Numerical Solutions of an Integro-differential Equation with Smooth and Singular Kernels

F. M. Alharbi

Common First Year Deanship
Umm Al-Qura University, Mecca, Saudi Arabia

Abstract

The paper is concerned to derive a Fredholm integral equation of second kind (FIESK) from an integro-differential equation with smooth or singular kernels (IDE) in the space $L_2[-1,1]$. Then, we used the modified Simpson’s rule and collection method to obtained the solution of IDE with smooth kernel. On the other hand, Toeplitz matrix method and the product Nyström method are used to find the numerical solution of IDE when the kernel take a singular form. Many of numerical applications computed and the errors, in each case are calculated. Moreover, many important special and new cases can be established from the work.

Mathematics Subject Classification: 45B05, 45G10

Keywords: Integro-differential equations (IDE), Fredholm integral equation of second kind (FIESK), singular kernel, Numerical methods

1 Introduction

Scientific progress in various medical, physical and engineering fields required to studies of ideas to integro-differential models, see [4]. The system of IDEs as a model of Tumor-immune cells competition is displayed in [13]. The main role of IDEs in a glass-forming process and nano-hydrodynamics field respectively
are discussed in [15]. The second-order integro-differential nonlocal theory of elasticity is established as an extension of the Eringen nonlocal integral model in [2]. In the recent years, there are many different methods that established to discuss the main solutions of the IDEs and discussion the results with respected to the kind of the modeling which it representing the references[4],[13],[15] and[2]. In this paper, our goal is to obtain the solution of IDE when the kernel takes the smooth or singular forms by using the theory of integral equation (ID).

2 IDE and ID

Consider the IDE:

\[ y''(s) + A(s)y'(s) + p(s)y(s) = Q(s) - \int_a^b L(s,u)y(u)du, \quad a \leq s \leq b \]  

under the boundary conditions,

\[ y(a) = q_1, \quad y(b) = q_2 \]  

where \( y(s) \) is the unknown function, \( L(s,u) \) is the kernel of the IDE may be continuous or singular but at least satisfies Fredholm condition. \( A(s), p(s) \) and \( Q(s) \) are known continuous functions in the class \( C[a,b] \) with its derivatives.

Integrating Eq.(2.1), twice, then letting \( s = b \) to obtain

\[
y(s) - q_1 = \frac{(s-a)}{(b-a)} \left[ (q_2 - q_1) - \int_a^b (b-t)(Q(t)dt + \int_a^b \int_a^b (b-t)L(s,u)y(u)du dt \\
+ \int_a^b \{ A(t) - (b-t)[A'(t) - p(t)] \} y(t)dt \right] \\
+ \int_a^s (s-t)Q(t)dt - \int_a^s \int_a^s (s-t)L(s,u)y(u)dudt \\
- \int_a^b \{ A(t) - (s-t)[A'(t) - p(t)] \} y(t)dt].
\]  

So, Eq.(2.3) reduce to

\[ y(s) = f(s) + \int_a^b k(s,u)y(u)du, \]  

where,
Integro-differential equation with smooth and singular kernels

\[ f(s) = \begin{cases} 
q_1 + \frac{s-a}{b-a} \{ q_2 - q_1 - \int_a^b (b-t)Q(t)dt \}, & s < t \\
\frac{1}{b-a} [(s-a)q_2 + (b-s)q_1] + \int_a^b \frac{(b-s)(a-t)}{(b-a)} Q(t)dt, & s > t
\end{cases} \]

(2.5)

and,

\[ k(s, u) = \begin{cases} 
\frac{s-a}{b-a} \{ A(u) - (b-u)[A'(u) - p(u)] + \int_a^b (b-t)L(t, u)dt \}, & s < t \\
\frac{(s-b)}{(b-a)} \{ A(u) + (u-a)[A'(u) - p(u)] \int_a^b (t-a)L(t, u)dt \}, & s > t
\end{cases} \]

(2.6)

Eq.(2.4) represents a FIDSK in the linear case. The solution of the integral equations, especially if the kernel in continuous form, may be obtained analytically using many different methods, see [4],[12] and[15]. At the same time the sense of numerical methods take an important place in solving the linear integral equation with smooth and singular kernel, see [8],[9],[10],[11],[14] and[2].

3 IDE with smooth kernel

We will seek the numerical solution of Eq.(2.1) with smooth kernel under the condition

(2.2) by using the theory of IE. To display this, we choose the following methods:

3.1 Modified Simpson’s rule

To obtain the solution of Eq.(2.4) by using modified Simpson’s quadrature rule, we have

\[ y_i = f_i + \frac{h}{3} k_{i,0} y_0 + \frac{4h}{3} \sum_{j=0}^{n-1} k_{i,2j+1} y_{2j+1} + \frac{2h}{3} \sum_{j=1}^{n} \frac{1}{2} k_{i,2j} y_{2j} + \frac{h}{3} k_{i,n} y_n \\
+ \frac{h^4}{180} [D''_{i,0} y_0 + k_{i,0} y''_0 + 3D_{i,0} y''_0 - D''_{i,n} y_n - k_{i,n} y''_n - 3D'_{i,n} y'_n - 3D_{i,n} y''_n] \]

(3.1)

where, \( D^m = \frac{\partial^{m+1} k(s, u)}{\partial u^{m+1}} \), \( m = 0,1,2 \) must be exist, \( h = \frac{b-a}{n} \), \( 1 \leq j \leq n \).
Also by repeated Simpson’s quadrature we have,

\[ y_i^m = f_i^m + \frac{h}{3} D_i^{m-1} y_0 + \frac{4h}{3} \sum_{j=0}^{n-1} D_i^{m-1} y_{2j+1} + \frac{2h}{3} \sum_{j=0}^{n-1} D_i^{m-1} y_{2j} + \frac{h}{3} D_i^{m-1} y_n \]

, \( m = 1,2,3, i = 0,1 \).

(3.2)

So, we have system of \( n + 7 \) equations with \( n + 7 \) variables which is solving the FIE then the IDE.

3.2 Collocation method

To solve Eq.(2.4) by collocation method, we estimate unknown function \( y(s) \) with the function \( \psi_i(s) \) as,

\[ y(s) \approx \sum_{i=0}^{n} a_i \psi_i(s) \]  \hspace{1cm} (3.3)

Substituting Eq.(3.3) in Eq.(2.4) we get

\[ \sum_{i=0}^{n} a_i \psi_i(s) = f(s) + \int_{a}^{b} k(s,t) \sum_{i=0}^{n} a_i \psi_i(t) dt + R_n(s) \]  \hspace{1cm} (3.4)

Hence,

\[ R_n(s) = \sum_{i=0}^{n} a_i \psi_i(s) - \int_{a}^{b} k(s,t) \sum_{i=0}^{n} a_i \psi_i(t) dt - f(s) \]  \hspace{1cm} (3.5)

the unknown coefficients \( a_i \) are defined by several collocation points, \( s_j, j = 0,1, ..., n \), so that \( R_n(s) \to 0 \) for all \( j \).

Here we select the collocation points as,

\[ s_j = a + jh \hspace{1cm} h = \frac{b-a}{n} \hspace{1cm} n = 0,1, ..., n \]  \hspace{1cm} (3.6)

Thus Eq.(3.4) can be converted to a system of linear equations,

\[ A_n X = B_n \]  \hspace{1cm} (3.7)

where,
Integro-differential equation with smooth and singular kernels

\[ A_n = \left[ \psi_i(s) - \int_a^b k(s, t)\psi_i(t)dt \right]_{j=1}^n, i = 0, ..., n \]
\[ B_n = [f(s_j)], j = 0, ..., n \]
\[ X^T = [a_i]_{i=0}^n \] (3.8)

4 IDE with singular kernel

To discuss the solution of Eq.(2.1) when its kernel take a singular shape under the previous conditions by using the theory of IEs, we use the following methods:

4.1 Toeplitz matrix method

To discuss the solution of Eq.(2.1) by using Toeplitz matrix method we partition \([-a, a]\) to \(-N, -N + 1, ..., 0, ..., N - 1\), and write the integral term in the form, see [9-11]

\[ \int_{-a}^a k(|s - u|)y(u)du = \sum_{n=-N}^{N-1} \int_{nh}^{nh+h} k(|s - u|)y(u)du , \quad h = \frac{a}{N}. \] (4.1)

Then, approximate the sub integral to

\[ \int_{nh}^{nh+h} k(|s - u|)y(u)du = A_n(s) y(nh) + B_n(s) y(nh + h) + R_{N,n} \] (4.2)

where \(R_{N,n}\) is the estimate error. Using the principal of Toeplitz matrix method by assuming \(y(s) = 1\), \(s\) respectively to leads to \(R_{N,n} \to 0\), hence the function \(A_n(s)\) and \(B_n(s)\) take the following forms,

\[ A_n(s) = \frac{\left( (nh+h)l(s) - j(s) \right)}{h} \quad \text{and} \quad B_n(s) = \frac{\left( j(s) - nh \, l(s) \right)}{h} \] (4.3)

where:

\[ I(s) = \int_{nh}^{nh+h} k(|s - u|)du \quad , \quad f(s) = \int_{nh}^{nh+h} u k(|s - u|)du \] (4.4)

therefore Eq.(4.2) reduce to,

\[ \int_{-a}^a k(|s - u|)y(u)du = \sum_{n=-N}^N D_n(s) y(nh) \] (4.5)
where,

\[
D_n(s) = \begin{cases} 
A_{-N}(s) & , n = -N \\
A_n(s) + B_{n-1}(s) & , -N < n < N \\
B_{N-1}(s) & , n = N
\end{cases} \tag{4.6}
\]

After letting \( s = mh \), Eq.(2.1) we have

\[
y(mh) = f(mh) + \sum_{n=-N}^{N} \chi_{n,m} y(nh)
\tag{4.7}
\]

The last formula represents a linear system of algebraic equations, the elements of the matrix \( Y_{n,m} \) are given by

\[
\chi_{n,m} = P_{n-m} + P'_{n,m}
\tag{4.8}
\]

Where:

\[
P_{n-m} = A_n(mh) + B_{n-1}(mh) \quad , \quad -N < n < N.
\tag{4.9}
\]

The matrix \( P_{n-m} \) is the Toeplitz matrix of order \( 2N + 1 \), \(-N < n, m < N\), and the elements of the second matrix \( P'_{n,m} \) are zeros except the elements of the first and last rows. We can evaluate the values of the first rows by substituting \( n = -N, m = -N + i, 0 \leq i \leq 2N \), in \( B_{n-1}(mh) \), and the values of the last row by substituting \( n = -N, m = -N + i \), in \( A_n(mh) \).

Finally, the solution of the formula takes the form

\[
y(mh) = \left[ I - \chi_{n,m} \right]^{-1} f(mh) \quad , \quad \det(I - \chi_{n,m}) \neq 0
\tag{4.10}
\]

This method is said to be convergent of order \( r \) in \([-a, a]\), if for \( N \) sufficiently large, there exist a constant \( l > 0 \) independent of \( N \) such that,

\[
\|y(s) - y_N(s)\| \leq l N^{-r}
\tag{4.11}
\]

The estimating error \( R_{N,n} \) is determined from the following relation,

\[
R_{N,n} = \left| \int_{nh}^{nh+h} u^2 k(|s-u|)du - A_n(s)(nh)^2 - B_n(s)(nh + h)^2 \right|
\tag{4.12}
\]

### 4.2 The Product Nystrom Method

To discuss the solution of the FIESK using the product Nystrom method, see Delves and Mohamed [7] and Atkinson [6], consider the IE.
where \( p \) and \( \bar{k} \) are badly behaved and well behaved functions of their arguments, respectively. Divided the interval \([-a, a]\) in Eq.(4.13) into \( N \) equal subinterval, where \( s_i = u_i = a + ih, i = 0,1,2,..., N \) with \( h = \frac{2a}{N} \), \( N \) even. Also, we approximate the integral term by a product integration form of Simpson’s rule, where \( u = u_i \) we write

\[
\sum_{j=0}^{N} w_{ij} \bar{k}(|s_i - u_j|)y(u_j) = \sum_{j=0}^{N-2/2} \int_{u_{2j}}^{u_{2j+2}} p(s_i, u) \bar{k}(|s_i - u|)y(u)du
\]  

(4.14)

where \( w_{ij} \) are the weight determined completely in Delves and Mohamed[7]. If we approximate the non singular part by the second degree of Lagrange interpolation polynomial which interpolates it at the points \( u_{2j},u_{2j+1},u_{2j+2} \) over each interval \([u_{2j},u_{2j+2}]\) then after we introducing the change of variables \( u = u_{2j-2} + \zeta h, 0 \leq \zeta \leq 2 \) to have,

\[
\begin{align*}
\alpha_j(u_i) &= \frac{h}{2} \int_0^2 \zeta (\zeta - 1)p(u_i, u_{2j-2} + \zeta h)d\zeta \\
\beta_j(u_i) &= \frac{h}{2} \int_0^2 (\zeta - 2)(\zeta - 1)p(u_i, u_{2j-2} + \zeta h)d\zeta \\
\gamma_j(u_i) &= \frac{h}{2} \int_0^2 (2 - \zeta)p(u_i, u_{2j-2} + \zeta h)d\zeta 
\end{align*}
\]  

(4.15)

if we define,

\[
\chi_k = \int_0^2 \zeta^k p(u_i - u_{2j-2} + \zeta h)d\zeta \quad , k = 0,1,2 \quad , u_i - u_{2j-2} = (i - 2j + 2)h 
\]  

(4.16)

and assume \( z = i - 2j + 2 \) to have,

\[
\begin{align*}
w_{i,0} &= h[2\chi_0(z) - 3\chi_1(z) + \chi_2(z)]/2 \quad , z = i \\
w_{i,2j+1} &= h[2\chi_1(z) - \chi_2(z)] \quad , z = i - 2j \\
w_{i,2j} &= h[\chi_2(z) - \chi_1(z) + 2\chi_0(z - 2) - 3\chi_1(z - 2) + \chi_2(z - 2)]/2 \quad , z = i - 2j + 2 \\
w_{i,N} &= h[\chi_2(z) - \chi_1(z)/2] \quad , z = i - N + 2 
\end{align*}
\]  

(4.17)

therefore, the integral equation (4.13) can be reduced to a system of linear algebraic equations,
\[ y_i(ih) - \sum_{j=0}^{N} w_{i,j} \bar{k}(|s_i - u_j|) y_j(jh) = f_i(ih) \quad , \]

which has the solution,

\[ Y = [I - W]^{-1} F \quad , \quad \det(I - W) \neq 0 \quad . \]

This method is said to be convergent of order \( r \) in \([-a, a]\), if and only if for \( N \) sufficiently large, there exist a constant \( C > 0 \) independent of \( N \) such that,

\[ \| y(s) - y_N(s) \| \leq C N^{-r} \quad . \]

## 5 Applications and numerical results

In this section, we intend to compare modified Simpson’s rule with the collocation method to solve the smooth IDE. While we make a comparing between Toeplitz matrix method and the product Nystrom method to obtain the numerical solution of the singular type. We solve all of applications by using Matlab program 2019 and compute the estimating errors.

### 5.1 Application 1 (Smooth kernel)

Consider the IDE:

\[ y''(s) + \sqrt{s} y(s) = Q(s) - \int_{-1}^{1} s^2 e^{u} y(u) du \quad , \quad -1 \leq s \leq 1 \quad (5.1) \]

under the boundary conditions,

\[ y(-1) = \frac{1}{2} \quad , \quad y(1) = \frac{2}{3} \quad (5.2) \]

The exact solution is: \( y(s) = 1 + \frac{11}{12} s \).

Eq.(5.1) reduce to Eq.(2.4) where:

\[ f(s) = \begin{cases} 
\left[ \frac{1}{2} + \frac{1}{6} - \int_{-1}^{1} (1 - t)Q(t)dt \right] s & s < t \\
\frac{1}{2} - \frac{1}{6} s - \left[ \int_{-1}^{1} Q(t)dt \right] (1 - s) & s > t
\end{cases} \quad (5.3) \]

and,
**Integro-differential equation with smooth and singular kernels**

\[ k(s, u) = \begin{cases} \frac{1}{12} \exp u + (1 - u)\sqrt{u}|s & s < t \\ \frac{1}{4} \exp u - \sqrt{u^3} (s - 1) & s > t \end{cases} \]

(5.4)

Numerical results by using modified Simpson’s rule and Collection method are shown in Table(1) and the estimating errors for both methods are display in Figure (1).

**Table (1)**

<table>
<thead>
<tr>
<th>( x )</th>
<th>Exact Solution</th>
<th>M.S.A. Solution / ( n = 10 )</th>
<th>C.A. Solution / ( n = 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0.083333</td>
<td>0.085778</td>
<td>0.086669</td>
</tr>
<tr>
<td>-0.77</td>
<td>0.294167</td>
<td>0.296049</td>
<td>0.29675</td>
</tr>
<tr>
<td>-0.55</td>
<td>0.495833</td>
<td>0.497178</td>
<td>0.497678</td>
</tr>
<tr>
<td>-0.33</td>
<td>0.6975</td>
<td>0.698307</td>
<td>0.698607</td>
</tr>
<tr>
<td>-0.11</td>
<td>0.899167</td>
<td>0.899436</td>
<td>0.899536</td>
</tr>
<tr>
<td>0.11</td>
<td>1.100833</td>
<td>1.100564</td>
<td>1.100464</td>
</tr>
<tr>
<td>0.33</td>
<td>1.3025</td>
<td>1.301693</td>
<td>1.301393</td>
</tr>
<tr>
<td>0.55</td>
<td>1.504167</td>
<td>1.502822</td>
<td>1.502322</td>
</tr>
<tr>
<td>0.77</td>
<td>1.705833</td>
<td>1.703951</td>
<td>1.70325</td>
</tr>
<tr>
<td>1</td>
<td>1.916667</td>
<td>1.914222</td>
<td>1.913312</td>
</tr>
</tbody>
</table>

**Figure (1)**

![Error of Modified Simpson's Approximate](image1)

![Error of Collection Approximate](image2)
5.2 Application 2 (Singular kernel)

Consider the IDE:

\[ y''(s) + sy'(s) + y(s) = Q(s) - \int_{-1}^{1} \ln|u - s|y(u)du \]

under the boundary conditions,

\[ y(-1) = y(1) = 0 \]

The exact solution is: \[ y(s) = \frac{s+1}{5} \].

Eq.(5.3) reduce to Eq.(2.4) where:

\[ f(s) = \begin{cases} \frac{s+1}{2} \left[ -\int_{-1}^{1} (1 - t)Q(t)dt \right] & s < t \\ -\frac{1}{4} (1 - s) \left[ \int_{-1}^{1} (1 + t)Q(t)dt \right] & s > t \end{cases} \]

and,

\[ k(s, u) = \frac{s+1}{2} \left[ u - 2 + (u^2 - u + \frac{1}{2}) \ln|u + 1| - (\frac{1}{2} - u^2 + u + \frac{1}{2}) \ln|u| - 1| + \ln|u| \right], \quad s < t \]

and,

\[ k(s, u) = \frac{s-1}{2} \left[ u - 2 - (u^2 + u - \frac{1}{2}) \ln|u + 1| + (u^2 + u + \frac{1}{2}) \ln|u - 1| + \ln|u| \right], \quad s > t \]

Numerical results by using Toeplitz matrix method and Product Nyström method are shown in Table(2) and the estimating errors for both methods are display in Figure(2).
Table (2)

<table>
<thead>
<tr>
<th>$x$</th>
<th>Exact Solution</th>
<th>T.M. Solution /</th>
<th>P.N. Solution /</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-0.8</td>
<td>0.04</td>
<td>0.0411332</td>
<td>0.041832</td>
</tr>
<tr>
<td>-0.6</td>
<td>0.32</td>
<td>0.3201143</td>
<td>0.3202242</td>
</tr>
<tr>
<td>-0.4</td>
<td>0.12</td>
<td>0.127232</td>
<td>0.127334</td>
</tr>
<tr>
<td>-0.2</td>
<td>0.16</td>
<td>0.1611532</td>
<td>0.1612532</td>
</tr>
<tr>
<td>0</td>
<td>0.2</td>
<td>0.24402</td>
<td>0.24413</td>
</tr>
<tr>
<td>0.2</td>
<td>0.24</td>
<td>0.2400132</td>
<td>0.240112</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6</td>
<td>0.622342</td>
<td>0.623004</td>
</tr>
<tr>
<td>0.6</td>
<td>0.32</td>
<td>0.3211541</td>
<td>0.32232</td>
</tr>
<tr>
<td>0.8</td>
<td>0.36</td>
<td>0.36245</td>
<td>0.36255</td>
</tr>
<tr>
<td>1</td>
<td>0.4</td>
<td>0.413382</td>
<td>0.413442</td>
</tr>
</tbody>
</table>

Figure (2)

Conclusion

1. The IDEs under the boundary conditions reduce to Fredholm integral equations of the second kind which have many different analytical and numerical methods to solved with respect to the kind of its kernel.
2. The Schrödinger equation in one dimension is the important special case of Eq. (2.1), which we obtained if $A(s), Q(s)$ and $L(s, u)$ are vanishes also $p(s) = \frac{2m}{\hbar^2} [E - U(s)]$ where $E$ and $U(s)$ are the total non relativistic and potential energies of a particle of mass $m$ respectively. If $E > U(s)$, the kinetic energy is positive (bound state and scattering) and if $E < U(s)$, then the kinetic energy is negative and not admissible classically (unbound state and tunnelling), see Figure (3). The Schrödinger equation is used to find the allowed energy levels of quantum mechanical systems (such as atoms or transistors). The associated wave function gives the probability of finding the particle of certain position.

![Figure (3)](image)

3. The work of Frankel,[5], when he discussed the singular IDE, 

$$y'(s) - f(s) = \int_{0}^{1} \frac{y(t)}{s - t} \, dt , \quad 0 < s < 1$$

is the main special case of Eq. (2.1). This equation has appeared in both combined infrared gaseous radiation and molecular conduction, and elastic contact studies.
References


Received: October 28, 2019; Published: December 29, 2019