < u, (x - c)^2 \theta_a^2 \Phi >
\]

\[
- 2 < u, (x - c) \theta_a \Phi > - 2 \lambda < u, (x - c) \theta_a B >.
\]

and

\[
< v, v(\theta_0 \tilde{B}_a) > = < v^2, \theta_0 \tilde{B}_a >
\]

\[
= < u^2 + 2u \delta_c + \lambda^2 \delta_a^2, \theta_0 (x - c)^2 B_a >
\]

\[
= < x^{-1} u^2, (x - c)^2 B_a > + 2 \lambda < u, \theta_c (x - c)^2 B_a > + \lambda^2 < \delta_c, (x - c)^2 B_a >
\]

\[
= < x^{-1} u^2, (x - c)^2 B_a > + 2 \lambda < u, (x - c) B_a > + \lambda^2 < \delta_c, (x - c) B_a >.
\]

Then, by (3.2) and (3.4), we obtain:

\[
< v, \theta_a \tilde{\Psi} + \theta_a^2 \tilde{\Phi} + v(\theta_0 \theta_a \tilde{B}) > = < (\theta_a \Phi u)' + \theta_a (\Psi + \theta_a \Phi) u + \theta_a B(x^{-1} u^2), (x - c)^2 >.
\]

So, if we put

\[
w = (\theta_a \Phi u)' + \theta_a (\Psi + \theta_a \Phi) u + \theta_a B(x^{-1} u^2),
\]

we get

\[
(x - a)w = (\Phi u)' + \Psi u + B(x^{-1} u^2) = 0.
\]

So, \( w = \gamma \delta_a \). But,

\[
< w, 1 > = < u, \theta_a \Psi + \theta_a^2 \Phi + u(\theta_0 \theta_a B) > \neq 0,
\]
because \( u \) is of class \( s \). Then, \( \gamma \neq 0 \) and \( <w,(x-c)^2> = \gamma(a-c)^2 \neq 0 \), therefore \( v \) is of class \( s + 2 \).

Case 2: \( \Phi(c) = B(c) = 0 \)

From (3.2) \( v \) is of class \( \tilde{s} \leq s + 1 \). We deduce that \( v \) verifies the functional equation (3.3), with

\[
\begin{align*}
\tilde{\Phi}(x) &= (x-c)\Phi(x) \\
\tilde{\Psi}(x) &= (x-c)\Psi(x) - \phi(x) - 2\lambda B(x) \\
\tilde{B}(x) &= (x-c)B(x).
\end{align*}
\]  

It’s obvious that \( \tilde{p} \leq \sup(p+1,t,r) \); \( \tilde{t} = t + 1 \); \( \tilde{r} = r + 1 \)

- if \( 0 \leq d \leq p \) then \( \tilde{p} = p + 1 \) and \( 0 \leq \tilde{d} \leq \tilde{p} \), so \( s_1 = \tilde{p} - 1 = p = s + 1 \).

- if \( d = p + 1 \) then \( \tilde{d} = p + 2 \), so \( \tilde{p} \leq p + 1 = \tilde{d} - 1 \) and we have the two following subcases:
  - if \( \tilde{d} = \tilde{p} + 1 \), then \( s_1 = \tilde{p} - 1 = \tilde{d} - 2 = p = s + 1 \).
  - if \( \tilde{d} \geq \tilde{p} + 2 \) then \( s_1 = \tilde{d} - 2 = p = s + 1 \).

- if \( d \geq p + 2 \), then \( \tilde{p} = d = \tilde{d} - 1 \) so \( s_1 = \tilde{p} - 1 = d - 1 = s + 1 \).

From (3.14) we find:

\[\tilde{B}(c) = \tilde{\Psi}(c) + \tilde{\Phi}'(c) = 0\]

and a simple calculus gives

\[<v,\theta_c\tilde{\Psi} + \theta_c^2\tilde{\Phi} + v(\theta_0\theta_c\tilde{B})(x)> = \lambda\left(\Psi(c) - \lambda B'(c)\right).\]  

(3.15)

Two cases to consider

a) if \( \Psi(c) - \lambda B'(c) = 0 \), then the form \( v \) is of class \( \tilde{s} \leq s \).

b) if \( \Psi(c) - \lambda B'(c) \neq 0 \), then using the same steps as in the proof of Case 1, we prove that \( v \) is of class \( s + 1 \).

Case 3: \( \Phi(c) \neq 0 \) and \( B(c) \neq 0 \)

If we suppose that \( \lambda = -\frac{\Phi(c)}{B(c)} \) is not a singular value to ensure the regularity of the form \( v \), with the condition \( \Psi(c) - \lambda B'(c) \neq 0 \), then \( v \) is also of class \( s + 1 \). Indeed \( \tilde{\Psi}(c) + \tilde{\Phi}'(c) = -2\lambda B(c) \neq 0 \).
Remark 3.1

In the four cases below, \( v = u + \lambda \delta_c \) is of class \( \bar{s} \leq s \) and verifies the functional equation \((\tilde{\Phi} v) + \tilde{\Psi}(x)v + \tilde{B}(x^{-1}v^2) = 0\), with \( \Phi(x) = \Phi(x) \), \( \Psi(x) = \Psi(x) - 2\lambda \theta_c B(x) \), \( \bar{B}(x) = B(x) \):

1) \( \Phi(c) = B(c) = \Psi(c) = B'(c) = 0 \)
2) \( \Phi(c) = B(c) \), \( B'(c) \neq 0 \) et \( \lambda = \frac{\Psi(c)}{B'(c)} \neq \lambda_n \)
3) \( \Phi(c) \neq 0 \), \( \Psi(c) = B'(c) = 0 \) et \( \lambda = \frac{\Phi(c)}{B(c)} \neq \lambda_n \)
4) \( \Phi(c) \neq 0 \), \( B'(c) \neq 0 \) et \( \lambda = \frac{\Phi(c)}{B(c)} = \frac{\Psi(c)}{B'(c)} \neq \lambda_n \).

References


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