Hyers-Ulam Stability of Functional Inequalities with Three Variables in Banach Spaces and Non-Archimedean Banach Spaces

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Abstract

In this paper we study to solve four functional inequalities with three variables and their Hyers-Ulam stability. Two first ones are investigated in Banach spaces and the two last ones are investigated in non-Archimedean Banach spaces. We will show that the solutions of the first and third inequalities are additive mappings and the solutions of the others are quadratic mappings. Then, the Hyers-Ulam stability of these inequalities are given and proven. These are the main results of this paper.

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1. Introduction

Let $X$ and $Y$ be a normed spaces on the same field $\mathbb{K}$, and $f : X \rightarrow Y$ be a mapping. We use the notation $\|\cdot\|$ for the norms on both $X$ and $Y$. In this paper, we investigate some functional inequalities when $X$ is a normed space and $Y$ is a Banach space or when $X$ is a non-Archimedean normed space and $Y$ is a non-Archimedean Banach space. In fact, when $X$ is a normed space and $Y$ is a Banach space we solve and prove the Hyers-Ulam stability of two following functional inequalities

$$
\left\| f \left( \frac{x + y}{2} + z \right) - f \left( \frac{x + y}{2} \right) - f(z) \right\| \leq \left\| f \left( \frac{x + y}{2} + \frac{z}{2} \right) - \frac{1}{2} f \left( \frac{x + y}{2} \right) - \frac{1}{2} f(z) \right\|,
$$

(1)

$$
\left\| f \left( \frac{x + y}{2} + z \right) + f \left( \frac{x + y}{2} - z \right) - 2f \left( \frac{x + y}{2} \right) - 2f(z) \right\|
\leq \left\| f \left( \frac{x + y}{2^2} + \frac{z}{2} \right) + f \left( \frac{x + y}{2^2} - \frac{z}{2} \right) - \frac{1}{2} f \left( \frac{x + y}{2} \right) - \frac{1}{2} f(z) \right\|,
$$

(2)

and when $X$ is a non-Archimedean normed space and $Y$ is a non-Archimedean Banach space we solve and prove the Hyers-Ulam stability of two following functional inequalities

$$
\left\| f \left( \frac{x + y}{2^2} + \frac{z}{2} \right) - \frac{1}{2} f \left( \frac{x + y}{2} \right) - \frac{1}{2} f(z) \right\| \leq \left\| f \left( \frac{x + y}{2} + z \right) - f \left( \frac{x + y}{2} \right) - f(z) \right\|,
$$

(3)

$$
\left\| f \left( \frac{x + y}{2^2} + \frac{z}{2} \right) + f \left( \frac{x + y}{2^2} - \frac{z}{2} \right) - \frac{1}{2} f \left( \frac{x + y}{2} \right) - \frac{1}{2} f(z) \right\|
\leq \left\| f \left( \frac{x + y}{2} + z \right) + f \left( \frac{x + y}{2} - z \right) - 2f \left( \frac{x + y}{2} \right) - 2f(z) \right\|.
$$

(4)

The notions of non-Archimedean normed space and non-Archimedean Banach space will remind in the next section. The Hyers-Ulam stability was first investigated for functional equation of Ulam in [12] concerning the stability of group homomorphisms. Then, Hyers [6] gave a first affirmative partial answer to the equation of Ulam in Banach spaces. After that, Hyers’Theorem was generalized by Aoki[1] for additive mappings and by Rassias [10] for linear mappings considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Gvruta [3] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias’ approach.

The Hyers-Ulam stability for functional inequalities have been investigated such as in [4, 11], Gilanyi showed that is if $f$ satisfies the functional inequality

$$
\left\| 2f(x) + 2f(y) - f(xy^{-1}) \right\| \leq \|f(xy)\|
$$

(5)

then $f$ satisfies the Jordan-von Neumann functional equation

$$
2f(x) + 2f(y) = f(xy) + f(xy^{-1}).
$$

(6)

\[
\|f(x + y) - f(x) - f(y)\| \leq \left\| f\left(\frac{x + y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y)\right\|, \tag{7}
\]

\[
\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\|
\leq \left\| f\left(\frac{x + y}{2}\right) + f\left(\frac{x - y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y)\right\|, \tag{8}
\]

in Banach spaces and the following functional inequalities

\[
\left\| f\left(\frac{x + y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y)\right\| \leq \|f(x + y) - f(x) - f(y)\|, \tag{9}
\]

\[
\left\| f\left(\frac{x + y}{2}\right) + f\left(\frac{x - y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y)\right\| \leq \|f(x + y) + f(x - y) - 2f(x) - 2f(y)\|. \tag{10}
\]

in Non-Archimedean Banach spaces.

In this paper, we solve and prove the Hyers-Ulam stability for four functional inequalities (1)-(4), i.e. the functional inequalities with three variables. We follow the approach of the authors in [7, 8]. Under suitable assumptions on spaces \(X\) and \(Y\), we will prove that the mappings satisfying the inequalities (1) or (3) are additive and those satisfying the inequalities (2) or (4) are quadratic. Thus, The results in this paper are generalization of those in [7, 8] for functional inequalities with three variables.

The paper is organized as follows: In Section 2 we remind some basic notations in [7, 8] such as Non-Archimedean field, Non-Archimedean normed space and Non-Archimedean Banach space. Then, we prove that the mappings (1) or (3) ((2) or (4)) are respectively additive (quadratic). Section 3 is devoted to prove the Hyers-Ulam stability of the functional inequalities (1) and (2) when \(X\) is a normed space and \(Y\) is a Banach spaces. In Section 3.2 we prove the Hyers-Ulam stability of the functional inequalities (2) and (4) when \(X\) is a non-Archimedean normed space and \(Y\) is a non-Archimedean Banach space. Finally, we give some remarks and conclusions in Section 4.

2. Preliminaries

2.1. Non-Archimedean normed and Banach spaces. In this subsection we recall some basic notations from [7, 8] such as non-Archimedean fields, non-Archimedean normed spaces and non-Archimedean Banach spaces.

A valuation is a function \(\|\cdot\|\) from a field \(\mathbb{K}\) into \([0, \infty)\) such that 0 is the unique element having the 0 valuation,

\[\|r \cdot s\| := \|r\| \cdot |s|, \forall r, s \in \mathbb{K}\]
and the triangle inequality holds, i.e.,

\[ |r + s| \leq |r| + |s|, \forall r, s \in \mathbb{K}. \]

A field $\mathbb{K}$ is called a valued field if $\mathbb{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuation. Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the strong triangle inequality is replaced by

\[ |r + s| \leq \max\{|r|, |s|\}, \forall r, s \in \mathbb{K}, \]

then the function $|\cdot|$ is called a non-Archimedean valuation. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1, \forall n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ talking everything except for 0 into 1 and $|0| = 0$. In this paper, we assume that the base field is a non-Archimedean field with $|2| \neq 1$, hence call it simply a field.

**Definition 1.** Let be a vector space over a field $\mathbb{K}$ with a non-Archimedean $|\cdot|$. A function $\|\cdot\| : X \to [0, \infty)$ is said a non-Archimedean norm if it satisfies the following conditions:

1. $\|x\| = 0$ if and only if $x = 0$;
2. $\|rx\| = |r|\|x\| (r \in \mathbb{K}, x \in X)$;
3. the strong triangle inequality hold.

Then, $(X, \|\cdot\|)$ is called a non-Archimedean normed space.

**Definition 2.**

1. Let $\{x_n\}$ be a sequence in a non-Archimedean normed space $X$. Then, sequence $\{x_n\}$ is called Cauchy if for any given $\epsilon > 0$ there a positive integer $N$ such that $\|x_n - x_m\| \leq \epsilon$ for all $n, m \geq N$.
2. Let $\{x_n\}$ be a sequence in a non-Archimedean normed space $X$. Then, sequence $\{x_n\}$ is called Cauchy if for any given $\epsilon > 0$ there a positive integer $N$ such that

\[ \|x_n - x\| \leq \epsilon \]

for all $n, m \geq N$. Then, we call $x \in X$ a limit of sequence $x_n$ and denote $\lim_{n \to \infty} x_n = x$.
3. If every Cauchy sequence in $X$ converges, then the non-Archimedean normed space $X$ is called a non-Archimedean Banach space.

### 2.2. Solutions of the inequalities

The functional equation

\[ f(x + y) = f(x) + f(y) \tag{11} \]

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping. The function equation

\[ f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{12} \]

is called the quadratic function equation. In particular, every solution of the quadratic function equation is said to be a quadratic mapping [7, 8].

Now, we first study the the solutions of (1) and (2). Note that for these inequalities, $X$ is a normed space and $Y$ is a Banach space. Under this setting, we can show that the
mapping satisfying (1) is additive and that satisfying (2) is quadratic. These results are
given in the following lemmas.

**Lemma 1.** A mapping \( f : X \to Y \) satisfies
\[
\left\| f \left( \frac{x+y}{2} + z \right) - f \left( \frac{x+y}{2} \right) - f(z) \right\| \leq \left\| f \left( \frac{x+y}{2^2} + \frac{z}{2} \right) - \frac{1}{2} f \left( \frac{x+y}{2} \right) - \frac{1}{2} f(z) \right\|
\]
for all \( x, y, z \in X \) if and only if \( f : X \to Y \) is additive.

**Proof.** Assume that \( f : X \to Y \) satisfies (13). Letting \( x = y = z = 0 \) in (13), we get \( \|f(0)\| \leq 0 \). It implies that \( f(0) = 0 \). Furthermore, letting \( x = y = z \) in (13), we get \( \|f(2x) - 2f(x)\| \leq 0 \) and so \( f(2x) = 2f(x) \) for all \( x \in X \). Thus \( f \left( \frac{z}{2} \right) = \frac{1}{2} f(x) \) for all \( x \in X \). Then, it follows from (13) that
\[
\left\| f \left( \frac{x+y}{2} + z \right) - f \left( \frac{x+y}{2} \right) - f(z) \right\| \leq \left\| f \left( \frac{x+y}{2^2} + \frac{z}{2} \right) - \frac{1}{2} f \left( \frac{x+y}{2} \right) - \frac{1}{2} f(z) \right\|
\]
\[
= \left\| f \left( \frac{x+y}{2^2} + \frac{z}{2} \right) - \frac{1}{2} f \left( \frac{x+y}{2} \right) - \frac{1}{2} f(z) \right\|
\]
\[
= \frac{1}{2} \left\| f \left( \frac{x+y}{2} + z \right) - f \left( \frac{x+y}{2} \right) - f(z) \right\|
\]
\[
= \frac{1}{2} \left\| f \left( \frac{x+y}{2} + z \right) - \frac{1}{2} f \left( \frac{x+y}{2} \right) - \frac{1}{2} f(z) \right\|
\]
and so
\[
f \left( \frac{x+y}{2} + z \right) = f \left( \frac{x+y}{2} \right) + f(z)
\]
for all \( x, y, z \in X \). This implies that \( f \) is additive. The converse is obviously true. \( \square \)

**Lemma 2.** A mapping \( f : X \to Y \) satisfies
\[
\left\| f \left( \frac{x+y}{2} + z \right) + f \left( \frac{x+y}{2} - z \right) - 2f \left( \frac{x+y}{2} \right) - 2f(z) \right\| \leq \left\| f \left( \frac{x+y}{2^2} + \frac{z}{2} \right) + f \left( \frac{x+y}{2^2} - \frac{z}{2} \right) - \frac{1}{2} f \left( \frac{x+y}{2} \right) - \frac{1}{2} f(z) \right\|
\]
for all \( x, y \in X \) if and only if \( f : X \to Y \) is quadratic.

**Proof.** Assume that \( f : X \to Y \) satisfies (14). Letting \( x = y = z = 0 \) in (14), we get \( \|2f(0)\| \leq \|f(0)\| \). As a result, \( f(0) = 0 \). Furthermore, letting \( x = y = z \) in (14), we get \( \|f(2x) - 4f(x)\| \leq 0 \) and so \( f(2x) = 4f(x) \) for all \( x \in X \). Therefore, \( f \left( \frac{x}{2} \right) = \frac{1}{4} f(x) \) for
all \( x \in X \). As a result, it follows from (14) that

\[
\left\| f \left( \frac{x + y}{2} + z \right) + f \left( \frac{x + y}{2} - z \right) - 2f \left( \frac{x + y}{2} \right) - 2f(z) \right\| \\
\leq \left\| f \left( \frac{x + y}{2} + \frac{z}{2} \right) + f \left( \frac{x + y}{2} - \frac{z}{2} \right) - \frac{1}{2} f \left( \frac{x + y}{2} \right) - \frac{1}{2} f(z) \right\| \\
= \left\| f \left( \frac{x + y}{2} + \frac{z}{2} \right) + f \left( \frac{x + y}{2} - \frac{z}{2} \right) - \frac{1}{2} f \left( \frac{x + y}{2} \right) - \frac{1}{2} f(z) \right\| \\
= \left\| \frac{1}{4} f \left( \frac{x + y}{2} + z \right) + \frac{1}{4} f \left( \frac{x + y}{2} - z \right) - \frac{1}{2} f \left( \frac{x + y}{2} \right) - \frac{1}{2} f(z) \right\| \\
= \frac{1}{4} \left\| f \left( \frac{x + y}{2} + z \right) + f \left( \frac{x + y}{2} - z \right) - 2f \left( \frac{x + y}{2} \right) - 2f(z) \right\| 
\]

and so

\[
f \left( \frac{x + y}{2} + z \right) + f \left( \frac{x + y}{2} - z \right) = 2f \left( \frac{x + y}{2} \right) + 2f(z)
\]

for all \( x, y, z \in X \). Thus, \( f \) is quadratic mapping. The converse is obviously true. \( \square \)

Next, we are going to study the solutions of (3) and (4). For these inequalities, the space \( X \) is a non-Archimedean normed space and the space \( Y \) is a non-Archimedean Banach space, and the field \( K \) satisfy \(|2| \neq 1\). Under this setting, we can show that any odd mapping satisfying (3) is additive and any even mapping satisfying (4) is quadratic. These results are given in the following lemmas.

**Lemma 3.** An odd mapping \( f: X \rightarrow Y \) satisfies

\[
\left\| f \left( \frac{x + y}{2} + \frac{z}{2} \right) - \frac{1}{2} f \left( \frac{x + y}{2} \right) - \frac{1}{2} f(z) \right\| \leq \left\| f \left( \frac{x + y}{2} + z \right) - f \left( \frac{x + y}{2} \right) - f(z) \right\| \tag{15}
\]

for all \( x, y, z \in X \) if and only if \( f: X \rightarrow Y \) is additive.

**Proof.** Assume that \( f: X \rightarrow Y \) satisfies (15). Letting \( x = -y \) in (15), we get

\[
\left\| f \left( \frac{z}{2} \right) - \frac{1}{2} f(z) \right\| \leq 0
\]

and so

\[
f \left( \frac{z}{2} \right) = \frac{1}{2} f(z)
\]
for all \( z \in X \). Therefore,

\[
\frac{1}{2} \left\| f \left( \frac{x + y}{2} + z \right) - f \left( \frac{x + y}{2} \right) \right\| \\
= \left\| \frac{1}{2} \left( f \left( \frac{x + y}{2} + z \right) - f \left( \frac{x + y}{2} \right) \right) \right\| \\
= \left\| f \left( \frac{x + y}{2^2} + \frac{z}{2} \right) - \frac{1}{2} f \left( \frac{x + y}{2} \right) - \frac{1}{2} f(z) \right\| \\
\leq \left\| f \left( \frac{x + y}{2} + z \right) - f \left( \frac{x + y}{2} \right) - f(z) \right\| 
\] (16)

for all \( x, y, z \in X \). Since \( |2| < 1 \),

\[
f \left( \frac{x + y}{2} + z \right) = f \left( \frac{x + y}{2} \right) + f(z)
\]

for all \( x, y, z \in X \). Letting \( x = y \) we imply that \( f \) is additive. The converse is obviously true.

**Lemma 4.** An even mapping \( f : X \to Y \) satisfies

\[
\left\| f \left( \frac{x + y}{2^2} + \frac{z}{2} \right) + f \left( \frac{x + y}{2^2} - \frac{z}{2} \right) - \frac{1}{2} f \left( \frac{x + y}{2} \right) - \frac{1}{2} f(z) \right\| \\
\leq \left\| f \left( \frac{x + y}{2} + z \right) + f \left( \frac{x + y}{2} - z \right) - 2 f \left( \frac{x + y}{2} \right) - 2 f(z) \right\| 
\] (17)

for all \( x, y, z \in X \) if and only if \( f : X \to Y \) is quadratic.

**Proof.** Assume that \( f : X \to Y \) satisfies (17). Let \( x = y = z = 0 \) in (17), we get

\[
\|f(0)\| \leq \|2f(0)\| = |2\|f(0)\|. 
\]

Since \( |2| \neq 1 \), it implies that \( f(0) = 0 \). Now, let \( x = -y \) in (17), we get

\[
\left\| f \left( \frac{z}{2} \right) + f \left( \frac{-z}{2} \right) - \frac{1}{2} f(z) \right\| = \left\| 2 f \left( \frac{z}{2} \right) - \frac{1}{2} f(z) \right\| \leq \left\| f(z) + f(-z) - 2 f(z) \right\| = 0
\]

So
$\|2f(\frac{z}{2}) - \frac{1}{2}f(z)\| \leq 0$ and so $f(\frac{z}{2}) = \frac{1}{4}f(z)$ for all $z \in X$. Thus in (17)

$$\frac{1}{2^2}\|f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x+y}{2} - z\right) - 2f\left(\frac{x+y}{2}\right) - 2f(z)\|$$

$$= \left\|\frac{1}{4}\left[f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x+y}{2} - z\right) - 2f\left(\frac{x+y}{2}\right) - 2f(z)\right]\right\|$$

$$= \frac{1}{4}\left|f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x+y}{2} - z\right) - \frac{1}{2}f\left(\frac{x+y}{2}\right) - \frac{1}{2}f(z)\right|$$

$$= \left|f\left(\frac{x+y}{2^2} + \frac{z}{2}\right) + f\left(\frac{x+y}{2^2} - \frac{z}{2}\right) - \frac{1}{2}f\left(\frac{x+y}{2}\right) - \frac{1}{2}f(z)\right|$$

$$\leq \left|f\left(\frac{x+y}{2^2} + \frac{z}{2}\right) + f\left(\frac{x+y}{2} - \frac{z}{2}\right) - 2f\left(\frac{x+y}{2}\right) - 2f(z)\right|$$

for all $x,y,z \in X$. Since $2 < 1$,

$$f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x+y}{2} - z\right) = 2f\left(\frac{x+y}{2}\right) + 2f(z)$$

for all $x,y,z \in X$. we imply that $f$ is a quadratic mapping. The converse is obviously true. □

### 3. Hyers-Ulam stability of the inequalities

#### 3.1. Functional inequalities in Banach spaces.

In this subsection we prove the Hyers-Ulam stability of the inequalities (1)-(4). Firstly, we study the inequality (1) in the normed space $X$ and the Banach space $Y$. The result is given in two following theorems.

**Theorem 1.** Let $r > 1$ and $\theta$ be a nonnegative real number, and let $f : X \to Y$ be a mapping such that

$$\left\|f\left(\frac{x+y}{2} + z\right) - f\left(\frac{x+y}{2}\right) - f(z)\right\| \leq r\|x\|^r + r\|y\|^r + r\|z\|^r$$

for all $x,y,z \in X$. Then, there exists a unique additive mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \leq \frac{3\theta}{2^r - 2}\|x\|^r$$

for all $x \in X$.

**Proof.** Letting $x = y = z$ in (18), we get

$$\|f(2x) - 2f(x)\| \leq 3\|x\|^r, \forall x \in X.$$ (20)
substituting \( x \) by \( x/2 \) in the last inequality, we have

\[
\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \leq \frac{3}{2^r} \theta \| x \|^r, \quad \forall x \in X.
\]

Hence

\[
\left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| \leq \sum_{j=l}^{m-1} 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \leq \frac{3}{2^r} \sum_{j=l}^{m-1} 2^j \frac{2^r}{2^{2j}} \theta \| x \|^r \quad (21)
\]

for all non-negative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (21) that the sequence \( \{2^n f\left(\frac{x}{2^n}\right)\} \) is a Cauchy sequence in \( Y \) for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{2^n f\left(\frac{x}{2^n}\right)\} \) converges.

Therefore, we can define the mapping \( h : X \to Y \) by \( h(x) := \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right) \) for every \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (21), we get (19).

It follows from (18) that

\[
\left\| h\left(\frac{x+y}{2} + z\right) - h\left(\frac{x+y}{2}\right) - h(z) \right\| = \lim_{n \to \infty} 2^n \left\| f\left(\frac{x+y}{2^{n+1}} + \frac{z}{2^n}\right) - f\left(\frac{x+y}{2^{n+1}}\right) - f\left(\frac{z}{2^n}\right) \right\|
\]

\[
\leq \lim_{n \to \infty} 2^n \left( f\left(\frac{x+y}{2^{n+2}} + \frac{z}{2^{n+1}}\right) - \frac{1}{2} f\left(\frac{x+y}{2^{n+1}}\right) - \frac{1}{2} f\left(\frac{z}{2^n}\right) \right)
\]

\[
+ \lim_{n \to \infty} \frac{2^n \theta}{2^{2n}} \left( \| x \|^r + \| y \|^r + \| z \|^r \right)
\]

\[
= \left\| h\left(\frac{x+y}{2^2} + \frac{z}{2}\right) - \frac{1}{2} h\left(\frac{x+y}{2}\right) - \frac{1}{2} h(z) \right\|
\]

for all \( x,y,z \in X \). Therefore,

\[
\left\| h\left(\frac{x+y}{2} + z\right) - h\left(\frac{x+y}{2}\right) - h(z) \right\| \leq \left\| h\left(\frac{x+y}{2^2} + \frac{z}{2}\right) - \frac{1}{2} h\left(\frac{x+y}{2}\right) - \frac{1}{2} h(z) \right\|.
\]

By Lemma 1 it implies that \( h \) is additive. Now, we will prove that the additive mapping \( h \) is unique. In fact, let \( T : X \to Y \) be an additive mapping satisfying (19). Then, we have

\[
\left\| h(x) - T(x) \right\| = 2^n \left\| h\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right) \right\|
\]

\[
\leq 2^n \left( \left\| h\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| \right)
\]

\[
\leq \frac{2^n}{(2^r - 2)2^{2n}} 3 \theta \| x \|^r,
\]

which tends to zero as \( n \to \infty \) for all \( x \in X \). Therefore, we can conclude that \( h(x) = T(x) \) for all \( x \in X \). This proves that the additive mapping \( h : X \to Y \) is a unique mapping satisfying (19). \( \square \)
Theorem 2. Let $r < 1$ and $\theta$ is a positive real number, and let $f : X \to Y$ be a mapping satisfying
\[
\left\| f \left( \frac{x + y}{2} + z \right) - f \left( \frac{x + y}{2} \right) - f(z) \right\| \leq \left\| f \left( \frac{x + y}{2^2} + \frac{z}{2} \right) - \frac{1}{2} f \left( \frac{x + y}{2} \right) - \frac{1}{2} f(z) \right\|
+ \theta (\| x \|^r + \| y \|^r + \| z \|^r) \tag{22}
\]
for all $x, y, z \in X$. Then, there exists a unique additive mapping $h : X \to Y$ such that
\[
\left\| f(x) - h(x) \right\| \leq \frac{3\theta}{2 - 2^r} \| x \|^r \tag{23}
\]
for all $x \in X$.

Proof. Letting $x = y = z$ in (22), we get
\[
\left\| f(2x) - 2f(x) \right\| \leq 3\theta \| x \|^r
\]
for all $x \in X$. So for all $x \in X$. Hence
\[
\left\| f(x) - \frac{1}{2} f(2x) \right\| \leq \frac{3\theta}{2} \| x \|^r
\]
for all $x \in X$. Hence
\[
\left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^m} f(2^m x) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x) \right\|
\leq \frac{3\theta}{2} \sum_{j=l}^{m-1} \frac{2^j}{2^{j+1}} \| x \|^r \tag{24}
\]
for all non-negative integer $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (24) that the sequence $\left\{ \frac{1}{2} f(2^n x) \right\}$ is a Cauchy sequence for all $x \in X$ converges. So one can define the mapping $h : X \to Y$ by $h(x) := \lim_{n \to \infty} \frac{1}{2} f(2^n x)$ for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (24), we get (23). The rest of the proof is similar to the proof of Theorem 1. $\square$

Note that for $r = 1$ we could not give any results on the Hyers-Ulam stability of (1) because the neither series (21) nor series (24) converges.

Next, we are going to study the inequality (2). Note that for this inequality $X$ is a normed space and $Y$ is a Banach space. The Hyers-Ulam stability of the inequality (2) is given in two next theorems.
Theorem 3. Let \( r > 2 \) and \( \theta \) be nonnegative real number, and let \( f : X \to Y \) be a mapping such that
\[
\left\| f\left( \frac{x+y}{2} + z \right) + f\left( \frac{x+y}{2} - z \right) - 2f\left( \frac{x+y}{2} \right) \right\| \\
\leq \left\| f\left( \frac{x+y}{2^2} + z \right) + f\left( \frac{x+y}{2^2} - z \right) - \frac{1}{2} f\left( \frac{x+y}{2^2} \right) \right\| \\
+ \theta (\|x\|^r + \|y\|^r + \|z\|^r)
\]
(25)
for all \( x, y, z \in X \). Then, there exists a unique quadratic mapping \( h : X \to Y \) such that
\[
\left\| f(x) - h(x) \right\| \leq \frac{3\theta}{2^r - 4} \|x\|^r
\]
(26)
for all \( x \in X \).

Proof. Letting \( x = y = z \) in (25), we get
\[
\left\| f(2x) - 4f(x) \right\| \leq 3\theta \|x\|^r
\]
for all \( x \in X \).

So
\[
\left\| f(x) - 4f\left( \frac{x}{2} \right) \right\| \leq \frac{3\theta}{2^r} \|x\|^r
\]
for all \( x \in X \). Hence
\[
\left\| 4^j f\left( \frac{x}{2^j} \right) - 4^m f\left( \frac{x}{2^m} \right) \right\| \leq \sum_{j=0}^{m-1} \left\| 4^j f\left( \frac{x}{2^j} \right) - 4^{j+1} f\left( \frac{x}{2^{j+1}} \right) \right\| \leq \frac{3\theta}{2^r} \sum_{j=k}^{l-1} 2^j \|x\|^r
\]
(27)
for all non-negative integers \( k \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (27) that the sequence \( \{4^n f(\frac{x}{2^n})\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{4^n f(\frac{x}{2^n})\} \) converges. Therefore, one can define the mapping \( h : X \to Y \) by
\[
h(x) := \lim_{n \to \infty} 2^n f\left( \frac{x}{2^n} \right)
\]
for all \( x \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (27), we get (26).

It follows from (25) that
\[
\left\| h\left( \frac{x+y}{2} + z \right) + h\left( \frac{x+y}{2} - z \right) - 2h\left( \frac{x+y}{2} \right) \right\| \\
= \lim_{n \to \infty} 4^n \left\| f\left( \frac{x+y}{2^{n+1}} + \frac{z}{2^n} \right) + f\left( \frac{x+y}{2^{n+1}} - \frac{z}{2^n} \right) - 2f\left( \frac{x+y}{2^{n+1}} \right) \right\| \\
\leq \lim_{n \to \infty} 4^n \left\| f\left( \frac{x+y}{2^{n+2}} + \frac{z}{2^{n+1}} \right) + f\left( \frac{x+y}{2^{n+2}} + \frac{z}{2^{n+1}} \right) - \frac{1}{2} f\left( \frac{x+y}{2^{n+2}} \right) \right\| \\
+ \lim_{n \to \infty} 4^n \theta \left( \|x\|^r + \|y\|^r + \|z\|^r \right)
\]
\[
= \left\| h\left( \frac{x+y}{2^2} + \frac{z}{2} \right) + h\left( \frac{x+y}{2^2} - \frac{z}{2} \right) - \frac{1}{2} h\left( \frac{x+y}{2} \right) \right\|
\]
for all $x, y, z \in X$.

So

$$
\left\| h\left(\frac{x+y}{2} + z\right) + h\left(\frac{x+y}{2} - z\right) - 2f\left(\frac{x+y}{2}\right) - 2h(z) \right\|
\leq
\left\| h\left(\frac{x+y}{2^2} + \frac{z}{2}\right) + h\left(\frac{x+y}{2^2} - \frac{z}{2}\right) - \frac{1}{2}h\left(\frac{x+y}{2}\right) - \frac{1}{2}h(z) \right\|
$$

for all $x, y, z \in X$.

By lemma 2, the mapping $h : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be mapping satisfy (26). Then, we have

$$
\|h(x) - T(x)\| = 4^n \left\| h\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right) \right\|
\leq 4^n \left( \left\| h\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| \right)
\leq \frac{6.4^n \theta}{(2^n - 4)2^n} \|x\|^r
$$

which tends to zero as $n \to \infty$ for all $x \in X$. Thus, we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves uniqueness of $h$. As a results the mapping $h : X \to Y$ is a unique quadratic mapping satisfying (26).

**Theorem 4.** Let $r < 2$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying

$$
\left\| f\left(\frac{x+y}{2} + z\right) + f\left(\frac{x+y}{2} - z\right) - 2f\left(\frac{x+y}{2}\right) - 2f(z) \right\|
\leq
\left\| f\left(\frac{x+y}{2^2} + \frac{z}{2}\right) + f\left(\frac{x+y}{2^2} - \frac{z}{2}\right) - \frac{1}{2}f\left(\frac{x+y}{2}\right) - \frac{1}{2}f(z) \right\|
+ \theta(\|x\|^r + \|y\|^r + \|z\|^r)
$$

for all $x, y, z \in X$. Then, there exists a unique quadratic mapping $h : X \to Y$ such that

$$
\|f(x) - h(x)\| \leq \frac{3\theta}{4 - 2r} \|x\|^r
$$

for all $x \in X$.

**Proof.** Letting $x = y = z$ in (28), we get

$$
\|f(2x) - 4f(x)\| \leq 3\|x\|^r
$$

for all $x \in X$. Thus,

$$
\left\| f(x) - \frac{1}{4}f(2x) \right\| \leq \frac{3}{4} \|x\|^r
$$
for all $x \in X$. The last inequality implies that
\[
\left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x) \right\| \leq \sum_{j=1}^{m-1} \left\| \frac{1}{4^j} f(2^j x) - \frac{1}{4^{j+1}} f(2^{j+1} x) \right\| \\
\leq \frac{3\theta}{4} \sum_{j=1}^{m-1} 2^{rj} \|x\|^r
\]
(30)
for all non-negative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (30) that the sequence $\{ \frac{1}{4^m} f(2^m x) \}$ is a Cauchy sequence for all $x \in X$. Since $Y$ is a Banach space, the sequence $\{ \frac{1}{4^m} f(2^m x) \}$ converges. Therefore, one can define the mapping $h : X \to Y$ by $h(x) := \lim_{n \to \infty} \frac{1}{4^n} f(2^n x)$ for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (30), we get (29).

The rest of the proof is similar to the proof of Theorem 3. □

3.2. Functional inequalities in Non-Archimedean Banach spaces. This subsection is devoted to prove the Hyers-Ulam stability of the functional inequalities (3) and (4). Note that for these inequalities $X$ is a non-Archimedean normed space and $Y$ is a non-Archimedean Banach space. Two following theorem are such results for (3).

Theorem 5. Let $r < 1$ and $\theta$ be nonnegative real number, and let $f : X \to Y$ be an old mapping such that
\[
\left\| f\left( \frac{x+y}{2^2} + \frac{z}{2} \right) - \frac{1}{2} f\left( \frac{x+y}{2} \right) - \frac{1}{2} f(z) \right\| \leq \left\| f\left( \frac{x+y}{2} \right) - f\left( \frac{x+y}{2} \right) - f(z) \right\| \\
+ \theta (\|x\|^r + \|y\|^r + \|z\|^r)
\]
(31)
for all $x, y, z \in X$. Then, there exists a unique additive mapping $A : X \to Y$ such that
\[
\left\| f(z) - A(z) \right\| \leq \theta \|z\|^r
\]
(32)
for all $z \in X$.

Proof. Letting $x = y = 0$ in (31), we get
\[
\left\| f\left( \frac{z}{2} \right) - \frac{1}{2} f(z) \right\| \leq \theta \|z\|^r
\]
(33)
for all $z \in X$. Therefore,
\[
\left\| f(z) - 2 f\left( \frac{z}{2} \right) \right\| \leq |2| \|z\|^r
\]
for all $z \in X$.

Using the last inequality, we imply that

$$
\begin{align*}
\left\| 2^lf\left(\frac{z}{2^l}\right) - 2^mf\left(\frac{z}{2^m}\right) \right\| & \leq \max\left\{ \left\| 2^lf\left(\frac{z}{2^l}\right) - 2^{l+1}f\left(\frac{z}{2^{l+1}}\right) \right\|, \ldots, \left\| 2^{m-1}f\left(\frac{z}{2^{m-1}}\right) - 2^mf\left(\frac{z}{2^m}\right) \right\| \right\} \\
& \leq \max\left\{ 2^l\left\| f\left(\frac{z}{2^l}\right) - 2f\left(\frac{z}{2^{l+1}}\right) \right\|, \ldots, \left\| f\left(\frac{x}{2^{m-1}}\right) - 2f\left(\frac{z}{2^m}\right) \right\| \right\} \\
& \leq \max\left\{ \frac{2^l}{|2^{l-r}|}, \ldots, \frac{2^{m-1}}{|2^{m-(m-1)}|} \right\} 2^l\theta\left\| z \right\|^r \\
& = \frac{2}{|2|(r-1)}\theta\| z \|^r 
\end{align*}
$$

for all non-negative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (34) that the sequence $\left\{ \frac{1}{2^k}f\left(2^kz\right) \right\}$ is a Cauchy sequence for all $z \in X$. Since $Y$ is a Archimedean Banach space, the sequence $\left\{ \frac{1}{2^k}f\left(2^kz\right) \right\}$ converges. Therefore, one can define the mapping $A : X \to Y$ by $A(z) := \lim_{n \to \infty} \frac{1}{2^k}f\left(\frac{z}{2^k}\right)$ for all $z \in X$. It is easy to show that $A$ is an odd mapping (since $f$ is odd). Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (34), we get (32).

Now, let $T : X \to Y$ be another additive mapping satisfying satisfying (32). Then we have

$$
\begin{align*}
\left\| A(z) - T(z) \right\| &= \left\| 2^q A\left(\frac{z}{2^q}\right) - 2^q T\left(\frac{z}{2^q}\right) \right\| \\
& \leq \max\left\{ 2^q \left\| A\left(\frac{z}{2^q}\right) - 2^q f\left(\frac{z}{2^q}\right) \right\|, \left\| A\left(\frac{x}{2^q}\right) - 2^q f\left(\frac{z}{2^q}\right) \right\| \right\} \\
& \leq \frac{2^q}{|2|^{(r-1)q}}\theta\| z \|^r,
\end{align*}
$$

which tend to zero as $q \to \infty$ for all $z \in X$. Therefore, $A(z) = T(z)$ for all $z \in X$. Using (31) we have

$$
\begin{align*}
\left\| A\left(\frac{x+y}{2} + \frac{z}{2}\right) - \frac{1}{2} A\left(\frac{x+y}{2}\right) - \frac{1}{2} A\left(\frac{z}{2}\right) \right\| \\
& = \lim_{n \to \infty} \left\| 2^n f\left(\frac{x+y}{2^{n+2}} + \frac{z}{2^{n+1}}\right) - \frac{1}{2} f\left(\frac{x+y}{2^{n+1}}\right) - \frac{1}{2} f\left(\frac{z}{2^n}\right) \right\| \\
& \leq \lim_{n \to \infty} \left\| 2^n f\left(\frac{x+y}{2^{n+1}} + \frac{z}{2^n}\right) - \frac{1}{2} f\left(\frac{x+y}{2^{n+1}}\right) - \frac{1}{2} f\left(\frac{z}{2^n}\right) \right\| + \lim_{n \to \infty} \frac{2^n}{|2|^{r-1}q}\theta\left(\| x \|^r + \| y \|^r + \| z \|^r \right) \\
& = \left\| A\left(\frac{x+y}{2} + z\right) - A\left(\frac{x+y}{2}\right) - A\left(\frac{z}{2}\right) \right\|
\end{align*}
$$

for all $x, y, z \in X$. By Lemma 3, the mapping $A : X \to Y$ is additive. □
Theorem 6. Let $r > 1$ and $\theta$ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying
\[
\left\| f\left( \frac{x + y}{2} + \frac{z}{2} \right) - \frac{1}{2} f\left( \frac{x + y}{2} \right) - \frac{1}{2} f\left( \frac{z}{2} \right) \right\| \leq \left\| f\left( \frac{x + y}{2} + z \right) - f\left( \frac{x + y}{2} \right) - f(z) \right\|
\]
\[\quad + \theta (\|x\|^r + \|y\|^r + \|z\|^r). \tag{35}\]
Then there exists a unique additive mapping $A : X \to Y$ such that
\[
\left\| f(z) - A(z) \right\| \leq 2\|\theta\|z\|^r \tag{36}\]
for all $z \in X$.

Proof. Letting $x = y = 0$ in (35), we get
\[
\left\| f\left( \frac{z}{2} \right) - \frac{1}{2} f\left( z \right) \right\| \leq \theta \|z\|^r \tag{37}\]
for all $z \in X$. So
\[
\left\| f(z) - \frac{1}{2} f(2z) \right\| \leq 2\|\theta\|z\|^r
\]
for all $z \in X$.

The rest of the proof is similar to the proof of Theorem 5. \qed

Before closing this subsection we move to study the Hyers-Ulam stability of (4). The results are given in the following theorems.

Theorem 7. Let $r < 2$ and $\theta$ be nonnegative real number, and let $f : X \to Y$ be an even mapping such that
\[
\left\| f\left( \frac{x + y}{2} + \frac{z}{2} \right) + f\left( \frac{x + y}{2} - \frac{z}{2} \right) - \frac{1}{2} f\left( \frac{x + y}{2} \right) - \frac{1}{2} f(z) \right\|
\]
\[\quad \leq \left\| f\left( \frac{x + y}{2} + z \right) + f\left( \frac{x + y}{2} - z \right) - 2f\left( \frac{x + y}{2} \right) - 2f(z) \right\| + \theta (\|x\|^r + \|y\|^r + \|z\|^r) \tag{38}\]
for all $x, y, z \in X$. Then, there exists a unique quadratic mapping $Q : X \to Y$ such that
\[
\left\| f(x) - Q(x) \right\| \leq 2\|\theta\|\|x\|^r \tag{39}\]
for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (38), we get $\|f(0)\| \leq \|2f(0)\| = 2\|f(0)\|$. So $f(0) = 0$. Letting $x = y$ and $z = 0$ in (38), we get
\[
\left\| 2f\left( \frac{x}{2} \right) - \frac{1}{2} f(x) \right\| \leq 2\|\theta\|\|x\|^r \tag{40}\]
for all $x \in X$. The last inequality can be rewritten as
\[
\left\| f(x) - 4f\left(\frac{x}{2}\right) \right\| \leq 2|2\|\theta\|x\|^r
\]
for all $x \in X$. Using the last inequality, we have
\[
\begin{align*}
\left\| 4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| &\leq max\left\{ \left\| 4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right) \right\|, \ldots, \left\| 4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| \right\} \\
&\leq max\left\{ \left| 4^l \right| \left\| f\left(\frac{x}{2^l}\right) - f\left(\frac{x}{2^{l+1}}\right) \right\|, \ldots, \left| 4^{m-1} \right| \left\| f\left(\frac{x}{2^{m-1}}\right) - f\left(\frac{x}{2^m}\right) \right\| \right\} \\
&\leq max\left\{ \frac{|4|^l}{|2|^{2l}}, \ldots, \frac{|4|^{m-1}}{|2|^{2(m-1)}} \right\} 2|2\|\theta\|x\|^r \\
&= \frac{2|2|}{|2|^{(r-2)l}}\theta\|x\|^r
\end{align*}
\]
for all non-negative integers $m$ and with $m > l$ and all $x \in X$. It follows from (41) that the sequence $\{4^k f\left(\frac{x}{2^k}\right)\}$ is Cauchy for all $x \in X$. Since $Y$ is a non-Archimedean Banach space, The sequence $\{4^k f\left(\frac{x}{2^k}\right)\}$ converges. So one can define the mapping $Q : X \to Y$ by $Q(x) := \lim_{k \to \infty} 4^k f\left(\frac{x}{2^k}\right)$ for all $x \in X$. It is easy to show that $A$ is even (since $f$ is even).

Moreover, letting $l = 0$ and passing the lim $n \to \infty$ in (41), we get (39). Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (41), we get (39).

Now, let $T : X \to Y$ be another quadratic mapping satisfying satisfying (39). Then, we have
\[
\begin{align*}
\left\| Q(x) - T(x) \right\| &= \left\| 4^q Q\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right) \right\| \\
&\leq max\left\{ 4^q \left\| A\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\|, 4^q \left\| T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\| \right\} \\
&\leq \frac{4|2|}{|2|^{(r-2)q}}\theta\|x\|^r,
\end{align*}
\]
which tends to zero as $q \to \infty$ for all $x \in X$. Therefore, $Q(x) = T(x)$ for all $x \in X$. Using (38) we have

$$\|Q\left(\frac{x+y+z}{2}\right) - \frac{1}{2}Q\left(\frac{x+y}{2}\right) - \frac{1}{2}Q(z)\| = \lim_{n \to \infty} \left|\left|4^n f\left(\frac{x+y+z}{4^n+1} + \frac{z}{4^n}\right) - \frac{1}{2}f\left(\frac{x+y}{4^n}\right) - \frac{1}{2}f\left(\frac{z}{4^n}\right)\right|\right|$$

$$\leq \lim_{n \to \infty} \left|\left|4^n f\left(\frac{x+y+z}{2} + \frac{z}{2}\right) - \frac{1}{2}f\left(\frac{x+y}{2}\right) - \frac{1}{2}f\left(\frac{z}{2}\right)\right|\right| + \lim_{n \to \infty} \frac{4^n}{4^n} \theta(\|x\|^r + \|y\|^r + \|z\|^r)$$

$$= \|Q\left(\frac{x+y+z}{2}\right) - Q\left(\frac{x+y}{2}\right) - Q(z)\|$$

for all $x,y,z \in X$. By Lemma 4, the mapping $Q : X \to Y$ is quadratic.

\[\square\]

**Theorem 8.** Let $r > 2$ and $\theta$ be nonnegative real number, and let $f : X \to Y$ be an even mapping satisfying

$$\left|\left|f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x+y}{2}\right) - f\left(\frac{x+y}{2}\right) - f\left(\frac{z}{2}\right)\right|\right| \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r)$$

(42)

Then, there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\left|\left|f(x) - Q(x)\right|\right| \leq 2\left|2\right|^{-1}\theta\|x\|^r$$

(43)

for all $x \in X$.

**Proof.** Letting $x = y = z = 0$ in (42), we get $\|f(0)\| \leq 2\|f(0)\| = 2\|f(0)\|$. So $f(0) = 0$. Letting $x = y$ and $z = 0$ in (42), we get

$$\left|\left|2f\left(\frac{x}{2}\right) - \frac{1}{2}f(x)\right|\right| \leq 2\theta\|x\|^r$$

(44)

for all $x \in X$.

So

$$\left|\left|f(x) - \frac{1}{4}f(2x)\right|\right| \leq 2\left|2\right|^{-1}\theta\|x\|^r$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 7. \[\square\]
4. Conclusion

In this paper, we have shown that the solutions of the first and third inequalities are additive mappings and the two others are quadratic mappings. The Hyers-Ulam stability for these inequalities are given from Theorem 1 to Theorem 8. These are the main results of the paper, which are the generalization of the results in [7, 8]. However, we could not give the answers for some specific cases. For example, The Hyers-Ulam stability for the first and third inequalities are still open for $r = 1$, that for the second and fourth inequalities are still open for $r = 2$. For these cases, we are still working on.

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