Bound on Poisson Approximation on the Length of Success Runs at least $k$ by Stein-Chen Method

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Abstract
The probability distribution of the number of success runs of the length $k$ ($k \geq 1$) in $n$ ($n \geq 1$) Bernoulli trials is obtained. It is noted that this distribution is a binomial distribution of the order $k$, and several open problems pertaining to it are stated. Let $W_n$ denotes the number of success runs with the length $k$ or more and we give bound for $W_n$ by Poisson distribution via Stein-Chen coupling method.

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1 Introduction
It is obviously known that, success runs are important in applied probability and statistical inference. Many recent studies on success runs in Bernoulli trials are applied the framework contained in the fundamental book of Feller [4] which the definition of run is particularly defined as a recurrent pattern. According to this definition, two consecutive success runs may not be separated by any failure. As shown in the example, the sequence SSSSSSSSS (where the symbol S denotes a success) can be interpreted as.. including 4 success runs of the length of 2 or 2 success runs of length 4. Practically, if we search for runs of $k$ length, the counting of consecutive successes must be restarted when the desired value $k$ is reached (see Feller, [4]).
Consider the sequence of \( n \) two-state (success/failure) trials which is arranging on a line or on a circle, the success run is defined as a sequence of consecutive successes (S) preceded and succeeded by failures (F) or by nothing. The number of successes in a (success) run is referred to its length. The study of the number of success runs consisting of the number of runs of a specified length using several counting schemes, the waiting time for the occurrence of a pre-specified number of runs, and the shortest and the longest success run length have attracted the interest of many authors conducted many studies to consider the numbers of success runs of length which is exactly equal to and greater than or equal to a threshold length; see Mood [7], Fu and Koutras [5], Muselli [8] and Eryilmaz and Demir [3], while Fu and Lou [6] studied the statistic denoting of the sum of the lengths of the success runs (i.e. the total number of successes in all the success runs) of length greater than or equal to a pre-specified length, and the waiting time for the first time that the above mentioned statistic equals or exceeds a predetermined level (Antzoulakos et al. [1]). These authors studied the sequences of trials which is arranged on a line by exploiting a Markov-Chain embedding technique; see Fu and Koutras [5], and Fu and Lou [6].

We are, consequently, interested in studying the following scheme: the ball which is randomly drawn from an urn initially containing \( w \) white balls and \( b \) black balls, their colors are observed, and they are, then, returned to the urn along with \( s \) additional balls of the same color as the ball drawn. Drawing a white ball is considered as success (S), while drawing a black ball is referred to failure (F). This sampling scheme is repeated \( n \) times and a binary sequence is derived by reducing to a Bernoulli sequence (independent and identically distributed binary trials) for \( s = 0 \). What is the probability for the number of success runs of length \( k \) or longer in \( n \) independent trials with success probability \( p \) (\( 0 < p < 1 \))?

For each \( i \in \{1, 2, 3, \ldots, n\} \), we define the indicator random variable \( X_i \), as follows:

\[
X_i = \begin{cases} 
1 & \text{if the event that a success run of length } k \text{, or longer begins at position } i \\
0 & \text{otherwise.}
\end{cases}
\]

Let \( W_n \) be the number of success runs of length \( k \) or longer in \( n \) independent trials with success probability \( p \) (\( 0 < p < 1 \)) and \( q = 1 - p \). Therefore,

\[
P(X_i = 1) = \sum_{a=k}^{w} \binom{w}{a} p^a q^{w-a}.
\]

And set

\[
W_n = \sum_{i=1}^{n} X_i.
\]
For \( n \) beginning sufficiently large, it is logical to approximate the distribution of \( W_n \) by Poisson distribution with mean

\[
\lambda = E(W_n) = \sum_{i=1}^{n} \sum_{a=k}^{w} \binom{w}{a} p^a q^{w-a}.
\]

Our article is organized as follows. In section 2, we introduced Stein-Chen coupling method and in section 3, we give a bound for Poisson approximation of the number of success runs of length \( k(\geq 1) \) or longer in \( n(\geq 1) \) independent trials with success probability \( p(0 < p < 1) \) by using Stein-Chen coupling method. The following theorem is our main result.

**Theorem 1.1.** Let \( W_n \) be the number of success runs of length \( k(\geq 1) \) or longer in \( n(\geq 1) \) independent trials with success probability \( p(0 < p < 1) \). Then we have

\[
|P(W_n \in A) - \text{Poi}_{\lambda}(A)| \leq C_{\lambda,w,k} \frac{(n-1)(q-p)^{w-k}(p)^{2k}}{(p-q)q^{2k-1}} \{\left(\frac{p}{q}\right)^{w-k+1} - 1\}
\]

where \( C_{\lambda,w,k} = \frac{w!(k+1)}{k!(k!)^2} \min\left\{1, \lambda, \frac{\Delta(\lambda)}{M_A+1}\right\}, \)

\[
\Delta(\lambda) = \left\{ \begin{array}{ll}
\lambda + 1 - \lambda & \text{if } \lambda^{-1}(e^\lambda - 1) \leq M_A, \\
2(e^\lambda - 1) & \text{if } \lambda^{-1}(e^\lambda - 1) > M_A,
\end{array} \right.
\]

and

\[
M_A = \left\{ \begin{array}{ll}
\max\{w \mid C_w \subseteq A\} & \text{if } 0 \in A, \\
\min\{w \mid w \in A\} & \text{if } 0 \notin A
\end{array} \right.
\]

when \( C_w = \{0, 1, \ldots, w-1\} \).

2 **Poisson Approximation via Stein-Chen Method**

The Stein-Chen Method of Poisson Approximation provides a powerful technique for computing an error bound when approximating probabilities by a Poisson distribution.

Stein [9] introduced a new powerful technique for the obtaining the rate of convergence to standard normal distribution. Chen [11] modified Stein’s method so as to obtain approximation results for the Poisson distribution, therefore the method is often referred to as Stein-Chen method. Our starting point is the Stein equation for Poisson distribution, which gives,

\[
I_A(j) - \text{Poi}_{\lambda}(A) = \lambda g_{\lambda,A}(j+1) - j g_{\lambda,A}(j)
\]

(2.1)
$\lambda > 0$, $j \in \mathbb{N} \cup \{0\}$, $A \subseteq \mathbb{N} \cup \{0\}$ and $I_A : \mathbb{N} \cup \{0\} \to \mathbb{R}$ be defined by

$$I_A(w) = \begin{cases} 1 & ; w \in A, \\ 0 & ; w \notin A. \end{cases}$$

The solution $g_{\lambda,A}$ of (2.1) is the form

$$g_{\lambda,A}(w) = \begin{cases} (w - 1)! \lambda^{-w} e^{\lambda} [P_{\lambda}(I_{A \cap C_{w-1}}) - P_{\lambda}(I_A)] & ; w \geq 1, \\ 0 & ; w = 0 \end{cases}$$

where

$$P_{\lambda}(I_A) = e^{-\lambda} \sum_{l=0}^{\infty} I_A(l) \frac{\lambda^l}{l!}$$

and

$$C_{w-1} = \{0, 1, \ldots, w - 1\}.$$

By substituting $j$ and $\lambda$ in (2.1) by any integer-valued random variable $W$ and $\lambda = E W$, we have

$$P(W_n \in A) - \text{Poi}_{\lambda,A} = E(\lambda g_{\lambda,A}(W_n + 1)) - E(W_n g_{\lambda,A}(W_n)).$$

(2.2)

In the case where the dependence between the instances of $X_i$ is global, there is an alternative approach to approximating the distribution of $W_n$. This approach is referred to as The Coupling approach, which was first proposed by Barbour ([2] 1982). This approach is particularly useful when it is possible to construct a random variable $W_{n,i}$ for each $i$ on a common probability space with $W_n$ such that $W_{n,i}$ is distributed as $W_n - X_i$ conditional on the event $X_i = 1$.

There have been a number of successful applications of this method, Barbour ([2] 1982), Janson ([15] 1994), Lange ([12] 2003).

**Theorem 2.1.** If $W = \sum_{i=1}^{n} X_i$, $p_i = E(X_i) = P(X_i = 1)$, $\lambda = E(W_n)$ and for each $i$, $W_i$ be the random variable on the same probability space as $W$ such that the distribution $\mathcal{L}(W - X_i|X_i = 1)$. Then

$$|P(W_n \in A) - \text{Poi}_{\lambda,A}| \leq \|g_{\lambda,A}\| \sum_{i=1}^{n} p_i E|W_n - W_{n,i}|$$

(2.3)

where $\|g_{\lambda,A}\| := \sup_w [g_{\lambda,A}(w + 1) - g_{\lambda,A}(w)]$.

Many authors would like to determine a bound of $\|g_{\lambda,A}\|$. For $A \subseteq \mathbb{N} \cup \{0\}$, Chen ([11], 1975) prove that

$$\|g_{\lambda,A}\| \leq \min\{1, \lambda^{-1}\}$$
and Janson ([15], 1994) showed that
\[ \| g_{\lambda,A} \| \leq \lambda^{-1}(1 - e^{-\lambda}). \] (2.4)

In case of non-uniform bound, Neammanee ([10], 2003) showed that
\[ \| g_{\lambda,A} \| \leq \min\left\{ 1, \frac{1}{w_0}, \lambda^{-1}\right\} \]
and Teerapabolarn and Neammanee ([13], 2005) gave bound of
\[ \| g_{\lambda,A} \| \]
where
\[ A = \{0, 1, \ldots, w_0\} \] in the terms of
\[ \| g_{\lambda,A} \| \leq \lambda^{-1}(1 - e^{-\lambda}) \min\left\{ 1, \frac{e^{\lambda}}{w_0 + 1}\right\}. \]

In general case for any subset \( A \) of \( \{0, 1, \ldots, n\} \), Santiwipanont and Teerapabolarn ([14], 2006) gave a bound in the form of
\[ \| g_{\lambda,A} \| \leq \lambda^{-1} \min\left\{ 1, \lambda, \frac{\triangle(\lambda)}{M_A + 1}\right\} \] (2.5)
where
\[ \triangle(\lambda) = \begin{cases} e^\lambda + \lambda - 1 & \text{if } \lambda^{-1}(e^\lambda - 1) \leq M_A, \\ 2(e^\lambda - 1) & \text{if } \lambda^{-1}(e^\lambda - 1) > M_A, \end{cases} \]
and
\[ M_A = \begin{cases} \max\{w \mid C_w \subseteq A\} & \text{if } 0 \in A, \\ \min\{w \mid w \in A\} & \text{if } 0 \notin A. \end{cases} \]

The difficult part in applying Theorem 2.1 is to find \( W_{n,i} \) which make
\( E|W_n - W_{n,i}| \) small enough. This has not the solution in general. For the case of \( X_1, \ldots, X_n \) are independent, we let \( W_{n,i} = W_n - X_i \). Then \( E|W_n - W_{n,i}| = p_i \) and from (2.3), we have
\[ |P(W_n \in A) - \text{Poi}_\lambda(A)| \leq \| g_{\lambda,A} \| \sum_{i=1}^{n} p_i^2. \]

The problem of the construction of \( W_{n,i} \) is difficult in the case of dependent indicator summand.

In next section, we will use Theorem 2.1 to prove our main result by constructing the random variable \( W_{n,i} \) which make \( E|W_n - W_{n,i}| \) small.

### 3 Proof of Main Result

**Proof. Of Theorem 1.1.** For each \( j \in \{1, 2, 3, \ldots, n\} \), such that \( j \neq i \), the indicator random variable \( X_j^i \) is defined in the proceed condition: if the length
of success run which are started in the first $i^{th}$ position has $k - 1$ success runs, the $k - 1$ success run which has landed in the $i^{th}$ positions is taken, the $i^{th}$ position is removed and this procedure is independently duplicated into positions. Consequently, the random indicator variable $X_j^{i}$ is defined as follow,
\[
X_j^{i} = \begin{cases} 
1 & \text{if the } j^{th} \text{ position is starting at the length of success at least } k \\
0 & \text{otherwise.} 
\end{cases}
\]

Let $W_{n,i} = \sum_{i=1,j\neq i}^{n} X_j^{i}$ be the number of positions which start at the length of success runs at least $k$ after we remove the $i^{th}$ position that is starting at the length of success runs at least $k$.

Suppose that $\{j_s | s = 1, 2, \ldots, w_0\}$ is the set of $r$ positions containing the length of success runs at least $k$, $r < n$. So for each $w_0 \in \{0, 1, 2, \ldots, w - k\}$, we get
\[
P(W_{n,i} = w_0) = \prod_{s=1}^{w_0} \sum_{i=0}^{k} \sum_{a=k}^{w-k_i} \left( w - k_i \choose a \right) (p_{j_s}^{a})(q_{j_s}^{w-k_i-a}).
\]

and
\[
P(W_n - X_i = w_0 | X_i = 1) = \frac{P(W_n - X_i = w_0, X_i = 1)}{P(X_i = 1)}
= \frac{P(W_n = w_0 + 1, X_i = 1)}{P(X_i = 1)}
= \frac{\sum_{a=k_i}^{w} \left( w \choose a \right) p^{a} q^{w-k_i} \prod_{s=1}^{w_0} \sum_{k_i=0}^{k} \sum_{a=k_i}^{w-k_i} \left( w - k_i \choose a \right) (p_{j_s}^{a})(q_{j_s}^{w-k_i-a})}{\sum_{a=k_i}^{w} \left( w \choose a \right) p^{a} q^{w-k_i}}
= \prod_{s=1}^{w_0} \sum_{i=0}^{k} \sum_{a=k}^{w-k_i} \left( w - k_i \choose a \right) (p_{j_s}^{a})(q_{j_s}^{w-k_i-a}).
\]

Therefore, $W_{n,i}$ has the same distribution as $W_n - X_i$ condition on $X_i = 1$. ($W_{n,i}$ which is satisfying Theorem 2.1). In order to bound $E|W_n - W_{n,i}|$, we observe that

- In case of $X_j^{i} = 1$, we have the $i^{th}$ position which starts at the length of success run at least $k$. Thus, the number of success runs that are starting at the length of success runs at least $k$ after removing the $i^{th}$, equals to the number of such success runs starting in the first $n$ positions minus 1, that is
\[ W_{n,i} = W_n - 1. \quad (3.1) \]

- In case of \( X_j^i = 0 \), the number of positions that are starting at the length of success runs at least \( k \) after removing the \( i^{th} \) position which starts \( k \) success runs and we repeated them again as defined, equals to the number of such success runs at least \( k \) starting in the first \( n \) positions minus the sum of number of the \( j^{th} \) positions which start success runs at least \( k \) in the first-repeated and they are less than \( k \) success runs after we repeat them again, that is

\[ W_{n,i} = W_n - \sum_{j=1, i\neq j}^{n} X_j^i Y_j^i. \quad (3.2) \]

For each \( j \in \{0, 1, 2, \ldots, n\}, j \neq i \), we define the indicator random variable \( Y_j^i \) as follow:

\[ Y_j^i = \begin{cases} 
1 & \text{if the } j^{th} \text{ position is starting success run less than } k, \\
0 & \text{otherwise.}
\end{cases} \]

We know that

\[ E | W_n - W_{n,i} | = E(W_n - W_{n,i})^+ + E(W_n - W_{n,i})^-, \]

where

\[ (W_n - W_{n,i})^+ = \max\{W_n - W_{n,i}, 0\}, \]

and

\[ (W_n - W_{n,i})^- = -\min\{W_n - W_{n,i}, 0\}. \]

Form (3.1) and (3.2).

- In case of \( X_j^i = 1 \), we have \( (W_n - W_{n,i})^+ = 1 \) and \( (W_n - W_{n,i})^- = 0 \)

- In case of \( X_j^i = 0 \), we have \( (W_n - W_{n,i})^+ = \sum_{i,j=1, i\neq j}^{n} X_j^i Y_j^i \) and \( (W_n - W_{n,i})^- = 0 \).

Therefore,

\[ (W_n - W_{n,i})^+ = \sum_{i,j=1, i\neq j}^{n} X_j^i Y_j^i \]
and \((W_n - W_{n,i})^- = 0\).

By the fact, the probability of such success runs starting into \(j^{th}\) position after removing the \(i^{th}\) position is \(p_j = \frac{p}{1 - p} \neq p\). We have

\[
E(W_n - W_{n,i})^+ \leq E\{ \sum_{i,j=1, i \neq j}^n X_j Y_j^i \} \]

\[
= \sum_{i,j=1, i \neq j}^n E\{X_j Y_j^i \} \]

\[
= \sum_{i,j=1, i \neq j}^n P(X_j = 1, Y_j = 1) \]

\[
= \sum_{i,j=1, i \neq j}^n P(X_j = 1)P(Y_j = 1) \]

\[
= \sum_{i,j=1, i \neq j}^n \sum_{a=k_i, k_i=0}^{w} \sum_{k=0}^{w-k} \binom{w-k}{k} p^a q^{w-k-a} \binom{w}{k_i} \binom{q}{k_i} (1 - \frac{q}{p})^{w-k-k_i} \]

\[
= \sum_{i,j=1, i \neq j}^n \sum_{a=k_i, k_i=0}^{w} \sum_{k=0}^{w-k} \frac{w!}{(w-k-a)!a!(k_i-a)!(k_i-k)!} \binom{p}{k_i} q^{k_i} (q-p)^{w-k-k_i} \]

\[
\leq (n-1) \sum_{a=k_i}^{w} \frac{w!(q-p)^{w-k-a}}{(a-k)!a!(b-k)!} \binom{p}{k_i} q^{k_i} (q-p)^{w-k} \]

\[
\leq (n-1)(q-p)^{w-k} \frac{k!}{w-k + 1} \frac{p^{2k}}{k!} \frac{1}{(p-q)^{2k-1}} \{ \frac{q}{p} \}^{w-k+1} - 1 \}. \tag{3.3}
\]

Therefore,

\[
E(W_n - W_{n,i}) \leq (n-1)(q-p)^{w-k} \frac{k!}{w-k + 1} \frac{p^{2k}}{k!} \frac{1}{(p-q)^{2k-1}} \{ \frac{q}{p} \}^{w-k+1} - 1 \}. \tag{3.4}
\]

Hence, by (2.3), (2.4) and (3.4), we have

\[
|P(W_n \in A) - Pois(\lambda(A))| \leq C_{\lambda,w,k} \frac{(n-1)(q-p)^{w-k}(p)^{2k}}{(p-q)^{2k-1}} \{ \frac{q}{p} \}^{w-k+1} - 1 \};
\]

where \(\lambda = \sum_{i=1}^n \sum_{a=k}^w \binom{w}{a} p^a q^{w-a}\) and \(C_{\lambda,w,k} = \frac{w!(k+1)}{k!(k+1)!} \frac{1}{(p-q)^{2k-1}} \{ \frac{q}{p} \}^{w-k+1} - 1 \).

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