On a Generalized Mittag-Leffler Function

Luciano Leonardo Luque

Faculty of Exact Sciences
National University of the Northeast
Av. Libertad 5540 (3400); Corrientes, Argentina

This article is distributed under the Creative Commons by-nc-nd Attribution License. Copyright © 2019 Hikari Ltd.

Abstract

In this short paper is introduce and investigate a new generalized Mittag-Leffler function, that it will allows us to define a generalized Exponential function and fractional Trigonometric functions.

Mathematics Subject Classification: 33E12, 33B10, 26A33

Keywords: Fractional Trigonometric Functions, Generalized Exponential Function

1 Introduction and Preliminaries

In recent years new generalizations of the Mittag-Leffler function have been introduced, and many of their properties have been studied, mainly analyzed as solutions of some fractional differential equations, as can be seen, for example, in [1] [4], [6], [7], [12], [14], [15] and [16].

There are many generalizations of the trigonometric functions; some of the ones that were taken as reference for this article are found in [3], [1], [17] and [9].

Recently, the fractional trigonometry has acquired a great interest, for example, in [9] it is use for the modeling of different phenomena that respond to the behavior of spirals. The generalized trigonometric functions are also used in the solution of fractional differential equations, as can be seen, for example, in [10] and [11].
The Mittag-Leffler functions $E_\alpha(x)$ and $E_{\alpha,\beta}(x)$ are defined by the following series:

$$E_\alpha(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(\alpha j + 1)} \quad (x \in \mathbb{C}; \operatorname{Re}(\alpha) > 0),$$

and

$$E_{\alpha,\beta}(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(\alpha j + \beta)} \quad (x \in \mathbb{C}; \operatorname{Re}(\alpha), \operatorname{Re}(\beta) > 0),$$

respectively; where $\Gamma(x)$ is the classical Gamma function. These functions are extensions of the exponential function, since $E_{\alpha,1} = E_\alpha$ and $E_1(\lambda x) = e^{\lambda x}$ ($\lambda \in \mathbb{C}$) (see, for example, [7, 8]). Based on $E_{\alpha,\beta}(x)$ it is defined the called $\alpha$-Exponential Function:

$$e^{\lambda x}_\alpha = x^{\alpha-1} E_{\alpha,\alpha} (\lambda x^\alpha),$$

with $x \in \mathbb{C} \setminus \{0\}$, $\operatorname{Re}(\alpha) > 0$, $\lambda \in \mathbb{C}$, and it verified that

$$\lim_{x \to 0} [x^{1-\alpha} e^{\lambda x}_\alpha] = \frac{1}{\Gamma(\alpha)} \quad (\operatorname{Re}(\alpha) > 0).$$

The $\alpha$-Exponential Function is an extension of the exponential function, since $e^{\lambda x}_1 = e^{\lambda x}$.

If $x > a$ and $\lambda = b + ic$ ($b, c \in \mathbb{R}$), then the real part and imaginary part of $e^{\lambda x}_\alpha$ are defined as

$$\cos_\alpha [\lambda(x - a)] = \operatorname{Re} [e^{i\lambda(x-a)}_\alpha]$$

and

$$\sin_\alpha [\lambda(x - a)] = \operatorname{Im} [e^{i\lambda(x-a)}_\alpha].$$

Prabhakar introduce in [16] the Mittag-Leffler type function

$$E_{\alpha,\beta}^\gamma(x) = \sum_{j=0}^{\infty} \frac{(\gamma)_j x^j}{\Gamma(\alpha j + \beta) j!},$$

with $\alpha, \beta, \gamma \in \mathbb{C}; \operatorname{Re}(\alpha), \operatorname{Re}(\beta) > 0$, and $x \in \mathbb{C}$; where $(\gamma)_j$ is the Pochhammer symbol (see, for example [8]), with $x \in \mathbb{C}$, defined by

$$(\gamma)_0 = 1 \quad \text{and} \quad (\gamma)_n = \gamma(\gamma+1)...(\gamma+n-1) \quad (n \in \mathbb{N}).$$

And it verified $E_{\alpha,\beta}^1 = E_{\alpha,\beta}$.

If a function $f(x)$ defined in $[a,b]$ the following expresion (see [8])

$$(D_{\alpha}^\alpha f)(x) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dx} \right)^n \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt, \quad n = [\alpha] + 1, \quad x > a;$$
it is called Fractional Derivative of order \( \alpha \). In [8, Chapter 2], it is prove that if \( \alpha, \beta \in \mathbb{C}, \Re(\alpha), \Re(\beta) > 0 \), then

\[
(D_+^{\alpha}(t-a)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)}(x-a)^{\beta-\alpha-1} \quad (\Re(\alpha) \geq 0).
\]  

(10)

The following relationship is obtained from (9) and (10):

\[
(D_+^{\alpha}(t-a)^{\beta-1}E_{\mu,\beta}[\lambda(t-a)^{\mu}](x) = (x-a)^{\alpha+\beta-1}E_{\mu,\alpha+\beta}[\lambda(x-a)^{\mu}]
\]  

(11)

with \( \lambda \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\mu) > 0 \). In particular, if \( \beta = \mu = \alpha \), and taking into account that \( \lim_{x \to 0} \frac{1}{\Gamma(x)} = 0 \) from (11) the following result is obtain

\[
(D_+^{\alpha}e_\alpha^{\lambda(t-a)}(x) = \lambda e_\alpha^{\lambda(x-a)},
\]  

(12)

when \( \Re(\alpha) > 0 \), and \( \lambda \in \mathbb{C} \) (see, for example [8]).

## 2 Main results

In this section our results are proved.

**Definition 2.1** Let \( \alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0, \)

\( n \in \mathbb{N}_0 \) be. The L-Mittag-leffler function is defined by the following serie

\[
L_{\alpha,\beta}^{\gamma,n}(x) = \sum_{j=0}^{\infty} \frac{(\gamma)_{j+n}x^j}{\Gamma(\alpha j + \beta)(j+n)!}, \quad (x \in \mathbb{C}).
\]  

(13)

**Remark 1** We have the following particular cases: \( L_{\alpha,\beta}^{\gamma,0}(x) = E_{\alpha,\beta}^{\gamma}(x) \), \( L_{\alpha,\beta}^{1,0}(x) = E_{\alpha,\beta}(x) \), and \( L_{\alpha,1}^{1,0}(x) = E_{\alpha}(x) \).

**Theorem 2.2** The L-Mittag-Leffler function \( L_{\alpha,\beta}^{\gamma,n}(x) \) defined by (13) is an entire function.

**Proof:** We will make use of the asymptotic expansion for the Gamma function (see, for example [8]):

\[
\Gamma(x) = (2\pi)^{1/2}x^{x-1/2}e^{-x} \left[ 1 + O\left(\frac{1}{x}\right) \right] \quad (|x| \to \infty),
\]  

(14)

and the quotient expansion of two Gamma functions at infinity

\[
\frac{\Gamma(x+a)}{\Gamma(x+b)} = x^{a-b} \left[ 1 + O\left(\frac{1}{x}\right) \right] \quad (|\arg(x+a)| < \pi; \ |x| \to \infty).
\]  

(15)
We will also consider the following relationship between the Pochhammer Symbol and the Gamma function:

\[(\gamma)_n = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)} \quad (n \in \mathbb{N}_0).\]  

(16)

Let \( R \) denote the radius of convergence of the power series in (13), and we call

\[c_j^{n,\gamma,\alpha} = \frac{(\gamma)_{j+n}}{\Gamma(\alpha(j+1)) (j+n)!}.\]  

Since

\[R = \limsup_{n \to \infty} \left| \frac{c_j^{n,\gamma,\alpha}}{c_{j+1}^{n,\gamma,\alpha}} \right| .\]  

(18)

in view of the hypothesis, we can easily see that

\[
\left| \frac{(\gamma)_{j+n}}{\Gamma(\alpha(j+1)) (j+n)!} \right| = \left| \frac{(\gamma)_{j+n+1}}{\Gamma(\alpha(j+2)) (j+n+1)!} \right| = \left| \frac{\Gamma(\gamma+1)(\gamma)_{j+n}}{\Gamma(\alpha(j+1)) (j+n)!} \right| = (j+n+1) \left| \frac{\Gamma(\alpha(j+2))}{\Gamma(\alpha(j+1))} \right| \left| \frac{\Gamma(\gamma+n+j)}{\Gamma(\gamma+n+1)} \right| \sim |\alpha|^\beta \, j^{-\Re(\alpha)+1} \to \infty
\]

(19)

when \( j \to \infty \). Then \( L_{\alpha,\beta}^{\gamma,n}(x) = \sum_{j=0}^{\infty} c_j^{n,\gamma,\alpha} x^j \) is an entire function.

**Theorem 2.3** Let \( \alpha, \beta, \mu \in \mathbb{C} \) and \( \Re(\alpha) > 0, \Re(\beta) > \Re(\mu) > 0 \), then

\[D_{\alpha^+}^\mu \left[ (t-a)^{\beta-1} L_{\alpha,\beta}^{\gamma,n}(\lambda(t-a)^\alpha) \right] (x) = (x-a)^{\beta-\mu-1} L_{\alpha,\beta-\mu}^{\gamma,n}[\lambda(x-a)^\alpha], \quad (x > a).\]  

(20)

**Proof:** Taking into account (10), and the Definition 2.1, the proof is completed:

\[
D_{\alpha^+}^\mu \left[ (t-a)^{\beta-1} L_{\alpha,\beta}^{\gamma,n}(\lambda(t-a)^\alpha) \right] (x) = D_{\alpha^+}^\mu \left[ \sum_{j=0}^{\infty} \frac{(\gamma)_{n+j} \lambda^j (t-a)^{\alpha j + \beta - 1}}{\Gamma(\alpha j + \beta)(n+j)!} \right] (x)
\]

\[
= \sum_{j=0}^{\infty} \frac{(\gamma)_{n+j} \lambda^j}{\Gamma(\alpha j + \beta) (n+j)!} D_{\alpha^+}^\mu \left[ (t-a)^{\alpha j + \beta - 1} \right] (x)
\]

\[
= \sum_{j=0}^{\infty} \frac{(\gamma)_{n+j} \lambda^j}{\Gamma(\alpha j + \beta) \Gamma(\alpha j + \beta - \mu)} (x-a)^{\alpha j + \beta - \mu - 1}
\]

\[
= \sum_{j=0}^{\infty} \frac{(\gamma)_{n+j} \lambda^j}{\Gamma(\alpha j + \beta - \mu)} (x-a)^{\alpha j + \beta - \mu - 1} = (x-a)^{\beta-\mu-1} L_{\alpha,\beta-\mu}^{\gamma,n}[\lambda(x-a)].
\]  

(21)
Theorem 2.4 If \( \alpha, \beta \in \mathbb{C}, \) with \( \Re(\alpha), \Re(\beta) > 0, \) then
\[
D_{a+}^\beta [(t-a)^{\beta-1} L_{\alpha,\beta}^{\gamma,n}(\lambda(t-a)^{\alpha})] (x) = \lambda(x-a)^{\alpha-1} L_{\alpha,\alpha}^{\gamma,n+1}[\lambda(x-a)^{\alpha}] , \quad (x > a) .
\] (22)

Proof: Proceeding as above, from (10) and the Definition 2.1, we obtain :
\[
D_{a+}^\beta [(t-a)^{\beta-1} L_{\alpha,\beta}^{\gamma,n}(\lambda(t-a)^{\alpha})] (x) = \sum_{j=0}^{\infty} \frac{(\gamma)_{n+j+1} \lambda^{j+1} (x-a)^{\alpha(j+1)-1}}{\Gamma(\alpha(j+1))(n+j+1)!} D_{a+}^\beta [(t-a)^{\alpha j+\beta-1}] (x)
\]
\[
= \sum_{j=0}^{\infty} \frac{(\gamma)_{n+j+1} \lambda^{j+1} (x-a)^{\alpha(j+1)-1}}{\Gamma(\alpha(j+1))(n+j+1)!} = \lambda(x-a)^{\alpha-1} L_{\alpha,\alpha}^{\gamma,n+1}[\lambda(x-a)^{\alpha}] .
\] (23)

Corollary 2.5 If \( \alpha \in \mathbb{C}, \Re(\alpha) > 0, \) then
\[
D_{a+}^{\alpha} [(t-a)^{\alpha-1} L_{\alpha,\alpha}^{\gamma,n}(\lambda(t-a)^{\alpha})] (x) = \lambda(x-a)^{\alpha-1} L_{\alpha,\alpha}^{\gamma,n+1}(\lambda(x-a)^{\alpha}) , \quad (x > a) .
\] (24)

Proof: The thesis results from applying the Theorem 2.4, with \( \alpha = \beta. \)

2.1 The \( \gamma-\alpha-n \)-Exponential Function

In this section we define new fractional trigonometric functions based on the generalized exponential function introduced in (25).

Definition 2.6 If \( \lambda, \gamma \in \mathbb{C} \) with \( \Re(\gamma) > 0, \) and \( a \in \mathbb{R}, \) \( \alpha \in \mathbb{R}^+, \) \( n \in \mathbb{N}_0; \) the \( \gamma-\alpha-n \)-Exponential function will be defined as follows:
\[
\lambda_{\alpha,\gamma,n}^{\lambda(x-a)} = (x-a)^{\alpha-1} L_{\alpha,\alpha}^{\gamma,n}(\lambda(x-a)^{\alpha}) \quad (x > a) ,
\] (25)

where \( L_{\alpha,\beta}^{\gamma,n}(x) \) is the L-Mittag-Leffler function defined by (13).

Remark 2 We have the special cases: \( \lambda_{\alpha,1,n}^{\lambda(x-a)} = \lambda_{\alpha}^{\lambda(x-a)} \) and \( \lambda_{1,1,n}^{\lambda(x-a)} = e^{\lambda(x-a)}. \)

Theorem 2.7 The \( \gamma-\alpha-n \)-Exponential function is well defined.

Proof: Let \( j_0 \in \mathbb{N}_0 \) be such that \( j_0 = \min \{ j \in \mathbb{N}_0 : \alpha(j+1) - 1 \geq 0 \}. \) Then, since
\[
\lambda_{\alpha,\gamma,n}^{\lambda(x-a)} = \sum_{j=0}^{j_0-1} \frac{(\gamma)_{j+1} \lambda^{j+1} (x-a)^{\alpha(j+1)-1}}{\Gamma(\alpha(j+1))(j+1)!} + \sum_{j=j_0}^{\infty} \frac{(\gamma)_{j+n} \lambda^{j+n} (x-a)^{\alpha(j+n)-1}}{\Gamma(\alpha(j+1))(j+n)!} ,
\] (26)

the only possible singularity is given in the \( j_0 \) first terms in \( \lambda; \) therefore, for every \( x > a \) there exists \( \lambda_{\alpha,\gamma,n}^{\lambda(x-a)} \).
Theorem 2.8 The series that defines the function $e_{\alpha, \gamma, n}^{\lambda(x-a)}$ converges uniformly and absolutely in any compact set $A \subset (a, +\infty)$, $a \geq 0$.

Proof: Taking $c_{n, \gamma, \alpha}^{\lambda(x-a)} = \frac{(\gamma)_{j+n}}{\Gamma(\alpha(j+1)(j+n))}$, we can write:

$$
e_{\alpha, \gamma, n}^{\lambda(x-a)} = \sum_{j=0}^{\infty} c_{n, \gamma, \alpha}^{\lambda(x-a)} x^{j(x-a)^{\alpha(j+1)-1}}.
$$

(27)

Since $A$ is a compact set and the function $|x-a|^{\alpha(j+1)-1}$ is continuous on $A$, there exist $b \in A$ such that $|b-a|^{\alpha(j+1)-1} = \max_{x \in A} \{ |x-a|^{\alpha(j+1)-1} \}$. Then, we have that

$$
|c_{n, \gamma, \alpha}^{\lambda(x-a)} x^{j(x-a)^{\alpha(j+1)-1}}| \leq c_{n, \gamma, \alpha}^{\lambda(x-a)} |x-a|^{\alpha(j+1)-1}.
$$

(28)

On the other hand, proceeding as in (19) it result that

$$
\left| \frac{c_{j+1, \gamma, \alpha}^{\lambda(x-a)}}{c_{j, \gamma, \alpha}^{\lambda(x-a)}} \right| \frac{|x-a|^{\alpha(j+2)-1}}{|x-a|^{\alpha(j+1)-1}} = \left| \frac{c_{j+1, \gamma, \alpha}^{\lambda(x-a)}}{c_{j, \gamma, \alpha}^{\lambda(x-a)}} \right| \frac{|x-a|}{|x-a|^{\alpha}} \sim \frac{|x-a|}{|x-a|^{\alpha}} \rightarrow 0,
$$

(29)

that is, the series $\sum_{j=0}^{\infty} c_{j, \gamma, \alpha}^{\lambda(x-a)} |x-a|^{\alpha(j+1)-1}$ converges. Then, from (28) and (29), and by the Weierstrass Criterion, the proof is completed.

Theorem 2.9 Let $x \in A \subset (a, \infty)$ be, with $A$ a compact set, then

$$
\lim_{n \to \infty} \Gamma(\gamma) e_{\alpha, \gamma, n}^{\lambda(x-a)} = e_{\alpha}^{\lambda(x-a)}.
$$

(30)

Proof: From Theorem 2.8, we know that the series that defines the function converges uniformly in $A$, so we can exchange the limit with the sum, and since for large values of $n$:

$$
(\gamma)_{j+n} \bigg/ (j+n)! = \frac{\Gamma(\gamma+j+n)}{\Gamma(j+n+1)\Gamma(\gamma)} \sim \frac{\Gamma(n)}{\Gamma(n)\Gamma(\gamma)} = \frac{1}{\Gamma(\gamma)}.
$$

(31)

Then

$$
\lim_{n \to \infty} \Gamma(\gamma) e_{\alpha, \gamma, n}^{\lambda(x-a)} = \Gamma(\gamma) \sum_{j=0}^{\infty} \left[ \lim_{n \to \infty} \frac{(\gamma)_{j+n}}{(j+n)!} \right] \frac{\lambda^j(x-a)^{\alpha(j+1)-1}}{\Gamma(\alpha(j+1))} = \sum_{j=0}^{\infty} \frac{\lambda^j(x-a)^{\alpha(j+1)-1}}{\Gamma(\alpha(j+1))} = e_{\alpha}^{\lambda(x-a)}.
$$
Theorem 2.10 Let \( \lambda, \gamma \in \mathbb{C}, \Re(\gamma) > 0, n \in \mathbb{N}_0, N \in \mathbb{N} \) and \( 0 < \alpha \leq 1 \), then
\[
(D_a^{N\alpha} e^{\lambda(t-a)}) (x) = \lambda^N e^{\lambda(x-a)},
\] (32)
where \( y^{(k\alpha)} = (D_a^{k\alpha} y(x)) \) \( (k = 1, 2, ..., N) \) represents a sequential fractional derivative, introduced by Miller and Ross in [13]:
\[
D_a^\alpha = \mathcal{D}_a^\alpha \ (0 < \alpha \leq 1),
\]
\[
D_a^j = \mathcal{D}_a^\alpha \mathcal{D}_a^{(k-1)\alpha},
\] (33)
where \( \mathcal{D}_a^\alpha \) is the Riemann-Liouville fractional derivative: \( \mathcal{D}_a^\alpha = D_a^\alpha \).

Proof: By (10), we obtain
\[
[D_a^\alpha e^{\lambda(t-a)}] (x) = \sum_{j=0}^{\infty} \frac{(\gamma)_{j+n}\lambda^j}{\Gamma(\alpha(j+1))(j+n)!} [D_a^\alpha (t-a)^{\alpha(j+1)-1}] (x)
\]
\[
= \sum_{j=1}^{\infty} \frac{(\gamma)_{j+n}\lambda^j}{\Gamma(\alpha(j+1))(j+n)!} \frac{\Gamma(\alpha(j+1))}{\Gamma(\alpha j)} (x-a)^{\alpha j-1}
\]
\[
= \sum_{j=0}^{\infty} \frac{(\gamma)_{j+n+1}\lambda^{j+1}}{\Gamma(\alpha(j+1))(j+n+1)!} (x-a)^{\alpha j-1} = \lambda e^{\lambda(x-a)}.
\] (34)
If we suppose that (32) is verified, by definition (33), it results
\[
[D_a^{(N+1)\alpha} e^{\lambda(t-a)}] (x) = D_a^\alpha [D_a^{N\alpha} e^{\lambda(t-a)}] (x) = D_a^\alpha \left[ \lambda^N e^{\lambda(x-a)} \right] (x) = \lambda^N [D_a^\alpha e^{\lambda(x-a)}] (x).
\] (35)
Then, using (34) in (35) the following is obtained
\[
[D_a^{(N+1)\alpha} e^{\lambda(t-a)}] (x) = \lambda^N \left[ \lambda e^{\lambda(x-a)} \right] = \lambda^{N+1} e^{\lambda(x-a)}.
\] (36)
Finally, we can conclude that (32) is valid for all \( N \in \mathbb{N} \).

2.1.1 Generalized Fractional Trigonometric Functions

Now we will introduce the \( \gamma-\alpha-n \)-Trigonometric functions.

Definition 2.11 Let \( \lambda, \gamma \in \mathbb{C}, \Re(\gamma) > 0 \) and \( a \in \mathbb{R}, \alpha \in \mathbb{R}^+, n \in \mathbb{N}_0 \). We will call \( \gamma-\alpha-n \)-Cosine and \( \gamma-\alpha-n \)-Sine, to the functions
\[
\cos_n^{\alpha,\gamma}(\lambda(x-a)) = \Re \left\{ e_i^{\lambda(x-a)} \right\},
\] (37)
\[
\sin_n^{\alpha,\gamma}(\lambda(x-a)) = \Im \left\{ e_i^{\lambda(x-a)} \right\} \ (x > a),
\] (38)
respectively.
Remark 3 Taking into account the above definition we obtain the following decomposition of the $\gamma$-$\alpha$-$n$-Exponential function:

$$e^{i\lambda(x-a)}_{\alpha,\gamma,n} = \sum_{j=0}^{\infty} \frac{(\gamma)_{j+n}(i\lambda)^j(x-a)^{\alpha(j+1)-1}}{\Gamma(\alpha(j+1))(j+n)!} + i \sum_{j=0}^{\infty} \frac{(-1)^j(\gamma)_{2j+n+1}\lambda^{2j+1}(x-a)^{\alpha(2j+2)-1}}{\Gamma(\alpha(2j+2))(2j+n+1)!}$$

$$= \cos_n^{\alpha,\gamma}(\lambda(x-a)) + i \sin_n^{\alpha,\gamma}(\lambda(x-a)). \tag{39}$$

By its definition, and the properties of the function $\gamma$-$\alpha$-exponential, it is evident that the functions defined in (37) and (38) will have some similar behavior with respect to the classic trigonometric functions. In addition, since the relationship $e^{i\lambda(x-a)}_{\alpha,1,n} = e^{i\lambda(x-a)}_{\alpha,n}$ is verified, we obtain:

$$\cos_n^{\alpha,1}(\lambda(x-a)) = \cos_\alpha(\lambda(x-a)), \tag{40}$$

$$\sin_n^{\alpha,1}(\lambda(x-a)) = \sin_\alpha(\lambda(x-a)), \tag{41}$$

where $\sin_\alpha$ and $\cos_\alpha$ are given in (6) and (5), respectively.

Theorem 2.12 Let $N = 4q + r$ be, with $q \in \mathbb{N}_0$ and $0 \leq r < 4$. Then

$$\left(D_{a+}^{N\alpha} \cos_n^{\alpha,\gamma}[\lambda(x-a)] \right) (x) = \begin{cases} 
\lambda^N \cos_n^{\alpha,\gamma}(\lambda(x-a)) & \text{if } r=0, \\
-\lambda^N \sin_n^{\alpha,\gamma}(\lambda(x-a)) & \text{if } r=1, \\
-\lambda^N \cos_n^{\alpha,\gamma}(\lambda(x-a)) & \text{if } r=2, \\
\lambda^N \sin_n^{\alpha,\gamma}(\lambda(x-a)) & \text{if } r=3,
\end{cases} \tag{42}$$

and

$$\left(D_{a+}^{N\alpha} \sin_n^{\alpha,\gamma}[\lambda(x-a)] \right) (x) = \begin{cases} 
\lambda^N \sin_n^{\alpha,\gamma}(\lambda(x-a)) & \text{if } r=0, \\
\lambda^N \cos_n^{\alpha,\gamma}(\lambda(x-a)) & \text{if } r=1, \\
-\lambda^N \sin_n^{\alpha,\gamma}(\lambda(x-a)) & \text{if } r=2, \\
-\lambda^N \cos_n^{\alpha,\gamma}(\lambda(x-a)) & \text{if } r=3.
\end{cases} \tag{43}$$

Proof: If we apply the Teorema 2.10, it is obtain

$$\left(D_{a+}^{N\alpha} [\cos_n^{\alpha,\gamma}(\lambda(t-a)) + i \sin_n^{\alpha,\gamma}(\lambda(t-a)))] \right) (x) = \left(D_{a+}^{N\alpha} \cos_n^{\alpha,\gamma}(\lambda(t-a)) \right) (x) + i \left(D_{a+}^{N\alpha} \sin_n^{\alpha,\gamma}(\lambda(t-a)) \right) (x). \tag{44}$$
On a generalized Mittag-Leffler function

\[ = \left( D_{a+}^{N\alpha} e^{t(t-a)} \right) (x) = (i\lambda)^N e^{i\lambda(x-a)} \]

\[ = i^N \left[ \lambda^{N-1}\cos_{n+2}^{\alpha,\gamma}(\lambda(x-a)) + \lambda^{N-1} i \sin_{n+2}^{\alpha,\gamma}(\lambda(x-a)) \right] \]

Finally, taking into account that

\[ i^r = \begin{cases} 
1 & \text{if } r=0 \\
i & \text{if } r=1 \\
1 & \text{if } r=2 \\
-i & \text{if } r=3 
\end{cases} \]

the thesis results.

Remark 4 If in the Theorem 2.12 is \( q = 0 \), that is \( 0 \leq N \leq 3 \), \( N = r \), the expressions in (42) and (43) take the following form:

\[ (D_{a+}^{N\alpha} \cos_{n}^{\alpha,\gamma}[\lambda(x-a)]) (x) = \begin{cases} 
\cos_{n}^{\alpha,\gamma}(\lambda(x-a)) & \text{if } N=0, \\
-\lambda \sin_{n+1}^{\alpha,\gamma}(\lambda(x-a)) & \text{if } N=1, \\
-\lambda^2 \cos_{n+2}^{\alpha,\gamma}(\lambda(x-a)) & \text{if } N=2, \\
\lambda^3 \sin_{n+3}^{\alpha,\gamma}(\lambda(x-a)) & \text{if } N=3, 
\end{cases} \]

and

\[ (D_{a+}^{N\alpha} \sin_{n}^{\alpha,\gamma}[\lambda(x-a)]) (x) = \begin{cases} 
\sin_{n+1}^{\alpha,\gamma}(\lambda(x-a)) & \text{if } N=0, \\
\lambda \cos_{n+1}^{\alpha,\gamma}(\lambda(x-a)) & \text{if } N=1, \\
-\lambda^2 \sin_{n+2}^{\alpha,\gamma}(\lambda(x-a)) & \text{if } N=2, \\
-\lambda^3 \cos_{n+3}^{\alpha,\gamma}(\lambda(x-a)) & \text{if } N=3. 
\end{cases} \]

In addition, taking \( \lambda = 1 = N \) in Theorem 2.12, we obtain that

\[ (D_{a+}^{\alpha} \cos_{n+1}^{\alpha,\gamma}(t-a)) (x) = -\sin_{n+1}^{\alpha,\gamma}(x-a), \]

\[ (D_{a+}^{\alpha} \sin_{n+1}^{\alpha,\gamma}(t-a)) (x) = \cos_{n+1}^{\alpha,\gamma}(x-a). \]

If we take \( \alpha = \gamma = 1 \) in (49) and (50), by (40) and (41 ), we obtain the known relationships of the classical trigonometric functions and the first order derivative:

\[ \frac{d}{dx} \cos(x-a) = (D_{a+}^{1} \cos_{n+1}^{1}(t-a)) (x) = -\sin_{n+1}^{1}(x-a) = -\sin(x-a), \]

\[ \frac{d}{dx} \sin(x-a) = (D_{a+}^{1} \sin_{n+1}^{1}(t-a)) (x) = \cos_{n+1}^{1}(x-a) = \cos(x-a). \]
Corollary 2.13 Let \( x \in A \subset [a, \infty) \) be, with \( A \) a compact set, then

\[
\lim_{n \to \infty} \Gamma(\gamma) \cos^{\alpha}_{n}(\lambda(x - a)) = \cos_{\alpha}(\lambda(x - a)); \\
\lim_{n \to \infty} \Gamma(\gamma) \sin^{\alpha}_{n}(\lambda(x - a)) = \sin_{\alpha}(\lambda(x - a));
\]

where \( \sin_{\alpha} \) and \( \cos_{\alpha} \) are defined in (5) and (6).

Proof: The thesis results from applying the Definition 2.11 and the Theorem 2.9.

3 Conclusion

The new generalization of the known Mittag-Leffler function that was presented allowed to define an immediate extension of the exponential function that has its own characteristics as those presented in the Theorems 2.9 and 2.10. It is expected that this function will be useful in the study of the solution of sequential differential equations.

References


Received: April 5, 2019; Published: May 23, 2019