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On the Hyers-Ulam-Rassias Stability of an Additive-Cubic Functional Equation

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Abstract

In this paper, we investigate Hyers-Ulam-Rassias stability of the functional equation

$$f(x+ky) - f(ky-x) - k^2 f(x+y) + k^2 f(y-x) + 2(k^2-1)f(x) = 0.$$

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1 Introduction

Throughout this paper, let V, W be real vector spaces and k a fixed nonzero real number such that $|k| \neq 1$. For a given mapping $f : V \to W$, we use the following abbreviations:

$$\begin{aligned}
f_o(x) :=& \frac{f(x) - f(-x)}{2},\\
Af(x,y) :=& f(x+y) - f(x) - f(y),\\
Cf(x,y) :=& f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y),\\
D_k f(x,y) :=& f(x+ky) - f(ky-x) - k^2 f(x+y) + k^2 f(y-x) + 2(k^2-1)f(x)
\end{aligned}$$

for all $x, y \in V$. Every solution of functional equation $Af(x,y) = 0$ and $Cf(x,y) = 0$ are called an additive mapping and a cubic mapping, respectively. If a mapping can be expressed by sum of a cubic mapping and an additive mapping, then we call the mapping an additive-cubic mapping. A functional equation is called an additive-cubic functional equation provided that each solution of that equation is an additive-cubic mapping and every additive-cubic mapping is a solution of that equation.

T. Z. Xu, J. M. Rassias, and W. X. Xu [14, 16, 17, 15] investigated the stability of the additive-cubic functional equation

$$f(kx+y)+f(kx-y)-kf(x+y)-kf(x-y)-2f(kx)+2kf(x)=0$$

on the various spaces when k is a fixed natural number. M. E. Gordji [5, 8, 13] investigated the stability of the above functional equation on the various spaces when $k = 2$. Many mathematicians investigated the stability of various types of the additive-cubic functional equations [3, 5, 7, 8, 9, 11, 13]. M. Arunkumar etc. [1] investigated the stability of the additive-cubic functional equation $D_2f(x,y) = 0$.

In 1940, Ulam [12] questioned about the stability of group homomorphisms. In 1941, Hyers [6] solved this question for Cauchy additive functional equations. In 1978, Rassias [10] generalized Hyers' result and Găvruta [4] made Rassias' result more generalized. The concept of stability shown by Rassias is called 'Hyers-Ulam-Rassias stability'.

In this paper, I will show that the functional equation $D_rf(x,y) = 0$ is an additive-cubic functional equation for r is a rational number, and also investigate Hyers-Ulam-Rassias stability of that functional equation $D_kf(x,y) = 0$ for k is a real number.

2 Main theorems

The following theorem is a particular case of Baker's theorem [2].

Theorem 2.1 (Theorem 1 in [2]) *Suppose that V and W are vector spaces over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ and $\alpha_0, \beta_0, \ldots, \alpha_m, \beta_m$ are scalar such that $\alpha_j\beta_l - \alpha_l\beta_j \neq 0$ whenever $0 \leq j < l \leq m$. If $f_l : V \to W$ for $0 \leq l \leq m$ and*

$$\sum_{l=0}^{m} f_l(\alpha_l x + \beta_l y) = 0$$

for all $x, y \in V$, then each f_l is a "generalized" polynomial mapping of "degree" at most $m - 1$.

Baker [2] also states that if f is a "generalized" polynomial mapping of "degree" at most $m-1$, then f is expressed as $f(x) = x_0 + \sum_{l=1}^{m-1} a_l^*(x)$ for $x \in V$, where a_l^* is a monomial mapping of degree l and f has a property $f(rx) = x_0 + \sum_{l=1}^{m-1} r^l a_l^*(x)$ for $x \in V$ and $r \in \mathbb{Q}$. The monomial mapping of degree 1, 2, 3 and 4 are also called an additive mapping, a quadric mapping, a cubic mapping, and a quartic mapping, respectively. Therefore, if f, g, h and k are generalized polynomial mappings of degree at most 4 satisfying $f(rx) = rf(x)$, $g(rx) = r^2 g(x)$, $h(rx) = r^3 h(x)$ and $k(rx) = r^4 h(x)$ for all $x \in V$ when r is a rational number with $r \neq 0, \pm 1$, then f, g, h and k are an additive mapping, a quadric mapping, a cubic mapping, and a quartic mapping, respectively.

In summary, the following corollary can be obtained from Baker's theorem.

Corollary 2.2 *Let V and W are vector spaces over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, and $r \in \mathbb{Q} - \{0, \pm 1\}$. Suppose that $n_1, \ldots, n_m$ are natural numbers, and $c_{l_i}, d_{l_i}, \alpha_0, \beta_0, \ldots, \alpha_m, \beta_m$ are scalar such that $\alpha_j \beta_l - \alpha_l \beta_j \neq 0$ whenever $0 \leq j < l \leq m$. If a mapping $f : V \to W$ satisfies the equality $f(rx) = r^k f(x)$ for all $x \in V$ and the inequality*

$$f(\alpha_0 x + \beta_0 y) + \sum_{l=1}^{m} \sum_{i=1}^{n_l} c_{l_i} f(d_{l_i}(\alpha_l x + \beta_l y)) = 0$$

for all $x, y \in V$, then f is a monomial mapping of degree k.

Proof. Put $f_0(\alpha_0 x + \beta_0 y) = f(\alpha_0 x + \beta_0 y)$, $f_l(\alpha_l x + \beta_l y) = \sum_{i=1}^{n_l} c_{l_i} f(d_{l_i}(\alpha_l x + \beta_l y))$ whenever $1 \leq l \leq m$. Then $f_0, \ldots, f_m$ satisfy the conditions in Theorem 2.1. So f is a generalized polynomial mapping of degree at most $m-1$. Since f satisfies the equality $f(rx) = r^k f(x)$ for all $x \in V$, f is a monomial mapping of degree k. □

Theorem 2.3 *Let k be a real number such that $k \neq 0, \pm 1$. If a mapping f satisfies the functional equation $D_k f(x, y) = 0$ for all $x, y \in V$, then f is an additive-cubic mapping.*

Proof. Assume that a mapping $f : V \to W$ satisfies the functional equation $D_k f(x, y) = 0$ for all $x, y \in V$ and g, h are the mappings defined by $g(x) = \frac{-f(2x)+8f(x)}{6}$ and $h(x) = \frac{f(2x)-2f(x)}{6}$. Then $D_k g(x, y) = 0$ and $D_k h(x, y) = 0$ for all $x, y \in V$ and $f = g + h$. Since f is a generalized polynomial mapping of degree at most 3 by Theorem 2.1, we obtain that g and h are generalized polynomial mappings of degree at most 3. Let $\Lambda f : V \to W$ be the mapping defined by

$$\begin{aligned}\Lambda f(x) :=& \frac{1}{k^4-k^2}\big((4k^2-3)D_kf_o(x,x) - 2k^2D_kf_o(2x,x) + 2k^2D_kf_o(x,2x)\\ &- 2D_kf_o((k+1)x,x) + 2D_kf_o((k-1)x,x) - k^2D_kf_o(2x,2x)\\ &+ D_kf_o(x,3x) - D_kf_o((2k+1)x,x) + D_kf_o((2k-1)x,x)\big)\\ &+ \frac{1}{2(k^2-1)}\big(D_kf(4x,0) - 10D_kf(2x,0) + 16D_kf(x,0)\big)\end{aligned}$$

for all $x \in V$. From the equality

$$f(4x) - 10f(2x) + 16f(x) = \Lambda f(x) \tag{2.1}$$

for all $x \in V$, we know that g satisfies $g(2x) = 2g(x)$ for all $x \in V$ and h satisfies $h(2x) = 2^3h(x)$ for all $x \in V$. As mentioned in the previous sentence above this theorem, g is an additive mapping and h is a cubic mapping, i.e., f is an additive-cubic mapping. □

Now I will show that the functional equation $D_rf(x,y) = 0$ is an additive-cubic functional equation when r is a rational number such that $r \neq 0, \pm 1$.

Theorem 2.4 *Let r be a rational number such that $r \neq 0, \pm 1$. A mapping f satisfies the functional equation $D_rf(x,y) = 0$ for all $x, y \in V$ if and only if f is an additive-cubic mapping.*

Proof. If a mapping $f : V \to W$ satisfies the functional equation $D_rf(x,y) = 0$ for all $x, y \in V$, then f is an additive-cubic mapping by Theorem 2.3.

Conversely, assume that f is an additive-cubic mapping, i.e. there exist an additive mapping g and a cubic mapping h such that $f = g+h$. Notice that the equalities $g(rx) = rg(x)$, $g(x) = -g(-x)$, $h(rx) = r^3h(x)$, and $h(x) = -h(-x)$ for all $x \in V$ and $r \in \mathbb{Q}$. First we know that $D_rg(x,y) = 0$ follows from the equality

$$D_rg(x,y) = Ag(rx,y) - r^2Ag(x,y)$$

for all $x, y \in V$. Let us first prove $D_nh(x,y) = 0$ if n is a natural number. Using mathematical induction, the equalities $D_nh(x,y) = 0$ follows from the equalities

$$\begin{aligned}&D_2h(x,y) = Ch(x,y) - Ch(x-y,y),\\ &D_3h(x,y) = D_2h(x+y,y) + D_2h(x-y,y) + 4D_2h(x,y),\\ &D_nh(x,y) = D_{n-1}h(x+y,y) + D_{n-1}h(x-y,y) - D_{n-2}h(x,y) + (n-1)^2D_2h(x,y)\end{aligned}$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. Let us now prove $D_rh(x,y) = 0$ if r is a rational number such that $r \neq 0, \pm 1$. Notice that if $r \in \mathbb{Q}$, then there exist

$m, n \in \mathbb{N}$ such that $r = \frac{n}{m}$ or $r = \frac{-n}{m}$. Since the equalities $D_{\frac{n}{m}}h(x,y) = 0$ and $D_{\frac{-n}{m}}h(x,y) = 0$ follow from the equalities

$$\begin{aligned} D_{\frac{n}{m}}h(x,y) =& D_n h\left(x, \frac{y}{m}\right) - \frac{n^2}{m^2} D_m h\left(x, \frac{y}{m}\right), \\ D_{\frac{-n}{m}}h(x,y) =& D_{\frac{n}{m}}h(x,y) \end{aligned}$$

for all $x, y \in V$ and $n, m \in \mathbb{N}$, we get $D_r h(x,y) = 0$ for all $x, y \in V$. $\square$

Let X be a real normed space and Y a real Banach space. For a given mapping $f : X \to Y$, let $J_n f : X \to Y$ be the mappings defined by

$$J_n f(x) = \begin{cases} \frac{4\cdot 8^n}{3}\big(f(2^{-n}x) - 2f(2^{-n-1}x)\big) - \frac{2^n}{3}\big(f(2^{-n}x) - 8f(2^{-n-1}x)\big) & \text{if } \ p > 3, \\ -\frac{2^{n-1}}{3}\big(f(2^{-n+1}x) - 8f(2^{-n}x)\big) + \frac{f(2^{n+1}x) - 2f(2^n x)}{6\cdot 8^n} & \text{if } \ 1 < p < 3, \\ \frac{8f(2^n x) - f(2^{n+1}x)}{6\cdot 2^n} + \frac{f(2^{n+1}x) - 2f(2^n x)}{6\cdot 8^n} & \text{if } \ p < 1 \end{cases}$$

for all $x \in X$ and all nonnegative integers n. Then, by the definition of $J_n f$ and (2.1), the equality

$$J_n f(x) - J_{n+1} f(x) = \begin{cases} \frac{4\cdot 8^n}{3}\Lambda f\big(2^{-n-2}x\big) - \frac{2^n}{3}\Lambda f\big(2^{-n-2}x\big) & \text{if } p > 3, \\ \frac{1}{48\cdot 8^n}\Lambda f\big(2^n x\big) - \frac{2^{n-1}}{3}\Lambda f\big(2^{-n-1}x\big) & \text{if } 1 < p < 3, \\ \frac{1}{12\cdot 2^n}\Lambda f\big(2^n x\big) - \frac{1}{48\cdot 8^n}\Lambda f\big(2^n x\big) & \text{if } p < 1 \end{cases} \tag{2.2}$$

holds for all $x \in X$ and all nonnegative integers n. Therefore, together with the equality $f(x) - J_n f(x) = \sum_{i=0}^{n-1}(J_i f(x) - J_{i+1} f(x))$ for all $x \in X$, we obtain the following lemma.

Lemma 2.5 *If $f : X \to Y$ is a mapping such that*

$$D_k f(x,y) = 0$$

for all $x, y \in X$, then

$$J_n f(x) = f(x)$$

for all $x \in X$ and all positive integers n.

From Lemma 2.3-Lemma 2.5, we can prove the following stability theorem.

Theorem 2.6 *Let $p \neq 1, 3$ be a nonnegative real number. Suppose that $f : X \to Y$ is a mapping such that*

$$\|D_k f(x,y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{2.3}$$

for all $x, y \in X$ (and $f(0) = 0$ for $p = 0$). Then there exists a unique solution mapping F of the functional equation $D_kF(x,y) = 0$ such that

$$\|f(x) - F(x)\| \le \begin{cases} \frac{K\|x\|^p}{3\cdot 2^p}\left(\frac{4}{2^p-8} - \frac{1}{2^p-2}\right) & \text{if} \quad p > 3, \\ \frac{K\|x\|^p}{6}\left(\frac{1}{8-2^p} + \frac{1}{2^p-2}\right) & \text{if} \quad 1 < p < 3, \\ \frac{K\|x\|^p}{6}\left(\frac{1}{2-2^p} - \frac{1}{8-2^p}\right) & \text{if} \quad p < 1 \end{cases} \tag{2.4}$$

for all $x \in X$, where

$$K = \frac{20k^2 + 13 + 11k^2 2^p + 3^p + \frac{k^2}{2}4^p + 2|k-1|^p + 2|k+1|^p + |2k-1|^p + |2k+1|^p}{|k^4 - k^2|}.$$

Proof. It follows from the definition of $\Lambda f(x)$ and (2.3) that

$$\begin{aligned} \|\Lambda f(x)\| = & \Big\| \frac{1}{k^4 - k^2}\big((4k^2-3)D_kf_o(x,x) - 2k^2D_kf_o(2x,x) + 2k^2D_kf_o(x,2x) \\ & - 2D_kf_o((k+1)x,x) + 2D_kf_o((k-1)x,x) - k^2D_kf_o(2x,2x) \\ & + D_kf_o(x,3x) - D_kf_o((2k+1)x,x) + D_kf_o((2k-1)x,x)\big) \\ & + \frac{1}{2(k^2-1)}\big(D_kf(4x,0) - 10D_kf(2x,0) + 16D_kf(x,0)\big)\Big\| \\ \le & K\|x\|^p \end{aligned}$$

for all $x \in X$. It follows from (2.2) and (2.3) that

$$\|J_nf(x) - J_{n+1}f(x)\| \le \begin{cases} \frac{2^n(4^{n+1}-1)}{3\cdot 2^{(n+2)p}}K\|x\|^p & \text{if } p > 3, \\ \left(\frac{2^{np}}{6\cdot 8^{n+1}} + \frac{2^n\theta\|x\|^p}{6\cdot 2^{(n+1)p}}\right)K\|x\|^p & \text{if } 1 < p < 3, \\ \frac{(4^{n+1}-1)2^{np}}{6\cdot 8^{n+1}}K\|x\|^p & \text{if } p < 1 \end{cases}$$

for all $x \in X$. Together with the equality $J_nf(x) - J_{n+m}f(x) = \sum_{i=n}^{n+m-1}(J_if(x) - J_{i+1}f(x))$ for all $x \in X$, we get

$$\|J_nf(x) - J_{n+m}f(x)\| \le \begin{cases} \sum_{i=n}^{n+m-1} \frac{2^i(4^{i+1}-1)}{3\cdot 2^{(i+2)p}}K\|x\|^p & \text{if } p > 3, \\ \sum_{i=n}^{n+m-1} \left(\frac{2^{ip}}{6\cdot 8^{i+1}} + \frac{2^i\theta\|x\|^p}{6\cdot 2^{(i+1)p}}\right)K\|x\|^p & \text{if } 1 < p < 3, \\ \sum_{i=n}^{n+m-1} \frac{(4^{i+1}-1)2^{ip}}{6\cdot 8^{i+1}}K\|x\|^p & \text{if } p < 1 \end{cases} \tag{2.5}$$

for all $x \in X$. It follows from (2.5) that the sequence $\{J_nf(x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{J_nf(x)\}$ converges for all $x \in X$. Hence we can define a mapping $F : X \to Y$ by

$$F(x) := \lim_{n\to\infty} J_nf(x)$$

for all $x \in X$. Moreover, letting $n = 0$ and passing the limit $n \to \infty$ in (2.5) we get the inequality (2.4). For the case $1 < p < 3$, from the definition of F, we easily get

$$\begin{aligned}\|D_kF(x,y)\| = \lim_{n\to\infty} \Bigg\| &\frac{2^n}{6}\left(-D_kf\left(\frac{2x}{2^n},\frac{2y}{2^n}\right) + 8D_kf\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right) \\ &+ \frac{D_kf\left(2^{n+1}x, 2^{n+1}y\right) - 2D_kf\left(2^nx, 2^ny\right)}{6\cdot 8^n}\Bigg\| \\ \le \lim_{n\to\infty} &\left(\frac{2^n(2^p+8)}{6\cdot 2^{np}} + \frac{2^{np}(2^p+2)}{6\cdot 8^n}\right)\theta(\|x\|^p + \|y\|^p) \\ =0 &\end{aligned}$$

for all $x, y \in X$. Also we easily show that $D_kF(x,y) = 0$ by the similar method for the other cases, either $0 < p < 1$ or $3 < p$. To prove the uniqueness of F, let $F' : X \to Y$ be another solution mapping satisfying (2.4). Instead of the condition (2.4), it is sufficient to show that there is a unique mapping that satisfies condition $\|f(x) - F(x)\| \le \frac{K\|x\|^p}{6}\left(\frac{1}{|8-2^p|} + \frac{1}{|2-2^p|}\right)$ simply. By Lemma 2.5, the equality $F'(x) = J_nF'(x)$ holds for all $n \in \mathbb{N}$. For the case $p > 3$, we have

$$\begin{aligned}\|J_nf(x) - F'(x)\| =&\|J_nf(x) - J_nF'(x)\| \\ \le&\frac{2^{3n+2}-2^n}{3}\|(f-F')(2^{-n}x)\| + \frac{2^{3n+3}-2^{n+3}}{3}\|(f-F')(2^{-n-1}x)\| \\ \le&\left(\frac{2^{3n+2}-2^n}{3\cdot 2^{np}} + \frac{2^{3n+3}-2^{n+3}}{3\cdot 2^{(n+1)p}}\right)\frac{K\|x\|^p}{6}\left(\frac{1}{|8-2^p|} + \frac{1}{|2-2^p|}\right) \\ \le&\frac{2^{3n+4}}{3\cdot 2^{np}}\frac{K\|x\|^p}{6}\left(\frac{1}{|16-2^p|} + \frac{1}{|4-2^p|}\right)\end{aligned}$$

for all $x \in X$ and all positive integer n. Taking the limit in the above inequality as $n \to \infty$, we can conclude that $F'(x) = \lim_{n\to\infty} J_nf(x)$ for all $x \in X$. For the other cases, either $0 < p < 1$ or $1 < p < 3$, we also easily show that $F'(x) = \lim_{n\to\infty} J_nf(x)$ by the similar method. This means that $F(x) = F'(x)$ for all $x \in X$.

□

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