Mild Projection of Surfaces with Boundaries

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Abstract

We classify simple singularities of projections to a plane of pairs of surfaces with boundaries embedded into three-space and with respect to mild equivalence relation. Also, we classify simple hypersurfaces equipped with fixed boundaries with respect to semi-mild equivalence relation. A brief description of the main theorems is given in the next section. Promising applications of classifications are discussed in the conclusion of the paper.

Mathematics Subject Classification: 58K05, 58K40, 53A15

Keywords: pairs of hypersurfaces with boundaries, mild projection, mild equivalence, semi-mild equivalence

1 Introduction

In [11], a non-standard equivalence relation was introduced and it was related to projections of submanifolds. Namely, consider the trivial bundle \( \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p; (x, y) \mapsto y \), then two varieties \( V_1 \) and \( V_2 \) embedded in \( \mathbb{R}^n \times \mathbb{R}^p \) are called pseudo-equivalent if there is a diffeomorphism \( \theta \) of the ambient space \( \mathbb{R}^n \times \mathbb{R}^p \) such that \( V_1 = \theta(V_2) \), the set of critical points of \( V_2 \) is mapped onto the set of critical points of \( V_1 \), and the differential of \( \theta \) at any critical point maps
the direction of the projection to that at the image of the point. The pseudo-equivalence relation does not satisfy the properties of a geometrical subgroup of equivalences in J. Damon sense [6]. In particular, the versatility theorem can fail. After a modification on pseudo-equivalence relation to get better properties with respect to parameter dependence, the so called quasi-equivalence relation was introduced and the classification of the respective simple singularities (in Arnond’s sense) of analytic hypersurfaces \( V = \{(x, y) : f(x, y) = 0\} \) given by a single equation \( f = 0 \) in low dimensions \( (n = 1; p = 2, 3) \) was obtained. Later, in [3], the study of the quasi equivalence of projections to a place of surfaces embedded into three-space with boundaries (that is curves embedded into surfaces) was considered. In particular, simple quasi singularities of boundaries on generic singularities of surfaces in \( \mathbb{R}^3 \) were obtained (see [10, 8] for the corresponding results for the standard \( O \)-equivalence relation).

In the current paper and based on the quasi idea, we introduce another new two non-standard equivalence relations related to projections of surfaces with boundaries. In particular, They are defined as follows:

1. First equivalence relation: Two pairs \((V_1, B_1)\) and \((V_2, B_2)\) are called pseudo mild equivalent if there exists a diffeomorphism of the ambient space which sends one pair to the other such that its differential preserve the direction of the projection only at the boundary points lying on the boundary points which are also critical points of the projection.

2. Second equivalence relation: Consider the space of all hypersurfaces equipped with fixed boundaries determined by. Then, two hypersurfaces \( V_1 \) and \( V_2 \) are called pseudo semi-mild equivalent if there exists a diffeomorphism of the ambient space which sends one of the hypersurfaces to the other such that its differential preserve the direction of the projection only at the boundary points lying on the boundary points which are also critical points of the projection (the boundaries are not necessarily preserved).

After a modification on 1. and 2. which is similar to that in the quasi case, we get the definitions of the mild equivalence relation and semi-mild equivalence relation, respectively.

For the mild equivalence relation, we consider surfaces with boundaries \((V, B)\) embedded in \((\mathbb{R}^3, 0)\) with the trivial bundle \( \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2, (x, y_1, y_2) \mapsto (y_1, y_2) \). Moreover, \( V \) will be only a Whitney singularity, that is \( V = V_{reg} = \{(x, y_1, y_2) : f = x + y_2 = 0\} \) or \( V = V_{fold} = \{(x, y_1, y_2) : f = \frac{1}{2}x^2 + y_1x + y_2 = 0\} \) or \( V = V_{pleat} = \{(x, y_1, y_2) : f = \frac{1}{2}x^3+y_1x+y_2 = 0\} \). Let \((f, g)\) be the pair representing \((V, B) \subset (\mathbb{R}^3, 0)\) such that \( V \in \{V_{reg}, V_{fold}, V_{pleat}\} \). Our main results in this case is as follows:
Assume that \((f, g)\) is simple with respect to mild equivalence. Then, \((f, g)\) is mild equivalent to one of \((f, g^*)\), described in the following table.

<table>
<thead>
<tr>
<th>Surface Type</th>
<th>(f(x, y_1, y_2))</th>
<th>(g^*(x, y_1))</th>
<th>Restrictions</th>
<th>Codimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regular point</td>
<td>(x + y_2)</td>
<td>(A_n : x^2 + y_1^{n+1})</td>
<td>(n \geq 0)</td>
<td>(n)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(D_n : y_1^2 x \pm x^{n-1})</td>
<td>(n \geq 4)</td>
<td>(n)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(E_6 : y_1^3 \pm x^4;)</td>
<td></td>
<td>6</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(E_7 : y_1^2 + y_1 x^3)</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(E_8 : y_1^3 + x^5)</td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>Fold</td>
<td>(\frac{1}{2} x^2 + xy_1 + y_2)</td>
<td>(x)</td>
<td>(n \geq 2)</td>
<td>(n+1)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(y_1)</td>
<td></td>
<td>4</td>
</tr>
<tr>
<td>Pleat</td>
<td>(\frac{1}{3} x^3 + xy_1 + y_2)</td>
<td>(x)</td>
<td>(n \geq 3)</td>
<td>(n + 2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(y_1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(x^2 \pm y_1^n)</td>
<td></td>
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</tr>
<tr>
<td></td>
<td></td>
<td>(xy_1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(y_1^2 \pm x^n)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

On the other hand, for the semi-mild equivalence relation, For simplicity, we consider regular surfaces with smooth boundaries determined by the equation \(g = x\). Our main results in this case is as follows:

Let a regular surface germ \(V = \{f = 0\}\) be simple with respect to semi-mild equivalence relation. Then the projection of \(V\) is semi-mild equivalent to the projection of one of the regular surfaces \(\tilde{V} = \{(x, y) : \tilde{f}(x, \tilde{y}) + y_p = 0\}\), described in the following table.

<table>
<thead>
<tr>
<th>Notation</th>
<th>(\tilde{f})</th>
<th>Restrictions</th>
<th>codimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A_0^*)</td>
<td>(x)</td>
<td>(y^2 = \sum_{i=2}^{p-1} \pm y_i^2, k \geq 1)</td>
<td>(k)</td>
</tr>
<tr>
<td>(A_1)</td>
<td>(x^2)</td>
<td>(y^2 = \sum_{i=3}^{p-1} \pm y_i^2k \geq 4)</td>
<td>(k)</td>
</tr>
<tr>
<td>(A_k)</td>
<td>(x(y_1^{k+1} + y^2))</td>
<td>(y^2 = \sum_{i=2}^{p-1} \pm y_i^2, k \geq 1)</td>
<td>(k)</td>
</tr>
<tr>
<td>(D_k)</td>
<td>(x(y_1^2 y_2 + y_2^{k-1} + y^2))</td>
<td>(y^2 = \sum_{i=3}^{p-1} \pm y_i^2k \geq 4)</td>
<td>(k)</td>
</tr>
<tr>
<td>(E_6)</td>
<td>(x(y_1^3 + y_2^3 + y^2))</td>
<td>(y^2 = \sum_{i=3}^{p-1} \pm y_i^2)</td>
<td>6</td>
</tr>
<tr>
<td>(E_7)</td>
<td>(x(y_1^3 + y_1 y_2^3 + y^2))</td>
<td>(y^2 = \sum_{i=3}^{p-1} \pm y_i^2)</td>
<td>7</td>
</tr>
<tr>
<td>(E_8)</td>
<td>(x(y_1^3 + y_2^3 + y^2))</td>
<td>(y^2 = \sum_{i=3}^{p-1} \pm y_i^2)</td>
<td>8</td>
</tr>
</tbody>
</table>
2 Mild equivalence relation of hypersurfaces with boundaries

Consider a trivial bundle $\mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}^p$; $(x, y) \mapsto y$ and a subvariety $V$ in $\mathbb{R}^n \times \mathbb{R}^p$. Consider germs of $C^\infty$ functions $f : (\mathbb{R}^n \times \mathbb{R}^p, 0) \rightarrow \mathbb{R}$ at the origin. Denote by $C_{x,y}$ the ring of all such germs at the origin and by $\mathcal{M}_{x,y}$ the maximal ideal in $C_{x,y}$. Sometimes, we set the notation $w = (x, y)$ for the whole set of coordinates on $\mathbb{R}^{n+p} = \mathbb{R}^n \times \mathbb{R}^p$ if the distinction between $x$ and $y$ is not important.

We will assume that the fibers are one-dimensional: $x \in \mathbb{R}, n = 1$. Also, we will study regular and analytic hypersurfaces $V = \{(x, y) : f(x, y) = 0\}$ given by a single equation $f = 0$, equipped with boundaries $B$ which are subvarieties of $V$ of codimension 1. The pair $(V, B)$ will be called a hypersurface with a boundary.

Up to permutations of $y_1, y_2, \ldots, y_{p-1}$ and $y_p$, and up to a multiplication by a non vanishing factor the equation of $V$ will take the form

$$f(x, y) = \tilde{f}(x, \tilde{y}) + y_p,$$

where $\tilde{y} = (y_1, y_2, \ldots, y_{p-1})$ and $\tilde{f} \in \mathcal{M}_{x,\tilde{y}}$. Therefore, we may assume that

$$B = \{(x, y) : f(x, y) = g(x, \tilde{y}) = 0\}.$$

**Definition 2.1** Two pairs $(V_1, B_1)$ and $(V_2, B_2)$ are called pseudo mild equivalent if there exists a diffeomorphism $\theta : \mathbb{R} \times \mathbb{R}^p \rightarrow \mathbb{R} \times \mathbb{R}^p$ such that the following are satisfies:

1. $V_1 = \theta(V_2), B_1 = \theta(B_2)$,

2. the set of critical points of $V_2$ is mapped by $\theta$ onto the set of critical points of $V_1$, and

3. the differential of $\theta$ preserves the direction of the projection only at the boundary points lying on $B_2$ which are also critical points of the projection of $V_2$.

A similar definition can be introduced for germs of hypersurfaces with boundaries.
Remarks 2.2

1. Let \((f_i, g_i)\) be the pair of function germs defining \((V_i, B_i)\). Then, the constraint 1. in Definition 2.1 implies that \((f_1, g_1) = \left(Kf_2(\theta), Pf_2(\theta) + Qg_2(\theta)\right)\), where \(K, P\) and \(Q\) are smooth functions with \(KQ \neq 0\).

2. If \((V_1, B_1)\) and \((V_2, B_2)\) are equivalent with respect to the standard equivalence relation of the action of the diffeomorphisms which preserve the bundle structure defined in [?, ?] then they will be called \(O\)-equivalent.

3. Clearly, if two pairs are \(O\)-equivalent then they are pseudo mild equivalent.

Now, suppose that all germs of hypersurfaces with boundaries at the origin in a smooth family \(G_t = (f_t, g_t)\) are pseudo mild equivalent to the germ \(G_1 = (f_1, g_1)\), that is

\[
(f_1, g_1) = \left(K_t f_t(\theta_t), P_t f_t(\theta_t) + Q_t g_t(\theta_t)\right), \quad t \in [1, 2],
\]

with respect to a smooth family \(\theta_t : (\mathbb{R} \times \mathbb{R}^p, 0) \to (\mathbb{R} \times \mathbb{R}^p, 0)\) of germs of diffeomorphisms and smooth families \(K_t, Q_t, P_t\) such that \(\theta_1 = id_{\mathbb{R} \times \mathbb{R}^p}, K_1 = Q_1 = 1, P_1 = 0, K_t(0)Q_t(0) \neq 0\), and \(t \in [1, 2]\). Then we have the homological equation:

\[
- \left[ \frac{\partial f_t}{\partial y_i} \right] t = \begin{bmatrix}
K_t f_t(\theta_t) + K_t \left(\frac{\partial f_t}{\partial x} \dot{X}(t) + \sum_{i=1}^p \frac{\partial f_t}{\partial y_i} \dot{Y}_i(t)\right)
+ & f_t \dot{P}_t + P_t \left(\frac{\partial f_t}{\partial x} \dot{X}(t) + \sum_{i=1}^p \frac{\partial f_t}{\partial y_i} \dot{Y}_i(t)\right) + g_t \dot{Q}_t + Q_t \left(\frac{\partial f_t}{\partial y_i} \dot{Y}_i(t) + \sum_{i=1}^p \frac{\partial f_t}{\partial y_i} \dot{Y}_i(t)\right)
\end{bmatrix} t,
\]

where \(\dot{K}_t = \frac{\partial K_t}{\partial t}, \dot{P}_t = \frac{\partial P_t}{\partial t}, \dot{Q}_t = \frac{\partial Q_t}{\partial t}\) and the vector field

\[
v_t = \dot{X}(t) \frac{\partial}{\partial x} + \sum_{i=1}^p \dot{Y}_i(t) \frac{\partial}{\partial y_i}
\]

generates the phase flow \(\theta_t\).

Denote by \(J_{G_t}\) the ideal in \(C_{x,y}\) generated by \(f_t, g_t\) and \(\frac{\partial f_t}{\partial x}\). Based on the proof of Proposition 2.2 in [11], we have
Proposition 2.3 The components of $v_t$ satisfy the following:

$$
\dot{X}(t) \in C_{x,y} \quad \text{and} \quad \dot{Y}_i(t) \in IRad(J_G_t).
$$

Remark 2.4 Recall that, the radical $Rad(J)$ of the ideal $J$ in $C_{x,y}$ is defined as the set of all elements in $C_{x,y}$, vanishing on the set of common zeros of germs from $J$:

$$Rad(J) = O(T(J)),$$

where

$$T(J) = \left\{ (x, y) : h(x, y) = 0 \text{ for all } h \in J \right\},$$

and

$$O(T(J)) = \left\{ \psi \in C_{x,y} : \psi(x, y) = 0 \text{ for all } (x, y) \in T(J) \right\}.$$

The integral radical $IRad(J)$ of the ideal $J$ is defined to be the module of function germs $\varphi \in C_{x,y}$ such that $\frac{\partial \varphi}{\partial x} \in Rad(J)$. Similar definitions can be introduced when we replace $C_{x,y}$ by the space $\mathbb{R}[x, y]$ of all real polynomials in the variables $x$ and $y$.

To have a better parameter dependence, we modify the pseudo mild equivalence relation (see [11] for more details). Namely, we replace $IRad(J_G_t)$ by the integral ideal $IJ_G_t$, which is defined to be the module of function germs $\varphi \in C_{x,y}$ such that $\frac{\partial \varphi}{\partial x} \in J_G_t$ and consider such replacement in the equivalence definition. This provides us with the notion of mild equivalence relation.

Definition 2.5 Two pairs $(f_1, g_1)$ and $(f_2, g_2)$ are called mild equivalent if there exists a family of diffeomorphisms $\theta_t : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R} \times \mathbb{R}^p$ continuously and piecewise smoothly depending on parameter $t \in [1, 2]$ and non vanishing families of smooth functions $f_t, g_t, K_t, P_t$ and $Q_t$ also depending on parameter $t \in [1, 2]$ with $\theta_1 = id_{\mathbb{R} \times \mathbb{R}^p}$, $K_1 = Q_1 = 1$, $P_1 = 0$ and $K_t Q_t \neq 0$ such that: $(f_1, g_1) = \left( K_t f_t(\theta_t), P_t f_t(\theta_t) + Q_t g_t(\theta_t) \right)$ and the components of the vector field $v_t = \dot{X}(t) \frac{\partial}{\partial x} + \sum_{i=1}^p \dot{Y}_i(t) \frac{\partial}{\partial y_i}$ generating $\theta_t$ on each segment of smoothness satisfy the following: $\dot{X}(t) \in C_{x,y}$ and $\dot{Y}_i(t) \in IJ_G_t$.

Remarks 2.6

1. In Definition 2.5, the family $\theta_t$ of diffeomorphisms generated by the vector field $v_t$ as well as the vector field itself will be called admissible for the families $f_t$ and $g_t$ with respect to mild equivalence.
2. The mild equivalence of \((f_1, g_1)\) and \((f_2, g_2)\) implies their pseudo mild equivalence since \(IJ_{G_t} \subset IRad(J_{G_t})\), for all \(t \in [1, 2]\).

3. A function germ \(\dot{Y}_i(t)\) belongs to \(IJ_{G_t}\) if it admits a decomposition

\[
\dot{Y}_i(t) = e_i + \int_0^x \left[ f_ia_i + \frac{\partial f_i}{\partial x}b_i + g_ic_i \right] \, dx,
\]

where \(a_i, b_i, c_i \in C_{x,y}\) and \(e_i \in C_y\). Therefore, all function germs independent of \(x\) are in \(IJ_{G_t}\). Hence, if \((f_1, g_1)\) and \((f_2, g_2)\) are \(O\)-equivalent then they are quasi mild equivalent.

4. The tangent space \(TQM_G\) to the mild equivalence class of \(G = (f, g)\) has the following description:

\[
TQM_G = \left\{ \begin{bmatrix} fA_1 + \frac{\partial f}{\partial x}K + \sum_{i=1}^p \frac{\partial f}{\partial y_i}H_i \\ fA_2 + gA_3 + \frac{\partial g}{\partial x}K + \sum_{i=1}^p \frac{\partial g}{\partial y_i}H_i \end{bmatrix} : A_1, A_2, A_3, K \in C_{x,y}, H_i \in IJ_G \right\}.
\]

Let \(J^*_G\) be the ideal generated by \(\frac{\partial f}{\partial x}\) and \(g\), and \(J^{**}_G\) be the ideal generated by \(f\) and \(g\).

**Proposition 2.7** The tangent space \(TQM_G\) can be described equivalently as

\[
TQM_G = \left\{ \begin{bmatrix} fA_1 + \frac{\partial f}{\partial x}K + \sum_{i=1}^p \frac{\partial f}{\partial y_i}H_i \\ fA_2 + gA_3 + \frac{\partial g}{\partial x}K + \sum_{i=1}^p \frac{\partial g}{\partial y_i}H_i \end{bmatrix} : A_1, A_2, A_3, K \in C_{x,y}, H_i \in IJ^*_G \right\},
\]

or

\[
TQM_G = \left\{ \begin{bmatrix} fA_1 + \frac{\partial f}{\partial x}K + \sum_{i=1}^p \frac{\partial f}{\partial y_i}H_i \\ fA_2 + gA_3 + \frac{\partial g}{\partial x}K + \sum_{i=1}^p \frac{\partial g}{\partial y_i}H_i \end{bmatrix} : A_1, A_2, A_3, K \in C_{x,y}, H_i \in IJ^{**}_G \right\}.
\]

**Proof.** A vector \(S\) belongs to \(TQM_G\) if it admits a decomposition

\[
S = \begin{bmatrix} fA_1 + \frac{\partial f}{\partial x}K + \sum_{i=1}^p \frac{\partial f}{\partial y_i}H_i \\ fA_2 + gA_3 + \frac{\partial g}{\partial x}K + \sum_{i=1}^p \frac{\partial g}{\partial y_i}H_i \end{bmatrix},
\]
where $A_1, A_2, A_3, K \in C_{x,y}$, and $H_i = e_i + \frac{x}{0} (f a_i + gb_i + \frac{\partial f}{\partial x} c_i) \ dx$ with $e_i \in C_y$ and $a_i, b_i, c_i \in C_{x,y}$.

Applying integration by parts, we have

$$\int_0^x f a_i \ dx = f \int_0^x a_i \ dx - \int_0^x \frac{\partial f}{\partial x} \left( \int_0^x a_i \ dx \right) \ dx.$$

Thus, (1) can be written as

$$S = \begin{bmatrix}
 f \tilde{A}_1 + \frac{\partial f}{\partial x} K + \sum_{i=1}^{p} \frac{\partial f}{\partial y_i} \tilde{H}_i \\
 f \tilde{A}_2 + g A_3 + \frac{\partial g}{\partial x} K + \sum_{i=1}^{p} \frac{\partial g}{\partial y_i} \tilde{H}_i
\end{bmatrix},$$

where $\tilde{H}_i = e_i + \frac{x}{0} \left[ gb_i + \frac{\partial f}{\partial x} (c_i - \frac{x}{0} a_i \ dx) \right] \ dx$, $\tilde{A}_1 = A_1 + \sum_{i=1}^{p} \frac{\partial f}{\partial y_i} \left( \int_0^x a_i \ dx \right)$ and $\tilde{A}_2 = A_2 + \sum_{i=1}^{p} \frac{\partial g}{\partial y_i} \left( \int_0^x a_i \ dx \right)$. The first result of the proposition follows.

Similarly, applying integration by parts, we see that

$$\int_0^x \frac{\partial f}{\partial x} c_i \ dx = f c_i - \int_0^x f \frac{\partial c_i}{\partial x} \ dx.$$

Hence,

$$\int_0^x (f a_i + gb_i + \frac{\partial f}{\partial x} c_i) \ dx = f c_i + \int_0^x \left[ f(a_i - \frac{\partial c_i}{\partial x}) + gb_i \right] \ dx.$$

This formula provides the second result of the proposition. \qed

Following [1], we call a germ $(V, B)$ *simple* if its sufficiently small neighbourhood in the space of all germs of hypersurfaces with boundaries contains only a finite number of quasi mild equivalence classes.
3 Classifications of generic surface with boundaries in \( \mathbb{R}^3 \)

In the current section, we consider surfaces with boundaries \((V, B)\) embedded in \((\mathbb{R}^3, 0)\) with the trivial bundle \(\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2\), \((x, y_1, y_2) \mapsto (y_1, y_2)\). For simplicity, \(V\) will be only a Whitney singularity:

1. The surface has only regular points and therefore is diffeomorphic to the base of the bundle space, in which case we set \(V = V_{\text{reg}} = \{(x, y_1, y_2) : f = x + y_2 = 0\}\).

2. The surface is a fold singularity: \(V = V_{\text{fold}} = \{(x, y_1, y_2) : f = \frac{1}{2}x^2 + y_1x + y_2 = 0\}\).

3. The surface is a pleat singularity: \(V = V_{\text{pleat}} = \{(x, y_1, y_2) : f = \frac{1}{2}x^3 + y_1x + y_2 = 0\}\).

Recall that a vector field \(v\) preserves a hypersurface \(V = \{(x, y) : f(x, y) = 0\}\) if the Lie derivative \(L_v f\) belongs to the principal ideal generated by \(f\). The module \(S_V\) of all germs of \(C^\infty\) vector fields preserving a hypersurface germ \((V, 0) \subset (\mathbb{R}^{1+p}, 0)\) is the Lie algebra of the group of diffeomorphisms of \((\mathbb{R}^{1+p}, 0)\) preserving \((V, 0)\). The module \(S_V\) is called the stationary algebra of \((V, 0)\) and its elements are called the stationary vector fields.

We shall classify simple pairs \((V, B)\) in \((\mathbb{R}^3, 0)\) with respect to quasi mild equivalence relation, where \(V \in \{V_{\text{reg}}, V_{\text{fold}}, V_{\text{pleat}}\}\). This means that we need to calculate the stationary algebra of the admissible vector fields

\[
v = \dot{x} \frac{\partial}{\partial x} + \dot{y}_1 \frac{\partial}{\partial y_1} + \dot{y}_2 \frac{\partial}{\partial y_2},
\]

the flow of which provides a quasi weak equivalence of the surface \(V\) with itself. Then, we classify the orbits of its action on the equations \(g(x, y_1) = 0\) of the boundaries modulo the equation of the surface.

Thus, the stationary vector field \(v\) of \(V = \{(x, y_1, y_2) : \tilde{f}(x, y_1) + y_2 = 0\}\) satisfies

\[
(\tilde{f} + y_2)A + \frac{\partial \tilde{f}}{\partial x} \dot{x} + \frac{\partial \tilde{f}}{\partial y_1} \dot{y}_1 + \dot{y}_2 = 0,
\] (2)
where $A \in C_{x,y_1,y_2}$. The $\dot{x}$-component is arbitrary but $\dot{y}_1$-component and $\dot{y}_2$-component have the forms:

$$\dot{y}_1 = e_1 + \int_0^x \left( \frac{\partial \tilde{f}}{\partial x} h_1 + gh_2 \right) \, dx, \quad (3)$$

$$\dot{y}_2 = e_2 + \int_0^x \left( \frac{\partial \tilde{f}}{\partial x} k_1 + gk_2 \right) \, dx, \quad (4)$$

where $e_1, e_2 \in C_{y_1,y_2}$ and $h_i, k_i \in C_{x,y_1,y_2}$, for all $i$.

Denote by $W = \dot{X} \frac{\partial}{\partial x} + \dot{Y} \frac{\partial}{\partial y_1}$ the projection of the vector field $v$ to the $(x,y_1)$-coordinate plane.

**Lemma 3.1** The vector field $v$ is stationary if and only the Lie derivative of the critical locus $\frac{\partial \tilde{f}}{\partial x}$ along the vector field $W$ belongs to the ideal generated by $g$ and $\frac{\partial \tilde{f}}{\partial x}$, that is

$$\frac{\partial^2 \tilde{f}}{\partial x^2} \dot{X} + \frac{\partial^2 \tilde{f}}{\partial x \partial y_1} \dot{Y} = g \tilde{D} + \frac{\partial \tilde{f}}{\partial x} \tilde{E},$$

where $D, E, \dot{X} \in C_{x,y_1}$, and the $\dot{Y}$-component has the form

$$\dot{Y} = e + \int_0^x \left( \frac{\partial \tilde{f}}{\partial x} a + gb \right) \, dx,$$

with $e \in C_{y_1}$ and $a, b \in C_{x,y_1}$.

**Proof.** If we differentiate (2) with respect to $x$ we get

$$g \left( k_2 + \frac{\partial \tilde{f}}{\partial y_1} h_2 \right) + (\tilde{f} + y_2) \frac{\partial A}{\partial x} + \frac{\partial \tilde{f}}{\partial x} \left( A + k_1 + \frac{\partial \tilde{f}}{\partial y_1} h_1 + \frac{\partial \tilde{f}}{\partial x} \right) + \frac{\partial^2 \tilde{f}}{\partial x^2} \dot{x} + \frac{\partial^2 \tilde{f}}{\partial x \partial y_1} \dot{y}_1 = 0. \quad (5)$$

Restricting (5) to $V$ by substituting $y_2 = -\tilde{f}$ yields

$$g \tilde{D} + \frac{\partial \tilde{f}}{\partial x} \tilde{E} + \frac{\partial^2 \tilde{f}}{\partial x^2} \dot{X} + \frac{\partial^2 \tilde{f}}{\partial x \partial y_1} \dot{Y} = 0,$$

for some smooth functions $\tilde{D}(x,y_1)$ and $\tilde{E}(x,y_1)$. Here, $\dot{X} = \dot{x}(x,y_1,-\tilde{f})$ and $\dot{Y} = \dot{y}_1(x,y_1,-\tilde{f})$. 

Now, if we restrict (3) to \( V \) we obtain
\[
\dot{Y} = e_1(y_1, -\tilde{f}) + \int_0^x \left[ \frac{\partial \tilde{f}}{\partial x} h_1(x, y_1, -\tilde{f}) + gh_2(x, y_1, -\tilde{f}) \right] \, dx.
\]

Using the Hadamard Lemma, we write
\[
e_1(y_1, -\tilde{f}) = e_1(y_1, 0) + \tilde{f} \tilde{e}_1(y_1, \tilde{f}).
\]

Due to the chain rule we have
\[
\frac{\partial (\tilde{f} \tilde{e}_1)}{\partial x} = \frac{\partial (\tilde{f} \tilde{e}_1)}{\partial \tilde{f}} \frac{\partial \tilde{f}}{\partial x},
\]
which means that \( \tilde{f} \tilde{e}_1 \) belongs to \( \int_0^x \frac{\partial \tilde{f}}{\partial x} h_3 \, dx \), where \( h_3 \in \mathbf{C}_{x,y_1} \). Hence,
\[
\dot{Y} = e_* + \int_0^x \left( \frac{\partial \tilde{f}}{\partial x} \tilde{h}_1 + gh_2 \right) \, dx,
\]
fore some smooth functions \( \tilde{h}_1(x, y_1), \tilde{h}_2(x, y_1) \) and \( e_*(y_1) \), as required.

Obviously the converse is also true. \( \square \)

Lemma 3.1 implies the following.

**Lemma 3.2** The components of the vector field \( W \) are given by the following formulas:

1. For a regular point type \((\tilde{f} = x)\): \( \dot{X}, \dot{Y} \in \mathcal{M}_{x,y_1} \).

2. For a fold type \((\tilde{f} = \frac{1}{2} x^2 + y_1 x)\): \( \dot{X} = (x + y_1)E + x^2M - xN - y_1P + gD, \)
\[
\dot{Y} = x(x + 2y_1)M + x^2(x + y_1) \frac{\partial M}{\partial x} + y_1P + xN,
\]
where \( D, E, M \in \mathbf{C}_{x,y_1}, P \in \mathbf{C}_{y_1} \) and \( \int_0^x gb \, dx = xN \), with \( b, N \in \mathbf{C}_{x,y_1} \).

3. For a pleat type \((\tilde{f} = \frac{1}{3} x^3 + y_1 x)\): \( \dot{X} = \frac{1}{2} \left[ -x^2 + y_1 E + x^2M + xP + (xg_* - \bar{g})D - N \right], \)
\[
\dot{Y} = \frac{1}{2} x(x + 3y_1)M + \frac{1}{2} x^2(x + y_1) \frac{\partial M}{\partial x} + y_1P + xN,
\]
where \( D, E, M \in C_{x,y_1}, \ P \in C_{y_1} \) and \( \int_0^x gb \ dx = xN, \) with \( b, N \in C_{x,y_1} \) and 
\( N(0) + \tilde{g}(0)D(0) = 0. \)

**Proof. 1.** In the case of a regular point type, we have \( \tilde{f} = x. \) The derivative equation is

\[
gD + E = 0, \tag{6}
\]
where \( D, E \in C_{x,y_1} \) and the \( \dot{Y} \)-component has the form

\[
\dot{Y} = e + \int_0^x [a + gb] \ dx, \tag{7}
\]
where \( e \in C_{y_1} \) and \( a, b \in C_{x,y_1}. \) Clearly, relations (6) and (7) imply that
\( \dot{X}, \dot{Y} \in M_{x,y_1}. \)

2. Next, for a fold type, we have \( \tilde{f} = \frac{1}{2}x^2 + xy_1. \) The derivative equation is

\[
gD + (x + y_1)E + \dot{X} + \dot{Y} = 0,
\]
where \( D, E, \dot{X} \in C_{x,y_1} \) and \( \dot{Y} \)-component has the form

\[
\dot{Y} = e + \int_0^x [(x + y_1)a + gb] \ dx,
\]
with \( e \in C_{y_1} \) and \( a, b \in C_{x,y_1}. \) By integration by parts we can write

\[
\dot{Y} = e + (x + y_1)x\tilde{a} - x^2M + xN, \tag{8}
\]
where \( \int_0^x a \ dx = x\tilde{a}, \int_0^x x\tilde{a} \ dx = x^2M \) and \( \int_0^x gb \ dx = xN. \)

If we set \( x = y_1 = 0 \) in (8), we see that \( e(0) = 0. \) This means that we can write

\[
e(y_1) = y_1P
\]
for some \( P \in C_{y_1}. \) Hence, the derivative equation can be written in the form

\[
gD + (x + y_1)E + \dot{X} + y_1P + (x + y_1)x\tilde{a} - x^2M + xN = 0.
\]
Therefore, we have
\[ \dot{X} = (x + y_1) \tilde{E} + x^2 M - x N - y_1 P - g D, \]  
(9)
where \( \tilde{E} = -(E + x \tilde{a}) \).

Differentiating the relation \( \int_0^x x \tilde{a} \, dx = x^2 M \) with respect to \( x \) gives \( \tilde{a} = 2M + x \frac{\partial M}{\partial x} \). Thus, if we substitute \( \tilde{a} \) in (8), we obtain
\[ \dot{Y} = x(x + 2y_1)M + x^2(x + y_1) \frac{\partial M}{\partial x} + x N + y_1 P, \]
as required.

3. Finally, for a pleat type, we have \( \tilde{f} = \frac{1}{3} x^3 + xy_1 \). The derivative equation is
\[ gD + (x^2 + y_1)E + 2x \dot{X} + \dot{Y} = 0, \]
where \( D, E, \dot{X} \in C_{x,y_1} \) and the \( \dot{Y} \)-component has the form
\[ \dot{Y} = e + \int_0^x [(x^2 + y_1)a + gb] \, dx, \]
with \( e \in C_{y_1} \) and \( a, b \in C_{x,y_1} \). By integration by parts we have
\[ \dot{Y} = e + (x^2 + y_1)x \tilde{a} - x^3 M + x N, \]
(10)
where \( \int_0^x a \, dx = x \tilde{a}, \int_0^x 2x^2 \tilde{a} \, dx = x^3 M \) and \( \int_0^x gb \, dx = x N \).

Since \( e(0) = 0 \), we have \( e(y_1) = y_1 P \) with \( P \in C_{y_1} \). Hence, the derivative equation can be written as
\[ (y_1 g_* + x \tilde{g})D + (x^2 + y_1)\tilde{E} + 2x \dot{X} + y_1 P - x^3 M + x N = 0, \]
(11)
where \( \tilde{E} = E + x \tilde{a} \). Equivalently, (11) can be written as
\[ x \left[ 2 \dot{X} + x \tilde{E} - x^2 M + \tilde{g} D + N \right] + y_1 \left[ P + \tilde{E} + g_* D \right] = 0. \]
(12)
Therefore, (12) implies the existence of a smooth function \( Q(x, y_1) \) such that
\[2\dot{X} + xE - x^2 M + \tilde{g}D + N = -yQ \quad (13)\]

and
\[P + \tilde{E} + g_s D = xQ. \quad (14)\]

If we substitute \(\tilde{E}\) from (14) into (13) we obtain
\[\dot{X} = \frac{1}{2} \left[ - (x^2 + y_1)Q + xP + x^2 M + (xg_s - \tilde{g})D - N \right]. \quad (15)\]

Differentiating relation \(\int_0^x 2x^2 \tilde{a} \, dx = x^3 M\) with respect to \(x\) gives \(\tilde{a} = \frac{3}{2}M + \frac{1}{2}x\frac{\partial M}{\partial x}\).

Hence, if we substitute \(\tilde{a}\) in (10), we obtain
\[\dot{Y} = \frac{1}{2} x(x^2 + 3y_1)M + \frac{1}{2} x^2 (x^2 + y_1) \frac{\partial M}{\partial x} + xN + y_1 P,\]

Notice that if we set \(x = y_1 = 0\) in (15) then we get \(N(0) + \tilde{g}(0)D(0) = 0\), as required.

Knowing the stationary algebra \(\left\{ W = \dot{X} \frac{\partial}{\partial x} + \dot{Y} \frac{\partial}{\partial y_1} \right\}\) described in Lemma 3.2 and considering Lemma ??, the respective classifications of simple quasi mild classes \((f, g)\) are obtained by standard Arnold’s spectral sequence method [1] together with appropriate preliminary transformations.

**Theorem 3.3** Let \((f, g)\) be the pair representing \((V, B) \subset (\mathbb{R}^3, 0)\) such that \(V \in \{V_{\text{reg}}, V_{\text{fold}}, V_{\text{pleat}}\}\). Assume that \((f, g)\) is simple with respect to mild equivalence. Then, \((f, g)\) is mild equivalent to one of \((f, g^*)\), described in the following table.
<table>
<thead>
<tr>
<th>Surface Type</th>
<th>( f(x, y_1, y_2) )</th>
<th>( g^*(x, y_1) )</th>
<th>Restrictions</th>
<th>Codimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regular point</td>
<td>( x + y_2 )</td>
<td>( A_n : x^2 + y_1^{n+1} ) ( D_n : y_1^n x \pm x^{n-1} ) ( E_6 : y_1^3 + x^4 ) ( E_7 : y_1^3 + y_1 x^3 ) ( E_8 : y_1^3 + x^5 )</td>
<td>( n \geq 0 ) ( n \geq 4 )</td>
<td>( n ) ( n ) 6 7 8</td>
</tr>
<tr>
<td>Fold</td>
<td>( \frac{1}{2} x^2 + xy_1 + y_2 )</td>
<td>( x ) ( y_1 )</td>
<td></td>
<td>0 1</td>
</tr>
<tr>
<td>Pleat</td>
<td>( \frac{1}{3} x^3 + xy_1 + y_2 )</td>
<td>( x ) ( y_1 ) ( x^2 \pm y_1^n ) ( xy_1 ) ( y_1^2 \pm x^n )</td>
<td>( n \geq 2 ) ( n \geq 3 )</td>
<td>0 2 4 ( n+1 ) ( n+2 )</td>
</tr>
</tbody>
</table>

**Remarks 3.4**

1. For a regular point type, clearly the simple mild classes of the boundaries are just the standard Arnold’s simple singularities with respect to right equivalence relation.

2. For a fold type, the fencing mild class is \( x^2 + \alpha y_1^2 \) with \( \alpha \neq 0 \).

3. For a pleat type, the fencing mild class is \( x^3 + \beta y_1^3 \) with \( \beta \neq 0 \).

4. The graph of low codimension adjacencies is as follows:
   - For a fold type: \( x \leftarrow y_1 \leftarrow x^2 + \alpha y_1^2 \)
   - For a pleat type:

     \[
     x \leftarrow y_1 + x^2 \leftarrow x^2 \pm y_1^n \leftarrow xy_1 \leftarrow x^3 + \beta y_1^3
     \]
     \[
     \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow
     \]
     \[
     x^2 + y_1^3 \leftarrow x^3 + y_1^n \leftarrow x^4 + y_1^2 \leftarrow \ldots \leftarrow x^n + y_1^2
     \]
     \[
     \uparrow \quad \uparrow \quad \uparrow \quad \vdots \quad \vdots
     \]
4 Semi-mild equivalence relation

Consider two pairs of hypersurfaces with boundaries $G_i = (V_i, B_i)$, $i = 1, 2$ defined by $(f_i, g_i)$. Assume that boundaries are fixed, that is $g_1 = g_2 = g$. It is instructive to have a similar equivalence relation to that in Definition 2.5 in which the boundaries are not necessarily preserved.

Consider the space of all hypersurfaces equipped with boundaries determined by $g$.

**Definition 4.1** Two hypersurfaces $V_1 = \{f_1 = 0\}$ and $V_2 = \{f_2 = 0\}$ are called semi-mild equivalent if there exists a family of diffeomorphisms $\Theta_t : \mathbb{R} \times \mathbb{R}^p \to \mathbb{R} \times \mathbb{R}^p$ continuously and piecewise smoothly depending on parameter $t \in [1, 2]$ and non vanishing families of smooth functions $f_t$ and $K_t$ also depending on parameter $t \in [1, 2]$ with $\theta_1 = \text{id}_{\mathbb{R} \times \mathbb{R}^p}$, and $K_1 = 1$ such that:

$$f_1 = K_t f_t(\Theta_t)$$

and the components of the vector field $v_t = \dot{X}(t) \frac{\partial}{\partial x} + \sum_{i=1}^{p} \dot{Y}_i(t) \frac{\partial}{\partial y_i}$ generating $\Theta_t$ on each segment of smoothness satisfy the following: $\dot{X}(t) \in C_{x,y}$ and $\dot{Y}_i(t) \in IJ_{G_t}$, where $J_{G_t}$ is the ideal generated by $f_t, \frac{\partial f_t}{\partial x}$ and $g$.

**Remarks 4.2**

1. In the definition 4.1, the family of functions $f_t$ is a homotopy between $f_1$ and $f_2$ and it will be called admissible deformation.

2. Also, in the definition 4.1, the diffeomorphism $\Theta_t$ need not to preserve the boundary $B$ but only need to preserve the direction of the projection at the critical points of the projection lying on the boundary. Such a family $\Theta_t$ of diffeomorphisms generated by the vector field $v_t$ as well as the vector field itself will be called admissible for the families of function germs $f_t$ which themselves are also called admissible with respect to semi-mild equivalence relation.

3. A function $H \in IJ_G$ if it admits a decomposition: $H = e_i + \int_{0}^{x} (f a_i + \frac{\partial f}{\partial x} b_i + g c_i) \, dx$, where $a, b, c \in C_{x,y}$ and $e_i \in C_y$.

4. The tangent space $TSM_f$ to the quasi equivalence class of $f$ has the following description:

$$TSM_f = \left\{ fA + \frac{\partial f}{\partial x} K + \sum_{i=1}^{p} \frac{\partial f}{\partial y_i} H_i : A, K \in C_{x,y}, H_i \in IJ_G \right\}.$$
Recall that \( IJ_G^* \) and \( IJ_G^{**} \) are the submodules of functions \( h(x, y) \) such that \( \frac{\partial h}{\partial x} \) belongs to the ideal generated only by \( \frac{\partial f}{\partial x} \) and \( g \), and \( f \) and \( g \), respectively. We have

**Proposition 4.3** The tangent space \( TSM_f \) can be described equivalently as

\[
TSM_f = \left\{ fA + \frac{\partial f}{\partial x} K + \sum_{i=1}^{p} \frac{\partial f}{\partial y_i} H_i : A, K \in C_{x,y}, H_i \in IJ_G^* \right\}.
\]

or

\[
TSM_f = \left\{ fA + \frac{\partial f}{\partial x} K + \sum_{i=1}^{p} \frac{\partial f}{\partial y_i} H_i : A, K \in C_{x,y}, H_i \in IJ_G^{**} \right\}.
\]

### 4.1 Classifications of hypersurfaces with smooth boundaries

For simplicity, we consider regular surfaces with boundaries being smooth and transversal to the ray of the projection direction \( \frac{\partial}{\partial x} \), in which case we set \( g = x \).

Consider the space \( \Omega \) of all germs of hypersurfaces with a boundary determined by \( g = x \), then we call a germ of a surface *simple* if its sufficiently small neighbourhood in \( \Omega \) contains only a finite number of semi-mild equivalence classes.

**Theorem 4.4** Let a regular surface germ \( V = \{ f = 0 \} \) be simple with respect to semi-mild equivalence relation. Then the projection of \( V \) is semi-mild equivalent to the projection of one of the regular surfaces \( \tilde{V} = \{ (x, y) : \tilde{f}(x, \tilde{y}) + y_p = 0 \} \), described in the following table.
<table>
<thead>
<tr>
<th>Notation</th>
<th>$f$</th>
<th>Restrictions</th>
<th>codimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{A}_0^*$</td>
<td>$x$</td>
<td>$-$</td>
<td>0</td>
</tr>
<tr>
<td>$\mathcal{A}_0$</td>
<td>$x^2$</td>
<td>$-$</td>
<td>0</td>
</tr>
<tr>
<td>$\mathcal{A}_k$</td>
<td>$x(y_1^{k+1} + y^2)$</td>
<td>$y^2 = \sum_{i=2}^{p-1} \pm y_i^2$, $k \geq 1$</td>
<td>$k$</td>
</tr>
<tr>
<td>$\mathcal{D}_k$</td>
<td>$x(y_1^2y_2 + y_2^{k-1} + y^2)$</td>
<td>$y^2 = \sum_{i=3}^{p-1} \pm y_i^2k \geq 4$</td>
<td>$k$</td>
</tr>
<tr>
<td>$\mathcal{E}_6$</td>
<td>$x(y_1^3 + y_4^4 + y^2)$</td>
<td>$y^2 = \sum_{i=3}^{p-1} y_i^2$</td>
<td>6</td>
</tr>
<tr>
<td>$\mathcal{E}_7$</td>
<td>$x(y_1^3 + y_1y_2^3 + y^2)$</td>
<td>$y^2 = \sum_{i=3}^{p-1} y_i^2$</td>
<td>7</td>
</tr>
<tr>
<td>$\mathcal{E}_8$</td>
<td>$x(y_1^3 + y_2^5 + y^2)$</td>
<td>$y^2 = \sum_{i=3}^{p-1} y_i^2$</td>
<td>8</td>
</tr>
</tbody>
</table>

Remarks 4.5

1. Any germ of a surface is either simple (and hence semi-mild equivalent to one of the surfaces in the above theorem) or belongs to a subset of infinite codimension in the space of all germs of surfaces.

2. The class $\mathcal{A}_1 : x^2 + y_2$ can be written equivalently as $x(y_1 + y^2) + y_2$.

3. The graph of adjacencies in low codimension is in one to one correspondence to the simple classes with respect the standard equivalence relation:

   $\mathcal{A}_5^* \leftarrow \mathcal{A}_0 \leftarrow \mathcal{A}_1 \leftarrow \mathcal{A}_2 \leftarrow \mathcal{A}_3 \leftarrow \mathcal{A}_4 \leftarrow \ldots$

   $\uparrow$

   $\mathcal{D}_4$

   $\uparrow$

   $\mathcal{D}_5 \leftarrow \mathcal{E}_6 \leftarrow \mathcal{E}_7 \leftarrow \mathcal{E}_8$

   $\uparrow$

   $\mathcal{D}_6$

   $\uparrow$

   $\mathcal{D}_7$

   $\uparrow$

   $\vdots$

To prove Theorem 4.4, we need to consider the following prenormal forms of a hypersurface with respect to semi-mild equivalence relation.

Let $V_t = \{ f_t = \tilde{f}_t(x, \tilde{y}) + y_p = 0 \}$ be an admissible deformation.
Lemma 4.6 1. If \( \frac{\partial f}{\partial x} \neq 0 \) then \( f_t \) is semi-mild equivalent to \( x + y_p \), for any value of \( t \).

2. If \( \frac{\partial^2 f}{\partial x^2} \neq 0 \) then \( f_t \) is semi-mild equivalent to \( x^2 + y_p \), for any value of \( t \).

**Proof.** The Lemma follows from the standard \( O \)-equivalence since any family \( \Theta_t \) of diffeomorphisms of \( \mathbb{R}^3 \) preserving the fibration \( (x, y) \mapsto y \) at the origin is admissible.

Lemma 4.7 The family \( f_t \) is semi-mild equivalent to \( x\varphi_t(\tilde{y}) + y_p \), where \( \varphi_t \) is a smooth function depending on \( t \in [1, 2] \) with \( \varphi_t \in C_{\tilde{y}} \).

**Proof.** The reduction result follows from standard Arnold’s spectral sequence method.

Corollary 4.8 Let \( f = x\varphi(\tilde{y}) + y_p \), where \( \varphi, \Psi \in C_{\tilde{y}} \). If \( \varphi(0) \neq 0 \) then \( f \) is semi-mild equivalent to \( x + y_p \).

**Proof of Theorem 4.4.** We distinguish the following cases:

- If \( \frac{\partial f}{\partial x} \neq 0 \) then \( f \) is semi-mild equivalent to the class \( A^*_0 : x + y_p \) [Lemma (4.6)].

- If \( \frac{\partial f}{\partial x} = 0 \) but \( \frac{\partial^2 f}{\partial x^2} \neq 0 \) then \( f \) the class \( A_1 : x^2 + y_p \) [Lemma (4.6)].

- In the most degenerate case, Using Lemma (4.7), \( f \) is reduced to the form \( x\varphi(\tilde{y}) + y_p \), where \( \varphi(\tilde{y}) \in M_{\tilde{y}} \). Up to the permutation between the indices of \( y_i \), one can easily prove the remaining classification results via the standard right equivalence relation within the \( \tilde{y} \)-space.

\[ \square \]

5 Conclusion

Throughout the current paper, we have dealt with two different types of new and non-standard equivalence relations in projection theory and we obtain the list of all simple “in the sense of Arnold” classes of singularities of hypersurfaces with boundaries. They are weaker than the standard and quasi ones, called mild equivalence and semi-mild equivalence. The main idea of these relation is to preserve only the direction of the critical locus of the projection lying on the boundary.
In a comparison with several application in geometry and differential equations of the standard singularities of projections of surfaces with boundaries, the further study of the non-standard equivalence relations would give extra information in various optimization problems and problems in variations theory with constraints. For example, one of the interesting application for quasi projection equivalence relation is used in partial differential equations (PDE) with boundary value problems. Consider the characteristics method solving the simplest Cauchy problem for first order linear PDE: \( \sum a_i(x) \frac{\partial u}{\partial x_i} = 0 \), where \( u(x) \) is unknown function with \( x \in \mathbb{R}^n \) and \( a_i(x) \) are given functions. The problem includes the boundary hypersurface \( S \subset \mathbb{R}^n \) and the boundary values \( U|_S = U_0 \). Generically the characteristic vector field \( v = a_i \frac{\partial}{\partial x_i} \) is tangent to \( S \) at some points which are called characteristic. Outside the set \( K \) of characteristic points, the problem has a unique local solution. So the geometry of the set \( K \) is essential feature of the problem. If we rectify the vector field getting say \( \frac{\partial}{\partial x_1} \), then the problem to classify \( K \) is exactly to find critical points of the projection of \( S \) along parallel rays. Similarly in many other complicated PDE boundary value problems, mainly in continuum mechanics, the generalisation of Neumann boundary condition is used. The derivative of the unknown function is taken along a given vector field (for Neumann this is normals to the boundary surface). The locus of the points on the surface where the vector field is not transversal to the surface is of importance.

The further work which can be done beyond the present paper is related to the interesting question on the bifurcation diagrams of mild (or semi-mild) classes and the applications of the non-standard classification of projections with boundaries to the above mentioned boundary value problems.

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References


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