Gamma Function and k-Gamma Function for Two Variables

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Abstract

The main objective of this paper to introduce two variable gamma function. We present Gauss, Weierstrass representation and prove the some basic properties of gamma function for two variables. Furthermore, we establish some inequalities involving this new function.

Keywords: gamma function, k-gamma function, pochhammer symbol, convex function
1 Introduction

Special functions are particular mathematical functions which have more or less established names and notations due to their importance in mathematical analysis, functional analysis, physics, or other applications. The gamma function is one of the most significant special function which discovered in $18^{th}$ century. Gamma function is a continuous extension to the factorial function, which is only defined for the nonnegative integers. While there are other continuous extensions to the factorial function, it is the only one that is convex for positive real numbers (see[6, 7, 8]). The gamma function belongs to the class of the transcendental functions and some well-known mathematical constants are arising in its study. It has vast applications in physics, mathematics and engineering. Most modern mathematical packages contain the gamma function as one of their standard function. Application of gamma function in an analytical physics and engineering (see [1, 2, 3, 5]). The gamma function, $\Gamma(x)$ is defined as

$$\Gamma(x) = \int_{0}^{\infty} e^{-t}t^{x-1}dt$$  \hspace{1cm} (1)

The gamma function satisfies the recursive property

$$\Gamma(x) = (x - 1)\Gamma(x - 1).$$

For specific values of $x$, $\Gamma(x)$ exist. The gamma function evaluated at $x = \frac{1}{2}$ $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. The Weierstrass defined the $\Gamma(x)$ by

$$\frac{1}{\Gamma(x)} = xe^{\gamma x} \prod_{n=1}^{\infty} \left[(1 + \frac{x}{n})exp(-\frac{x}{n})\right],$$ \hspace{1cm} (2)

in which $\gamma$ is constant also called Euler constant.

2 Preliminaries

In this section, we briefly review some basic definitions and facts concerning the gamma function (see[4, 6, 7, 8, 9]).

Definition 2.1: The factorial function defined as $(x)_n = x(x+1)(x+2)...(x+n-1)$; $x \neq 0$ and $(x)_0 = 1$. The function $(x)_n$ is known as pochhammer symbol. In manipulations with $(x)_n$, it is important to keep in mind that $(x)_n$ is a product of $n$ factors, starting with $x$ and with each factor large by unity than the preceding factor.

Definition 2.2: Let $x \in C$ ($C$ is a set of complex numbers), the gamma function is defined by

$$\Gamma(x) = \lim_{n \to \infty} \frac{n!(n)^{x-1}}{(x)_n}.$$ \hspace{1cm} (3)
The relation between Pochhammer symbol and gamma function is given below

\[(x)_n = \frac{\Gamma(x + n)}{\Gamma(x)}. \tag{4}\]

**Definition 2.3:** The k-Pochhammer symbol is defined as \((x)_{n,k} = x(x+k)(x+2k)\ldots(x+(n-1)k); x \neq 0 k > 0 \text{ and } (x)_{0,k} = 1.\)

**Definition 2.4:** Let \(x \in C\), the k-gamma function is defined by

\[\Gamma_k(x) = \lim_{n \to \infty} n!k^n(nk^\frac{n}{2}-1)(x)_{n,k}. \tag{5}\]

The relation between k-Pochhammer symbol and k-gamma function is given below

\[(x)_{n,k} = \frac{\Gamma_k(x + nk)}{\Gamma_k(x)}. \tag{6}\]

### 3 Gamma function for two variables

The main goal of this paper to introduce classical gamma function and k-gamma function for two variables. For this let \(x, y \in C\), \(\text{Re}(x), \text{Re}(y) > 0\), then gamma function for two variables given by the integral

\[\Gamma(x, y) = \int_0^\infty ye^{-ty}t^{x-1}dt. \tag{7}\]

One may observe that for \(y = 1\), the relations (1) and (7) are identical. That is \(\Gamma(x, 1) = \Gamma(x)\) which is the classical gamma function.

**Gauss Representation :** For \(x, y \in C/\mathbb{Z}^-\) the function \(\Gamma(x, y)\) define as

\[\Gamma(x, y) = \lim_{n \to \infty} \frac{y^n n!(n)^{\frac{x}{2}-1}}{(x)_{n,y}}. \tag{8}\]

**Proof:** From the definition

\[\Gamma(x, y) = \int_0^\infty ye^{-ty}t^{x-1}dt = \lim_{n \to \infty} \int_0^{(n)^{\frac{x}{2}}} yt^{x-1} \left(1 - \frac{ty}{n}\right)^n dt, \tag{9}\]

Let \(M_{n,i}(x), i = 0, 1, 2, \ldots n, \) be given by

\[M_{n,i}(x) = \int_0^{(n)^{\frac{x}{2}}} yt^{x-1} \left(1 - \frac{ty}{n}\right)^i dt, \]

The following recursive formula is proven using integration by parts

\[M_{n,i}(x) = \frac{y.i}{n.x} M_{n,i-1}(x + 1), \]
Also,
\[ M_{n,i}(x) = \int_0^{(n)^{\frac{1}{n}}} y t^{x-1} dt = \frac{y(n)^{\frac{x}{n}}}{x}, \]
Therefore
\[ M_{n,n}(x) = \frac{y^n.n!(n)^{\frac{x}{n}}-1}{(x)_{n,y} (y + \frac{x}{n})}. \]
and
\[ \Gamma(x, y) = \lim_{n \to \infty} M_{n,n}(x) = \lim_{n \to \infty} \frac{y^n.n!(n)^{\frac{x}{n}}-1}{(x)_{n,y}}. \]
which complete the proof.

**Proposition 1.** The Gamma function \( \Gamma(x, y) \) satisfies the following properties
1. \( \Gamma(x + 1, y) = \left(\frac{x}{y}\right) \Gamma(x - y + 1, y). \)
2. \( \Gamma(x + 1, y + 1) = \left(\frac{x}{y+1}\right) \Gamma(x - y, y + 1). \)
3. \( \Gamma(x, y) = \Gamma\left(\frac{x}{y}\right). \)
4. \( \Gamma(x, x) = \Gamma(y, y) = 1. \)
5. \( (x)_{n,y} = \frac{y^n\Gamma(x+n,y)}{\Gamma(x,y)}, \) or \( \frac{\Gamma(x+n,y)}{\Gamma(x,y)} = \left(\frac{x}{y}\right)_n. \)
6. \( \Gamma(x, y) = y^n e^{-t^a} t^{x-1} dt. \)
7. \( \frac{1}{\Gamma(x,y)} = x y^{-1} e^{-t^a} \prod_{n=1}^{\infty} \left[1 + \frac{x}{n y}\right] e^{\frac{x}{n y}}. \)
8. \( \Gamma(x, y) \Gamma(1 - x, y) = \frac{\pi}{\sin\left(\frac{x}{y}\right)}. \)
9. \( \Gamma(x, y) \) is logarithmically convex, For \( x, y \in R. \)
10. \( \Gamma(x + z, y) = \left[\Gamma(ax, y)\right]^z \left[\Gamma(bz, y)\right]^\frac{1}{z}. \)

**Proof:** By definition of gamma function for two variables
\[ \Gamma(x + 1, y) = \int_0^{\infty} ye^{-t^y} t^x dt. \]
solve integration by parts
\[ = \left| \frac{ye^{-t^y} t^x}{-yt^{y-1}} \right|_0^{\infty} + \left(\frac{x}{y}\right) \int_0^{\infty} ye^{-t^y} t^{x-y} dt. \]
\[ = \left(\frac{x}{y}\right) \Gamma(x - y + 1, y). \]
Which satisfies property (1), similarly we can prove (2). Now by using equation (7)
\[ \Gamma(x, y) = \int_0^\infty ye^{-ty} t^{x-1} dt. \]
\[ = \int_0^\infty ye^{-ty} t^{x-y} y^{y-1} dt, \]
\[ = \int_0^\infty e^{-ty} t^{\frac{x}{y}-1} y^{y-1} dt, \]
Let \( s = ty \) Then, \( ds = yt^{y-1} dt \)
\[ = \int_0^\infty e^{-s} s^{\frac{x}{y}-1} ds = \Gamma\left(\frac{x}{y}\right). \]
which complete the proof of property 3). Properties 4), 5) and 6) follow directly from definition. Properties 7) and 8) follows from \( \Gamma(x, y) = \Gamma\left(\frac{x}{y}\right) \).

**Theorem:** The real valued gamma function \( \Gamma(x, y) \) is logarithmically convex. In other words, for \( x, y \) and \( z > 1, \) and \( \frac{1}{a} + \frac{1}{b} = 1, \) the inequality
\[ \Gamma\left(\frac{x}{a} + \frac{z}{b}, y\right) \leq \left[ \Gamma(x, y) \right]^a \left[ \Gamma(z, y) \right]^b. \] (10)
holds.

**Proof:** From equation (7)
\[ \Gamma\left(\frac{x}{a} + \frac{z}{b}, y\right) = \int_0^\infty ye^{-ty} t^{(\frac{x}{a} + \frac{z}{b})-1} dt, \]
Since \( \frac{1}{a} + \frac{1}{b} = 1, \)
\[ = \int_0^\infty ye^{-ty} t^{\left(\frac{1}{a} + \frac{1}{b}\right) - \left(\frac{1}{a} + \frac{1}{b}\right)} dt, \]
\[ = \int_0^\infty ye^{-ty} t^{\frac{x}{a} - \frac{1}{a}} ye^{-ty} t^{\frac{z}{b} - \frac{1}{b}} dt, \]
By using Holder inequality, we obtain
\[ \Gamma\left(\frac{x}{a} + \frac{z}{b}, y\right) \leq \left[ \int_0^\infty \left( ye^{-ty} t^{\frac{x}{a} - \frac{1}{a}} \right)^a dt \right]^\frac{1}{a} \left[ \int_0^\infty \left( ye^{-ty} t^{\frac{z}{b} - \frac{1}{b}} \right)^b dt \right]^\frac{1}{b}, \]
\[ = \left[ \Gamma(x, y) \right]^\frac{1}{a} \left[ \Gamma(z, y) \right]^\frac{1}{b}. \]
which complete the proof of property (7).

**Theorem:** For \( x, z \geq 0, a > 1 \) and \( \frac{1}{a} + \frac{1}{b} = 1, \) the inequality
\[ \Gamma(x + z, y) = \left[ \Gamma(ax, y) \right]^\frac{1}{a} \left[ \Gamma(bz, y) \right]^\frac{1}{b}. \] (11)
is satisfied.

**Proof:** For $x, z \geq 0$, $a > 1$ and $\frac{1}{a} + \frac{1}{b} = 1$. From (2)

$$
\Gamma(x + z, y) = \int_0^\infty ye^{-ty} t^{x+z-1} dt,
$$

$$
= \int_0^\infty ye^{-ty} t^{x-z+(\frac{1}{a}+\frac{1}{b})} dt,
$$

$$
= \int_0^\infty ye^{-ty} t^{x-\frac{1}{a}} \int_0^\infty ye^{-ty} t^{z-\frac{1}{b}} dt,
$$

By using Holder inequality, we obtain

$$
\Gamma(x + z, y) \leq \left[\int_0^\infty \left(ye^{-ty} t^{x-\frac{1}{a}}\right)^a dt\right]^\frac{1}{a} \left[\int_0^\infty \left(ye^{-ty} t^{z-\frac{1}{b}}\right)^b dt\right]^\frac{1}{b},
$$

$$
= \left[\Gamma(ax, y)\right]^\frac{1}{a} \left[\Gamma(bz, y)\right]^\frac{1}{b}.
$$

**Lemma:** For $x, z \geq 0$, $a > 1$ and $\frac{1}{a} + \frac{1}{b} = 1$, the inequality

$$
xz \leq \frac{x^a}{a} + \frac{z^b}{b}; \quad (12)
$$

is satisfied.

**Corollary:** For $x, z \geq 0$, $a > 1$ and $\frac{1}{a} + \frac{1}{b} = 1$, the inequality

$$
\Gamma(x + z, y) \leq \frac{\Gamma(ax, y)}{a} + \frac{\Gamma(bz, y)}{b}. \quad (13)
$$

is satisfied.

**Proof:** From (11), we know

$$
\Gamma(x + z, y) = \left[\Gamma(ax, y)\right]^\frac{1}{a} \left[\Gamma(bz, y)\right]^\frac{1}{b}.
$$

By using (12)

$$
\left[\Gamma(ax, y)\right]^\frac{1}{a} \left[\Gamma(bz, y)\right]^\frac{1}{b} \leq \frac{\Gamma(ax, y)}{a} + \frac{\Gamma(bz, y)}{b}
$$

which complete the proof.

Let $x, y \in C$, $Re(x), Re(y) > 0$ and $k > 0$, then k-gamma function for two variables given by the integral

$$
\Gamma_k(x, y) = \int_0^\infty ye^{-\frac{ky}{x}t} t^{x-1} dt. \quad (14)
$$

and Gauss representation of k-gamma function for two variables are

$$
\Gamma_k(x, y) = \lim_{n \to \infty} \frac{(ky)^n n!(nk)^{\frac{x-1}{nk}}}{(x)_{n,yk}}. \quad (15)
$$
Proposition 2. The Gamma function $\Gamma_k(x, y)$ satisfies the following properties

1. $\Gamma_k(x, y) = (k)^{\frac{x}{yk}} \Gamma\left(\frac{x}{yk}\right)$

2. $\Gamma_k(k^2, k) = 1$.

3. $(x)_{n, yk} = \frac{y^n \Gamma_k(x+n yk, y)}{\Gamma_k(x, y)}$, or $(\frac{x}{y})_{n,k} = \frac{\Gamma(x+nk,y)}{\Gamma(x,y)}$.

4. $\Gamma_k(x, y) = y^{\frac{x}{ky}} \int_0^\infty e^{-y^\frac{t}{ky}} k^{x-1} dt$.

5. $\frac{1}{\Gamma_k(x, y)} = xy^{-1}(k)^{\frac{x}{ky}} e^{\gamma_k} \prod_{n=1}^{\infty} \left[1 + \frac{x}{nyk}\right] \exp\left(-\frac{x}{nyk}\right)$.

6. $\Gamma_k(x, y)\Gamma_k(k - x, y) = \frac{\pi^{\frac{x}{ky}}}{\sin\left(\frac{x}{ky}\right)}$.

7. $\Gamma_k(x, y)$ is logarithmically convex, for $x, y \in R$.

8. $\Gamma_k(x + z, y) = [\Gamma_k(ax, y)]^{\frac{1}{a}} [\Gamma_k(bz, y)]^{\frac{1}{b}}$.

Since the proofs are traditional oriented, we list the properties relating the $k$-gamma function for two variables. The properties involve Gauss and Weierstrass representation, convexity and relation with $k$-pochhammer symbol.

Conclusion In this paper, gamma function for two variables are introduced and analysis the basic properties. Which are important tool of analysis and can be usefully extend to other family of special functions which is a problem for further research.

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References


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