Study of the $D$-Infinity Differential Module

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Abstract

In this work, we study some basic properties of a $D$-infinity differential module. Consequence, we in all directions variegated straightforward notations and theorems of the module of differential, with the filtration in the cohomology theory of spectral sequences. We prove some relations on the spectral sequence and the stable $D$-infinity module algebra.

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1. Introduction

The study of many algebraic structures on topological spaces, consider as one of mean problems in algebraic topology $A$-infinity structure. The study of topology space $(X,e)$ with multiplication such that $e$ is 1-homotopy ($H$-space) is studied in [6] by considering topologies monoid, he formed that from the topological view that monoid pommel space. At [8] introduce the concept of $A$-infinity space. The important fact is $A$-infinity space is homorphical equivalence monoid is one set of space. In the same time partial, Kadishilly introduce the concept of $A$-infinity algebra and morphisms between $A$-infinity algebra on $A$-infinity algebra also module on $A$-infinity algebra and morphisms between $A$-infinity modules over $A$-infinity algebra. In [1], given the concept of $A$-infinity algebra is given by considering the homology chain complexes by the following facts: (1) If $A$ is differential algebra on the field, to on its homology $H^*(A)$ there is the structure of graded $D$-infinity algebra and the homomorphism in the category of $D$-infinity algebra on.

(2) For an arbitrary $M$ be a differential graded module over differential graded algebra, then of $H^*(M)$, there is a structure of graded $A$-infinity module on graded
$A$-infinity algebra and morphism in $A$-infinity module category over $A$-infinity algebra establish the structure of $H^*(M)$. The structure of $A$-infinity algebra can be given by considering Massy high product. These operations are described also in the homology of topological space. The $A$-infinity algebra and $A$-infinity module is considered a trivial example of algebra over operad. The structure of algebra over operad $E$ is called $E$-infinity algebra it’s defined and studied in [5]. By the help of operad we get many results in algebra and topology [3], [4] and [7]. Through the study of differential algebra of the Hochschild complex is establish for algebra and related to differential algebra structure in [5]. The Hochschild complex is the study of the quantum algebraic structure [5]. By help Hochschild complex, we can study the different form in the regular algebra. Hochschild complex and its homology are Connes Hochschild homology which is the study of additive algebraic k-theory (cyclic homology) and hermetian $k$-theory (dihedral homology) see ([3] and [4]). Cyclic homology in the useful in the study of group compact manifold.

In [8] presents the fundamental revealed the meanings of $D$-infinity differential and $A$-infinity structure in the spectral sequences. In [6], Lapin likely given a multiplicative $A$-infinity structure as parts of spectral sequences for fibrations. And differential graded of algebra and derived $E$-infinity algebras have of a mind to [1]. Some homotopy invariance for perturbations of $D$-infinity differential modules introduced in [7], the chain realization of differential modules with $\infty$-simplicial faces and $B$-construction over $A$-infinity algebras have to appear and studied in [1]. In our hunt, something worries us about the $D$-infinity differential module, stable $D^{(s)}$-infinity module and its connection in the differential modules of filtration and spectral sequences. Now we give some important facts about $D$-infinity differential modules, and $D^{(s)}$-infinity differential module.

**Definition (1-1):**
The graded modules is the Pair $(X,d)$ where, $X = \{X_n\}$, $n \in \mathbb{Z}$ be an arrangement of space module, and the function, $d : X \rightarrow X_{-1}$, $d$ is the map of graded $(-1)$ such that: $d^2 = 0$.

**Definition (1-2):**
In the differential modules, the function $f : (X,d) \rightarrow (Y,d)$ be the basic map in the graded modules; $f : X \rightarrow Y$, of the degree zero which satisfy the relation: $df = fd$.

**Definition (1-3):**
The $D$ - infinity differential module characterized by the module $X$ with a class $\{d^i : X \rightarrow X_{-1}, i > 0\}$ of graded module homeomorphisms, which satisfy

$$\sum_{i+j=k} d^i d^j = 0,$$  \hspace{1cm} (1)
for each whole integer number \((i + j = k > 0)\). If \(i = 0\), \(d^0 d^0 = 0\) and \((X, d^0)\) is a regular differential module, at \((i =1)\) then, the fundamental connection; 
\[d^1 d^0 + d^0 d^1 = 0,\]
this means the maps \(d^0\) and \(d^1\) are anti-commuting. This gives the composition; 
\[d^1 d^0 : X \to X,\]
is an endomorphism map of differential module \((X, d^0)\). If put \(k = 2\), we get; 
\[d^2 d^0 + d^0 d^2 = 0 - d^1 d^1.\]
At that point the mapping \(d^2 : X \to X\) which is differential homotopy from zero maps to map in differential modules; 
\[d^1 d^0 : (X, d^0) \to (X, d^0).\]
Then, the map; \(d^1 : X \to X\) is differential with exactness to the homotopy.

**Definition (1-4):**
A map; \(h : X \to Y\) is an essential homotopy function of differential modules, 
\(f, g : (X, d) \to (Y, d)\). The map of graded modules \(h : X \to Y\) of degree one have: 
\[dh + hd = f - g.\]

**Definition (1-5):**
For any two morphisms of arbitrary category \(D^{-}\)-infinity modules 
\(f = \{f^i\} : X \to Y\) and \(g = \{g^i\} : Y \to Z\) is characterized. We choose a structure 
\((gf)\) of morphisms \(f\) and \(g\) is given by the rule; 
\[(gf)^i = \sum_{s \equiv i} g^s f^i : X \to Z.\]
The Identity morphism for the \(D^{-}\)-infinity module \((X, d^i)\) is 
\(1_x = \{1_x^i\} : X \to X,\)
where, \(1_x^0 = 0, i > 0,\) and \(1_x^0\) denote the identity map in module \(X\). By the same manner, the category \(D^{-}\)-infinity modules are characterized.

**Definition (1-6):**
\(D^{(s)}\)-infinity differential modules or \(D^{(s)}\)-infinity modules is a \(D^{-}\)-infinity differential module \((X, d^i)\), denoted as \((X, d^{i+s})\), for an integer \(s \geq 0\), where the conditions \(d^i = 0, \ i < s,\) hold. And its morphisms is defined as 
\[\{d^i_s : X, \to X_{s-1}\},\]
and satisfy the rule, 
\[\sum_{i + j = k} d^{i+s} d^{j+s} = 0.\]
If we put \(s = 0\), at that point we gain a category \(D^{(s)}\)-modules agree with the above particularize classification of \(D_{\infty}\)-modules, and for each fixed number \(s \geq 0\), the category \(D^{(s)}\)-infinity modules are characterized the complete subcategory from \(D^{(s)}\)-infinity modules.

**Proposition (1-7):**
1- For \(D^{(s)}\)-infinity modules \((X, d^{i+s})\), the condition \(d^s d^s = 0\) hold.
2- Specifically, at the point \(s = 1\) we have \(D^{(1)}\)-infinity module \((X, d^{i+1})\).
Definition (1-8):
The homotopy map among the maps $f, g : X \to Y$ which is the morphism of $D^{(-)}$--infinity module is a homeomorphisms;
\[ h = \{ h^{i,s} : X, \to Y, i \geq 0 \} \] (2)
which satisfy the relation: for the integer $k \geq 0$, and;
\[ \sum_{i+j=k} d^{i+s} h^{j,s} + h^{j,s} d^{j+s} = f^k - g^k. \] (3)
If we put $k=0$, then; $d^s h^s + h^s d^s = f^0 - g^0$, and the maps $h^s : X \to Y$ is homotopy between maps $f^0, g^0 : (X, d^s) \to (Y, d^s)$ of the differential modules.

For an arbitrary $D^{(s)}$--infinity modules $(X, d^{i+s})$ and $(Y, d^{i+s})$, obtained SDR-case of $D^{(s)}$--infinity modules: \{ $\eta : (X, d^s) \to (Y, d^s) : \xi, h \}$, if the following statement are hold:
\[ \sum_{i+j=k} \eta h^{i,s} = 0, \quad \sum_{i+j=k} h^{i,s} \xi = 0, \quad \sum_{i+j=k} h^{i,s} h^{j,s}, \quad k \geq 0. \]

Definition (1-9):
For a morphism in $D^{(s)}$--infinity modules $f : h \to g$ the set of a module function $F = \{ f_i : h_i \to g_i, i > 0 \}$ such that: for integer number: $z > 0$, then,
\[ \sum_{i+j=z} f^{i+s} = \sum_{i+j=z} d^{i+s} f^i. \] (4)

Definition (1-10):
The stable $D^{(s)}$--infinity module $(Y, h^{i+s})$ is defined if for an arbitrary $y \in Y$, there exists a $k \geq 0$ depend on $Y$ satisfying; $h^{i,s}(y) = 0$, for $i > k$.

Definition (1-11):
Let $H(Y)$ be the homology of $D^{(s)}$--infinity module $(Y, d^{i+s})$ is define as the homology for a module $Y$ relative to a differential $D_s : Y \to Y$, where $D_s$ is defined by the rule, $D_s = (d^s + d^{s+1} + \ldots + d^{s+i} + \ldots)$.

2. Main results

In the following part we give the main results in the filtration on the differential module. So we give theorems and application in the differential modules with $(1)$--filtrations and establish a relations between the differential modules with $(1)$--filtrations and stable $D^{(0)}$--infinity differential modules.

Definition (2-1):
For an arbitrary differential module $(y, h)$, the filtration \{ $y^n, n \in \mathbb{Z}$ of any set from the graded su-bmodules $y^n \subseteq Y$, with the relations:
... \subseteq Y_n \subseteq Y_{n+1} \subseteq \ldots, \bigcup_{n \in \mathbb{Z}} Y^n = Y, \bigcap_{n \in \mathbb{Z}} Y^n = 0, h(Y^n) \subseteq Y^n, n \in \mathbb{Z}. \quad (5)

In the differential modules with a filtration we define the map:

\[ f : (\mathcal{Y}, \mathcal{Y}^n) \rightarrow (\mathcal{Z}, \mathcal{Z}^n) \quad (6) \]

is defined as an element function \( f : Y \rightarrow Z \) in the differential modules convinces with: \( f(Y^n) \subseteq Z^n, n \in \mathbb{Z} \).

**Definition (2-2):**
For the functions of the differential modules with the filtrations,

\[ F, \varphi : (\mathcal{Y}, \mathcal{Y}^n) \rightarrow (\mathcal{Z}, \mathcal{Z}^n) \]. \quad (7)

The homotopy map \( K \) between them is \( K : Y \rightarrow Z \) satisfying; \( K(Y^n) \subseteq Z^n, n \in \mathbb{Z} \).

**Definition (2-3):**

The one filtration \((1)-filtration\) of the differential module \((X, d)\) is a filtration \(\{X^n\}\) of this differential module have a condition, \(d(X^n) \subseteq X^{n+1}, n \in \mathbb{Z}\). The function of differential modules with \((1)-filtrations\) are defined as a function of the differential modules with filtrations.

In this part we get the main relations between a class category of a module of the differential, with \((1)-filtrations\) and the category of the stable \(D^{(1)}\)-infinity module.

**Theorem (2-4):**

If we have the pair \((X, d)\) is a differential modules with a \((1)-filtration\) \(\{X^n\}\). Take \(Y^k_X\) is determine the submodule of graded module \(X^k\) such that; \(X^k = Y^k_X \oplus X^{k-1}\).

By the fact; \(d(X_k) \subseteq X_{k+1}\), determine a stable \(D^{(1)}\)-infinity module \((X, d^{i+1})\) by the form: \(d^{i+1} = \bigoplus_{k \in \mathbb{Z}} d_k^{i+1} : X^k \rightarrow X^{k+1}, i \geq 0\), where: \(d_k^{i+1}(Y^k_X) \rightarrow (Y^k_{X-1})_i \oplus \ldots \oplus (Y^k_{X-(i+1)})_{i-1} \oplus \ldots\) is an elemental of the diagram;

\[ d : Y^k_X \rightarrow X_{k-1}^{k+1} = \left( (Y^k_{X-1})_i \oplus \ldots \oplus (Y^k_{X-(i+1)})_{i-1} \oplus \ldots \right). \quad (8) \]

The \(D^{(1)}\)-infinity module \((X, d^{i+1})\) is stable \(D^{(1)}\)-infinity module has the relation \((X, D^{(1)}) = (X, D)\), where \(D^{(1)}\) is the comprehend differential of \(D^{(1)}\)-infinity module \((X, d^{i+1})\). Thus, any guide of the modules of the differential over a field with \((1)-filtrations\) and any homotopy between the maps on modules of differential over a defined field with \((1)-filtrations\) particularly characterize a morphism of \(D^{(1)}\)-infinity modules and the homotopy between a homeomorphisms of \(D^{(1)}\)-infinity modules, respectively.
Lemma (2.5):
In a differential module \((Y, D)\) with a \((1)\)-filtration exceptionally each map defines on the graded module \(Y\) is the structure of stable \(D^{(1)}\)-infinity module \((Y, d^{i+1})\) such that \((Y, D_1) = (Y, d)\), where \(D_1\) is the summed differential of \(D^{(1)}\)-infinity module \((Y, d^{i+1})\).

Notes: In addition, each SDR-data of differential modules over the field with \((1)\)-filtrations uniquely defines an SDR-data of stable \(D^{(1)}\)-infinity modules for which the ‘summed’ SDR-data of differential modules agree with the inventive SDR-data of differential modules.

Proposition (2-6):
The combine \(\{(E_s, d_s)\}_{s \geq 1}\) is a spectral module of the sequence of differential module, and define; \(E_{s+1} = H(E_s) = Ker d_s / Im d_s\).

Proof: At that point, \(s = 1\) we get the spectral module \((E_s, d_s)\) is equivalent to \((M, d)\).

From [7], a \(D\)-infinity module over the field which determines the spectral sequence, the facts which related to \(D\)-infinity module are:

I- For a pair \((X, d^i)\) the stable \(D\)-infinity module. We get an essential spectral sequence \(\{E_s, d_s\}\) of \(D\)-infinity modules where \(E_s = (X_s, d^{i+s}), i \geq 0\), and \(d_s^s = d_s\), determined by the \(D\)-infinity module \((X, d^i)\) converges to \(H(X)\). All terms \(E_s\) of this spectral sequence, considered as differential modules with summary differentials \(D_s : E_s \rightarrow E_s\), are homotopy which is equal to each other and equivalent to the differential module \((H(X), d = 0)\).

II- If \(s \geq 0\), the parts of \(E_s\) and \(E_{s+1}\) of the spectral sequence \(\{(E_s, d_s)\}_{s \geq 1}\), considered the \(D^{(s)}\)-infinity modules are homotopy equivalent.

Theorem (2-7):
For the spectral sequence of an absolute module \((X, d)\) of differential, \(\{(E_s, d_s)\}_{s \geq 1}\), we get;

I- For a category spectral \(\{(E_s, d_s)\}_{s \geq 0}\), at that point; \(s \geq 0\), we get the differential \(d_s : E_s \rightarrow E_{s-1}\) on a part \(E_s\) where the parallel homology module on the form;

\[ H(E_s, D_s) = Ker D_s / Im D_s \]  \hspace{1cm} (9)

is isomorphic for \(E\) – infinity of the spectral sequence \(\{E_s, d_s\}\).

II- Any term \((E_s, d_s)\) of the spectral sequence \(\{(E_s, d_s)\}_{s \geq 1}\) there are a main
structure of stable $D(s)$-infinity module $(X, d^{i+s})$ which is joined to the differential $d^s$ in this term by the relation, $d^s_s = d_s$.

**Theorem (2-8):**
The sequence $\{(X_s, d_s)\}_{s \geq 1}$ for a subjective differential module $(X, d)$. Let the (1)-filtration of a differential module bounded $X$, at that point for each, $1 < s$, there is the homology module $H(X_s) = \ker D_s / \text{Im} D_s$ of a stable $D(s)$-infinity module $(X, d^{i+s})$ isomorphic to a limit part $X$-infinity in the spectral $\{(X_s, d_s)\}_{s \geq 1}$ and isomorphic to a homology module, $H(X) = \ker d / \text{Im} d$.

**Applications (2-9):**
Let $\{(Z_s, h_s)\}_{s \geq 1}$ be the homology spectral sequence an arbitrary Serre fibration [2], $P : E \to B$. Then on each liking $(Z_s, h_s)$ of this spectral sequence, there is the form of the stable $D(s)$-infinity module $(Z, h^{i+s})$ which is connected with a differential $h_s$ by the equality, $h^s_s = h_s$.

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**References**


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