Abstract

The continuous wavelet transform of a finite dimensional random field with weakly stationary, stationary increments and weakly stationary increment via arbitrary dilation matrix are shown to be joint weakly stationary random fields for different dilation matrices whose cross correlation function and cross power spectral density function are determined. Moreover, the ergodicity are discussed.

Keywords: Random field, ergodic theorem, wavelet transform
1 Introduction

Wavelets are known to have intimate connections to several other parts of mathematics, notably phase-space analysis in signal processing, reproducing kernel Hilbert spaces, etc. A random field, following Childers [1], Grimmett and Stirzaker [2] and Ludeman [3], is a family of random variables \( X_t \) from a probability space \((\Omega, \mathcal{F}, P)\) to a Borel subset \( S \) of \( \mathbb{R} \) or \( \mathbb{C} \), indexed by \( t \in \mathbb{R}^d \). For \( d = 1 \), a random field \( \{X_t\}_{t \in \mathbb{R}} \) is usually called a stochastic process.

The connection of wavelets with stochastic processes was exploited over 20 year ago in [1], [2], [3], [7], [8], [11], [13], [16], [19], [21] and [23]. On the other hand, the connection of wavelets with general random fields began to be established more recently, over the last 10 years in [1], [10], [17] and [18].

A random field \( \{X_t\}_{t \in \mathbb{R}^d} \) is called weakly stationary if it has a constant mean and the autocorrelation

\[
R_X(t, t + \tau) = E[X_t X_{t+\tau}] \quad \forall t, \tau \in \mathbb{R}^d
\]

is shift invariant. Furthermore, it is said to have weakly stationary increments if its increments \( \Delta X(t : \tau) = X_{t+\tau} - X_t \) are weakly stationary. Two random fields \( \{X_t\}_{t \in \mathbb{R}^d} \) and \( \{Y_t\}_{t \in \mathbb{R}^d} \) are jointly weakly stationary if both \( X_t \) and \( Y_t \) are individually weakly stationary and the cross-correlation

\[
R_{XY}(t, t + \tau) = E[X_t Y_{t+\tau}] \quad \forall t, \tau \in \mathbb{R}^d
\]

is shift invariant. One observes that if \( \{X_t\}_{t \in \mathbb{R}^d} \) is a weakly stationary random field then the autocorrelation function \( R_X(t, t + \tau) \) is a function of position translation \( \tau \) only, say \( R_X(\tau) \). For this type of random field, characterization of autocorrelation in the frequency domain is available as introduced by Childer [4], Grimmett and Stirzaker [9] and Ludeman [12]. It is called the power spectral density function denoted by \( S_X(\omega) \) and defined as the generalized Fourier transform of the autocorrelation function by

\[
S_X(\omega) = \int_{\mathbb{R}^d} R_X(\tau) e^{-i\omega \cdot \tau} d\tau.
\]

Applying the inverse Fourier transform, one obtains the power spectral representation of the autocorrelation function (1) as

\[
R_X(\tau) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} S_X(\omega) e^{i\omega \cdot \tau} d\omega.
\]

Similarly, if random fields \( \{X_t\}_{t \in \mathbb{R}^d} \) and \( \{Y_t\}_{t \in \mathbb{R}^d} \) are jointly weakly stationary then the cross-correlation function \( R_{XY}(t, t + \tau) \) is a function depending only on \( \tau \), say \( R_{XY}(\tau) \). The cross power spectral density function is defined as

\[
S_{XY}(\omega) = \int_{\mathbb{R}^d} R_{XY}(\tau) e^{-i\omega \cdot \tau} d\tau.
\]
Applying the inverse Fourier transform, one obtains the cross power spectral representation of the cross-correlation function (2) as

$$R_{XY}(\tau) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} S_{XY}(\omega) e^{i\omega \cdot \tau} d\omega. \quad (6)$$

Masry [16] showed that the wavelet transform of a weakly stationary increments stochastic process is a weakly stationary stochastic process whose autocorrelation function and spectral density function can be determined. Also, Cambanis and Houdré [3] found a new proof that the wavelet transform of a stationary stochastic process as well as a stochastic process with stationary increments is a weakly stationary process, and then Averkamp and Houdré [1] extended this viewpoint to random fields and obtained that the continuous wavelet transform of weakly stationary or weakly stationary increment random field, via positive scaling parameter are jointly weakly stationary random fields with zero mean. Furthermore, Masry [18] determined the power spectrum. As an example of a process which itself is not stationary, but whose increments are, the fractional Brownian field $\{B^H_t\}_{t \in \mathbb{R}^d}$ with Hurst index $0 < H < 1$ is a random field with zero mean, and its autocorrelation (covariance) function is

$$R_{BH}(t, s) = \frac{V_H}{2} \left[ \|t\|^{2H} + \|s\|^{2H} - \|t - s\|^{2H} \right] \quad (7)$$

where $V_H = E[(B^H_1)^2]$, note that $\{B^1_t\}_{t \in \mathbb{R}}$ is a Brownian field and $\{B^H_t\}_{t \in \mathbb{R}}$ is fractional Brownian motion. Patrick Flandrin (1989) proposed how to obtain the power spectral representation

$$R_{CWa}^{BH}(\tau) = \frac{aV_H}{2} \frac{\Gamma(2H+1) \sin \pi H}{2H} \int_{\mathbb{R}^d} e^{-i\lambda \cdot \tau} \frac{\hat{\varphi}(a\lambda)^2}{|\lambda|^{2H+1}} d\lambda \quad (8)$$


Let $\{X_t\}_{t \in \mathbb{R}}$ be a random process. Viniotis (1998) showed that ergodic in the mean or in short mean ergodic saying that the estimate for the mean converges to the true mean in the mean square sense as

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T X_t \, dt = E[X_t]. \quad (9)$$

In addition, Grimmete and Stirzaker (1992) present the ergodic theorem for weakly stationary random process saying that if $\{X_t\}_{t \in \mathbb{R}}$ is a weakly stationary random process then there exists a random variable $Y$ such that $E[Y] = E[X_0]$ and

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T X_t \, dt = Y \quad \text{in mean square sense.} \quad (10)$$
Moreover, if \( \{X_t\}_{t \in \mathbb{R}} \) is a weakly stationary random process with zero mean and finite autocorrelation function \( R_X(\tau) \), then there exists a random variable \( Y \) satisfying (10), \( E[Y] = 0 \) and

\[
E[Y^2] = \lim_{T \to \infty} \frac{1}{2T} \int_{-2T}^{2T} R_X(\tau) \, d\tau.
\]

This paper is organized as follows. In section 2 the wavelet transform of random fields is discussed. Section 3 is devoted to the discussion of class of wavelet transform of random fields and determining of power spectral density function of wavelet transform of weakly stationary random fields, stationary increment random fields and weakly stationary increment random fields. Also present an example the power spectral density function of the wavelet transform of fractional Brownian fields. In section 4, the mean ergodic random fields and ergodic theorem for weakly stationary random fields are discussed and connected to the continuous wavelet transform of some classes random fields.

## 2 Continuous Wavelet Transform of Random Fields

Let \( M_d(\mathbb{R}) \) denote the space of real \( d \times d \) matrices. The general linear group is \( GL_d(\mathbb{R}) = \{ A \in M_d(\mathbb{R}) | A \text{ is invertible} \} \). We call a closed subgroup \( H \) of \( GL_d(\mathbb{R}) \) a matrix group. Let \( H \) be a matrix group. Fix \( \varphi \in L^2(\mathbb{R}^d) \) is called mother wavelet. For each \( a \in H \), say dilation parameter, and \( b \in \mathbb{R}^d \), say translation parameter, introduce a family of 2-parameter function \( \varphi_{a,b} \) by \( \varphi_{a,b}(x) = T_b D_a \varphi(x) \), where \( T_b \varphi(x) = \varphi(x-b) \) and \( D_a \varphi(x) = | \det a |^{-\frac{1}{2}} \varphi(a^{-1}x) \), that is

\[
\varphi_{a,b}(x) = | \det a |^{-\frac{1}{2}} \varphi(a^{-1}(x-b)).
\]

Define the wavelet transform of \( f \in L^2(\mathbb{R}^d) \) associated with \( \varphi \) by

\[
CW^a_f(b) = \langle f, \varphi_{a,b} \rangle,
\]

that is

\[
CW^a_f(b) = | \det a |^{-\frac{1}{2}} \int_{\mathbb{R}^d} f(x) \overline{\varphi(a^{-1}(x-b))} \, dx.
\]

In particular, the continuous wavelet transform of a random field \( \{X_t(\omega)\}_{t \in \mathbb{R}^d} \) is the random field

\[
CW^a_X(b, \omega) = | \det a |^{-\frac{1}{2}} \int_{\mathbb{R}^d} X_u(\omega) \overline{\varphi(a^{-1}(u-b))} \, du, \quad a \in H.
\]
Clearly, by definition of the continuous wavelet transform, the integral (14) exists with probability one if $E[\int_{\mathbb{R}^d} |X_t(\omega)|^2 \, dt] < \infty$ and $\varphi \in L^2(\mathbb{R}^d)$. However, if $E[X_t^2] < \infty$, say $X_t$ is a second order random field, and $\varphi \in L^1(\mathbb{R}^d)$ we have

$$E[|CW^a_x(b,\omega)|^2] = |\det a|^{-1} \left[ \int_{\mathbb{R}^d} E[X_\xi^2] \frac{1}{2} |\varphi(a^{-1}(\xi - b))| \, d\xi \right]^2 < \infty.$$ 

Hence the integral (14) exists with probability one.

Also, if $E[X_t^2] < \infty$ and $t$ is in compact set and $\varphi$ has compact support and is integrable we have

$$E[|CW^a_x(b,\omega)|^2] \leq |\det a|^{-1} \left[ \int_{\mathbb{R}^d} E[X_\xi^2] \frac{1}{2} |\varphi(a^{-1}(\xi - b))| \, d\xi \right]^2 < \infty,$$

that is the integral (14) exists with probability one.

The continuous wavelet transform of the random field $\{X_t(\omega)\}_{t \in \mathbb{R}^d}$ is the random (position-scale) field $\{CW^a_x(b,\omega)\}_{a \in H, b \in \mathbb{R}^d}$ provided the path integral is defined with probability one. The continuous wavelet transform at scale $a \in H$ is the random (position) field $\{CW^a_x(b,\omega)\}_{b \in \mathbb{R}^d}$ which is the $a$-section of the wavelet transform $\{CW^a_x(b,\omega)\}_{a \in H, b \in \mathbb{R}^d}$. As such the output $\{CW^a_x(b)\}_{b \in \mathbb{R}^d}$ inherits certain features of the input $X_t$ and here we focus primarily on how features of the input $X_t$ may be read off appropriate properties of the output, at fixed scale or at different scales.

If $E[|CW^a_x(b,\omega)|^2] < \infty$ then for fixed $a \in H$, the autocorrelation of $\{CW^a_x(b,\omega)\}_{b \in \mathbb{R}^d}$ is given by

$$R_{cw^a_x}(\tau) = E\left[ CW^a_x(b) \overline{CW^a_x(b+\tau)} \right],$$

and for fixed $a_1, a_2 \in H$, the cross-correlation of $\{CW^a_x(b)\}_{b \in \mathbb{R}^d}$ is

$$R_{cw^a_x,cw^{a_2}_x}(\tau) = E\left[ CW^{a_1}_x(b) \overline{CW^{a_2}_x(b+\tau)} \right].$$

In the following, we will assume that $E[X_t^2] < \infty$ and $\varphi \in L^1(\mathbb{R}^d)$ so that the integral (14) exists with probability one and $E[|CW^a_x(b,\omega)|^2] < \infty$.

### 3 Power Spectral Density of Wavelet Transform of Random Fields

Let a second order random field with desired properties, such as the autocorrelation function being bounded and continuous be given. The power spectral
density function of random field is usually defined for weakly stationary random fields. However, the continuous wavelet transform of weakly stationary, stationary increment or weakly stationary increment random field are weakly stationary random field. It was thought for a while that the power spectral density function of the continuous wavelet transform of them. These wavelet transform deal with the integrable mother wavelet function $\varphi$ such that $\hat{\varphi}(0) = 0$ with scaling parameter $a$ in matrix group $H$ and translation parameter $b \in \mathbb{R}^d$.

### 3.1 Spectral Density Function of Wavelet Transform of Weakly Stationary Random Fields

Let $\{X_t\}_{t \in \mathbb{R}^d}$ be a weakly stationary random field. Assume that the autocorrelation $R_X(\tau)$ is a bounded continuous function. Since $R_X(\tau)$ is a positive definite function, by Bochner’s theorem, it has the spectral representation

$$R_X(\tau) = \int_{\mathbb{R}^d} e^{i\tau \cdot \lambda} dF_X(\lambda)$$ (17)

where $F_X(\lambda)$ is a finite Borel measure on $\mathbb{R}^d$.

We determine the spectral density function of the wavelet transform of a weakly stationary random field via arbitrary dilation matrix by the following theorem.

**Theorem 3.1** Let $H$ be a matrix group and $a, a_1, a_2 \in H$. If $\{X_t\}_{t \in \mathbb{R}^d}$ is a weakly stationary random field then $\{CW_{X}^{a_1}(t)\}_{t \in \mathbb{R}^d}$ and $\{CW_{X}^{a_2}(t)\}_{t \in \mathbb{R}^d}$ are jointly weakly stationary random fields with zero means. Moreover, the cross-correlation function has the spectral representation

$$R_{CW_X^{a_1}CW_X^{a_2}}(\tau) = |\det a_1| |\det a_2|^\frac{1}{2} \int_{\mathbb{R}^d} e^{i\tau \cdot \lambda} \hat{\varphi}(a_1 \lambda) \hat{\varphi}(a_2 \lambda) dF_X(\lambda).$$ (18)

In particular, the autocorrelation function has the spectral representation

$$R_{CW_X^a}(\tau) = |\det a|^\frac{1}{2} \int_{\mathbb{R}^d} e^{i\tau \cdot \lambda} |\hat{\varphi}(a \lambda)|^2 dF_X(\lambda).$$ (19)

The cross power spectral density function is

$$S_{CW_X^{a_1}CW_X^{a_2}}(\lambda) = |\det a_1| |\det a_2|^\frac{1}{2} \hat{\varphi}(a_1 \lambda) \hat{\varphi}(a_2 \lambda) S_X(\lambda)$$ (20)

and the power spectral density function is

$$S_{CW_X^a}(\lambda) = |\det a| |\hat{\varphi}(a \lambda)|^2 S_X(\lambda).$$ (21)
Continuous wavelet transform of some classes of random field

Proof. Consider for each \( a \in H \),

\[
E[\text{CW}_X^a(t)] = |\det a|^{-\frac{1}{2}} \int_{\mathbb{R}^d} E[X_u] \varphi(a^{-1}(u-t)) \, du.
\]

Since \( \{X_t\}_{t \in \mathbb{R}^d} \) is weakly stationary, \( E[X_u] = E[X_0] \) for all \( u \in \mathbb{R}^d \) and as \( \hat{\varphi}(0) = 0 \), it follows that

\[
E[\text{CW}_X^a(t)] = |\det a|^{-\frac{1}{2}} \int_{\mathbb{R}^d} E[X_0] \varphi(a^{-1}(u-t)) \, du = 0.
\]

Also, for each \( a_1, a_2 \in H \)

\[
\begin{align*}
R_{\text{CW}_X^{a_1},\text{CW}_X^{a_2}}(t,t+\tau) & = E \left[ |\det a_1|^{-\frac{1}{2}} \int_{\mathbb{R}^d} X_{\xi} \varphi(a_1^{-1}(\xi-t)) \, d\xi \int_{\mathbb{R}^d} X_{\eta} \varphi(a_2^{-1}(\eta-t-\tau)) \, d\eta \right] \\
& = |\det a_1|^{-\frac{1}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} E[X_{\xi}X_{\eta}] \varphi(a_1^{-1}(\xi-t)) \varphi(a_2^{-1}(\eta-t-\tau)) \, d\xi \, d\eta.
\end{align*}
\]

Since \( E[X_{\xi}X_{\eta}] = R_X(\xi - \eta) \) and then use equation (17) we have by Fubini’s theorem

\[
\begin{align*}
R_{\text{CW}_X^{a_1},\text{CW}_X^{a_2}}(t,t+\tau) & = |\det a_1|^{-\frac{1}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{ix\lambda} \varphi(a_1^{-1}(\xi-t)) \varphi(a_2^{-1}(\eta-t-\tau)) \, d\xi \, d\eta \, dF_X(\lambda).
\end{align*}
\]

Changing variable, it follows that

\[
R_{\text{CW}_X^{a_1},\text{CW}_X^{a_2}}(t,t+\tau) = |\det a_1|^{-\frac{1}{2}} \int_{\mathbb{R}^d} e^{-i\tau \cdot \lambda} \varphi(a_1 \lambda) \varphi(a_2 \lambda) \, dF_X(\lambda).
\]

If \( a_1 = a_2 = a \), we have

\[
R_{\text{CW}_X}(t,t+\tau) = |\det a| \int_{\mathbb{R}^d} e^{i\tau \cdot \lambda} |\varphi(a \lambda)|^2 \, dF_X(\lambda).
\]

We can see that the cross-correlation function \( R_{\text{CW}_X^{a_1},\text{CW}_X^{a_2}}(t,t+\tau) \) and the autocorrelation function \( R_{\text{CW}_X}(t,t+\tau) \) is depend on position translation \( \tau \) only, say \( R_{\text{CW}_X^{a_1},\text{CW}_X^{a_2}}(\tau) \) and \( R_{\text{CW}_X}(\tau) \). It follows that the zero mean random fields \( \{\text{CW}_X^{a_1}(t)\}_{t \in \mathbb{R}^d} \) and \( \{\text{CW}_X^{a_2}(t)\}_{t \in \mathbb{R}^d} \) are jointly weakly stationary random fields and the cross power spectral representation of \( R_{\text{CW}_X^{a_1},\text{CW}_X^{a_2}}(\tau) \) and the power spectral density representation of \( R_{\text{CW}_X}(\tau) \) are given as above.

By Bochner’s theorem we have the cross power spectral density function as

\[
S_{\text{CW}_X^{a_1},\text{CW}_X^{a_2}}(\lambda) = |\det a_1|^{-\frac{1}{2}} \frac{1}{2} \varphi(a_1 \lambda) \varphi(a_2 \lambda) \, dF_X(\lambda)
\]

\[
= |\det a_1|^{-\frac{1}{2}} \frac{1}{2} \varphi(a_1 \lambda) \varphi(a_2 \lambda) S_X(\lambda),
\]
and the power spectral density function is
\[ S_{cwX}(\lambda) = |\det a||\hat{\phi}(a\lambda)|^2 dF_X(\lambda) = |\det a||\hat{\phi}(a\lambda)|^2 S_X(\lambda). \]

\[ \square \]

### 3.2 Spectral Density Function of Wavelet Transform of Stationary Increment Random Fields

Let \( \{X_t\}_{t \in \mathbb{R}^d} \) be a stationary increment random field with zero mean. Assume that the autocorrelation \( R_X(\tau) \) is a bounded continuous function. For \( d = 1 \), its autocorrelation function has the power spectral representation, see Kacha Dzhaparide (2005),

\[ R_X(t,s) = \int_{\mathbb{R}} (e^{i\lambda t} - 1)(e^{-i\lambda s} - 1) dF_X(\lambda) \quad s, t \in \mathbb{R} \tag{22} \]

for some finite Borel measure \( dF_X(\lambda) \) on \( \mathbb{R} \).

In the special case of fractional Brownian motion \( \{B_H^t\}_{t \in \mathbb{R}} \), we have

\[ R_{B^H}(t,s) = \frac{V_H}{2} \left( |t|^{2H} + |s|^{2H} - |t-s|^{2H} \right) \tag{23} \]

where \( V_H = E[(B_1^H)^2] \). The spectral function is known to be given by

\[ dF_{B^H}(\lambda) = C_H^2 \frac{d\lambda}{|\lambda|^{2H+1}} \]

for some positive constant \( C_H^2 \). Then

\[ R_{B^H}(t,s) = C_H^2 \int_{\mathbb{R}} (e^{i\lambda t} - 1)(e^{-i\lambda s} - 1) \frac{d\lambda}{|\lambda|^{2H+1}} \quad s, t \in \mathbb{R}. \tag{24} \]

Malyarenko, see Kacha Dzhaparide (2005), treats at once the multidimensional case, the autocorrelation of stationary increments random field characterized by the power spectral representation as

\[ R_X(t,s) = \int_{\mathbb{R}^d} (e^{i\lambda t} - 1)(e^{-i\lambda s} - 1) dF_X(\lambda) \quad \text{for all } s, t \in \mathbb{R}^d \tag{25} \]

where \( dF_X(\lambda) \) is a finite Borel measure on \( \mathbb{R}^d \).

In the special case of a fractional Brownian field \( \{B_H^t\}_{t \in \mathbb{R}^d} \) with

\[ R_{B^H}(t,s) = \frac{V_H}{2} \left( \|t\|^{2H} + \|s\|^{2H} - \|t-s\|^{2H} \right) \tag{26} \]
the spectral function is known to be given by

$$dF_X(\lambda) = C^2_H \frac{d\lambda}{|\lambda|^{2H+d}}$$

with a certain positive constant $C^2_H = \frac{2^{2H-1}\Gamma(H+\frac{d}{2})\Gamma(H+1)\sin(\pi H)}{\sqrt{\pi H}}$.

Then

$$R_{BH}(t,s) = C^2_H \int_{\mathbb{R}^d} (e^{i\lambda t} - 1)(e^{-i\lambda s} - 1) \frac{d\lambda}{|\lambda|^{2H+d}} \quad \text{for all } s, t \in \mathbb{R}^d. \quad (27)$$

**Theorem 3.2** Let $H$ be a matrix group and $a, a_1, a_2 \in H$. Then $\{CW^a_X(t)\}_{t \in \mathbb{R}^d}$ and $\{CW^{a_1}_X(t)\}_{t \in \mathbb{R}^d}$ are jointly weakly stationary random fields with zero means. Moreover, the cross-correlation has the spectral representations

$$R_{CW^a_X CW^{a_1}_X}(\tau) = |\det a_1 \det a_2|^{\frac{1}{2}} \int_{\mathbb{R}^d} e^{i\tau \cdot \lambda} \overline{\varphi(a_1 \lambda)} \varphi(a_2 \lambda) dF_X(\lambda). \quad (28)$$

In particular, the autocorrelation has the spectral representations

$$R_{CW^a_X}(\tau) = |\det a| \int_{\mathbb{R}^d} e^{i\tau \cdot \lambda} |\varphi(a \lambda)|^2 dF_X(\lambda). \quad (29)$$

The cross power spectral density function is

$$S_{CW^a_X CW^{a_1}_X}(\lambda) = |\det a_1 \det a_2|^{\frac{1}{2}} \overline{\varphi(a_1 \lambda)} \varphi(a_2 \lambda) dF_X(\lambda), \quad (30)$$

and the power spectral density function is

$$S_{CW^a_X}(\lambda) = |\det a||\varphi(a \lambda)|^2 dF_X(\lambda). \quad (31)$$

**Proof.** Consider for each scaling parameter $a \in H$,

$$E[CW^a_X(t)] = |\det a|^{-\frac{1}{2}} \int_{\mathbb{R}^d} E[X_{u}] \overline{\varphi(a^{-1}(u-t))} \, du.$$

Since $E[X_t] = 0$ for all $t \in \mathbb{R}^d$ it follows that $E[CW^a_X(t)] = 0$. Also for each $a_1, a_2 \in H$

$$R_{CW^a_X CW^{a_1}_X}(t,t+\tau)$$

$$= |\det a_1 \det a_2|^{-\frac{1}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} E[X_\xi X_\eta] \overline{\varphi(a_1^{-1}(\xi-t))} \varphi(a_2^{-1}(\eta-t-\tau)) \, d\xi \, d\eta.$$  

Following the power spectral representation of $R_X(\xi, \eta) = E[X_\xi X_\eta]$ as equation (22) we have by Fubini’s theorem

$$R_{CW^a_X CW^{a_1}_X}(t,t+\tau)$$

$$= |\det a_1 \det a_2|^{-\frac{1}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (e^{i\lambda \xi} - 1) \overline{\varphi(a_1^{-1}(\xi-t))} \varphi(a_2^{-1}(\eta-t-\tau)) \, d\xi \, d\eta \, dF_X(\lambda).$$
As $\hat{\varphi}(0) = 0$ and changing variables we have by Fubini’s theorem

$$R_{CW_X^1, CW_X^2}(t, t+\tau) = |\det a_1 \det a_2|^\frac{1}{2} \int_{\mathbb{R}^d} e^{-ir\cdot\lambda} \hat{\varphi}(a_1 \lambda) \hat{\varphi}(a_2 \lambda) \, dF_X(\lambda).$$

If $a_1 = a_2 = a$, we have

$$R_{CW_X}(\tau) = |\det a| \int_{\mathbb{R}^d} e^{ir\cdot\lambda} |\hat{\varphi}(a\lambda)|^2 \, dF_X(\lambda).$$

We can see that the cross-correlation function $R_{CW_X^1, CW_X^2}(t, t+\tau)$ and the autocorrelation function $R_{CW_X}(t, t+\tau)$ are depend on position translation only. It follows that the zero mean random fields $\{CW_X^1(t)\}_{t \in \mathbb{R}^d}$ and $\{CW_X^2(t)\}_{t \in \mathbb{R}^d}$ are jointly weakly stationary random fields.

By Bochner’s theorem we have the cross-power spectral density function

$$S_{CW_X^1, CW_X^2}(\lambda) = \frac{1}{2\pi} \hat{\varphi}(a_1 \lambda) \hat{\varphi}(a_2 \lambda) \, dF_X(\lambda),$$

thus the spectral density function is

$$S_{CW_X}(\lambda) = |\det a||\hat{\varphi}(a\lambda)|^2 \, dF_X(\lambda).$$

In the special case of a fractional Brownian field, details are given in the next theorem.

Let us choose an integrable mother wavelet function $\varphi$ such that $\hat{\varphi}(0) = 0$ and

$$\sup_{|\lambda| \leq \epsilon} \frac{|\hat{\varphi}(\lambda)|}{|\lambda|^{H+\frac{d}{2}}} \leq M < \infty \text{ for some } \epsilon > 0. \quad (32)$$

**Theorem 3.3** Let $H$ be a matrix group and $a, a_1, a_2 \in H$. Then $\{CW_{B^H}(t)\}_{t \in \mathbb{R}^d}$ and $\{CW_{B^H}(t)\}_{t \in \mathbb{R}^d}$ are jointly weakly stationary random fields with zero means. Moreover, the cross-correlation function has the power spectral representations as

$$R_{CW_X^1, CW_X^2}(\tau) = |\det a_1 \det a_2|^\frac{1}{2} C_H^2 \int_{\mathbb{R}} e^{-ir\cdot\lambda} \hat{\varphi}(a_1 \lambda) \hat{\varphi}(a_2 \lambda) \, d\lambda \frac{1}{|\lambda|^{2H+d}}. \quad (33)$$

In particular, the autocorrelation function has the power spectral representations as

$$R_{CW_X}(\tau) = |\det a| C_H^2 \int_{\mathbb{R}} e^{-ir\cdot\lambda} |\hat{\varphi}(a\lambda)|^2 \, d\lambda \frac{1}{|\lambda|^{2H+d}}. \quad (34)$$

The cross power spectral density function is

$$S_{CW_X^1, CW_X^2}(\lambda) = |\det a_1 \det a_2|^\frac{1}{2} C_H^2 \frac{\hat{\varphi}(a_1 \lambda) \hat{\varphi}(a_2 \lambda)}{|\lambda|^{2H+d}}. \quad (35)$$
and the power spectral density function is

$$S_{CW_X}(\lambda) = |\det a| C_H^2 \frac{|\hat{\varphi}(a\lambda)|^2}{|\lambda|^{2H+d}}.$$  \hspace{1cm} (36)

**Proof.** Since $E[B_t^H] = 0$ for all $t \in \mathbb{R}^d$ we have $E[\text{CW}_{BH}^a(t)] = 0$. Now we will consider the cross-correlation function as the following.

$$R_{\text{CW}_{BH}^a \text{CW}_{BH}^a}(t, t + \tau)$$

$$= |\det a_1 \det a_2|^{-\frac{1}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} E \left[ B_{\xi}^H B_{\eta}^H \right] \varphi^{-1}(a_1^{-1}(\xi - t)) \varphi(a_2^{-1}(\eta - t - \tau)) \, d\xi \, d\eta.$$

Applying equation (27) and Fubini’s theorem

$$R_{\text{CW}_{BH}^a \text{CW}_{BH}^a}(\tau) = C_H^2 |\det a_1 \det a_2|^{-\frac{1}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-i\lambda \cdot \xi} \varphi^{-1}(a_1^{-1}(\xi - t)) \, d\xi \, d\lambda$$

$$\int_{\mathbb{R}^d} e^{-i\lambda \cdot \eta} \varphi(a_2^{-1}(\eta - t - \tau)) \, d\eta \frac{d\lambda}{|\lambda|^{2H+d}}.$$

Since $\hat{\varphi}(0) = 0$ we have

$$R_{\text{CW}_{BH}^a \text{CW}_{BH}^a}(\tau) = C_H^2 |\det a_1 \det a_2|^{-\frac{1}{2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-i\lambda \cdot \xi} \varphi^{-1}(a_1^{-1}(\xi - t)) \, d\xi \, d\lambda$$

$$\int_{\mathbb{R}^d} e^{-i\lambda \cdot \eta} \varphi(a_2^{-1}(\eta - t - \tau)) \, d\eta \frac{d\lambda}{|\lambda|^{2H+d}}.$$

Changing variable then we have

$$R_{\text{CW}_{BH}^a \text{CW}_{BH}^a}(\tau) = C_H^2 |\det a_1 \det a_2|^{-\frac{1}{2}} \int_{\mathbb{R}^d} e^{-i\lambda \cdot \tau} \frac{\varphi(a_1 \lambda) \varphi(a_2 \lambda)}{|\lambda|^{2H+d}} \, d\lambda.$$

Let $\epsilon > 0$. By assumption 32, it follows that $\int_{|\lambda| \leq \epsilon} \frac{\varphi(a_1 \lambda) \varphi(a_2 \lambda)}{|\lambda|^{2H+d}} \, d\lambda < \infty$. On the other hand as $\varphi \in L^1(\mathbb{R})$ and $|\hat{\varphi}(\lambda)| \leq \|\varphi\|_{L^1}$ we have that

$$\int_{|\lambda| > \epsilon} \frac{\varphi(a_1 \lambda) \varphi(a_2 \lambda)}{|\lambda|^{2H+d}} \, d\lambda < \infty.$$

Hence, $\frac{\varphi(a_1 \lambda) \varphi(a_2 \lambda)}{|\lambda|^{2H+d}}$ is integrable for $0 < H < 1$, which by Bochner’s theorem implies that the cross-power spectral density of the random field $\{\text{CW}_{BH}^a(t)\}_{t \in \mathbb{R}^d}$ exists and is given by

$$S_{\text{CW}_{BH}^a \text{CW}_{BH}^a}(\lambda) = C_H^2 |\det a_1 \det a_2|^{-\frac{1}{2}} \frac{\varphi(a_1 \lambda) \varphi(a_2 \lambda)}{|\lambda|^{2H+d}}.$$
If \( a_1 = a_2 = a \), we have
\[
R_{CW_{BH}}(\tau) = |\det a|C_H^2 \int_{\mathbb{R}^d} e^{-i\lambda \cdot \tau} |\hat{\varphi}(a\lambda)|^2 \frac{d\lambda}{|\lambda|^{2H+d}}.
\]
Thus the spectral density function is
\[
S_{CW_{BH}}(\lambda) = |\det a|C_H^2 \frac{|\hat{\varphi}(a\lambda)|^2}{|\lambda|^{2H+d}}.
\]

3.3 The Spectral Density Function of the Wavelet Transform of Weakly Stationary Increment Random Fields

Let \( \{X_t\}_{t \in \mathbb{R}} \) be a real valued random process with mean zero and mean square continuous weakly stationary increments, and let
\[
R_{\Delta X}(t; \tau_1, \tau_2) = E[(X_{s+t+\tau_1} - X_{s+t})(X_{s+\tau_2} - X_s)]
\]
be the correlation function of the increments, which is independent on \( s \) by the stationary assumption.

Note that \( R_{\Delta X}(t; \tau_1, \tau_2) \) has the spectral representation, see Yaglom [25],
\[
R_{\Delta X}(t; \tau_1, \tau_2) = \int_{-\infty}^{\infty} e^{it\lambda}(1 - e^{i\tau_1\lambda})(1 - e^{-i\tau_2\lambda}) \frac{1 + \lambda^2}{\lambda^2} d\mu(\lambda)
\]
where \( \mu \) is a finite spectral measure on the Borel subsets of the real line.

Note that \( R(0; \tau_1, \tau_2) = E[(X_{\tau_1} - X_0)(X_{\tau_2} - X_0)] \).

Let \( \varphi \) be an integrable mother wavelet function with \( \hat{\varphi}(0) = 0 \) and
\[
\sup_{|\lambda| \leq \epsilon} \frac{|\hat{\varphi}(\lambda)|}{|\lambda|} \leq M < \infty \text{ for some } \epsilon > 0.
\]
Then the wavelet transform of weakly stationary increment random process follows by the following theorem.

**Theorem 3.4** Let \( a_1, a_2 \neq 0 \). Then \( \{CW_{X_{a_1}}(t)\}_{t \in \mathbb{R}} \) and \( \{CW_{X_{a_2}}(t)\}_{t \in \mathbb{R}} \)
are jointly weakly stationary random processes with zero means. Moreover, the cross-correlation function has the cross power spectral representation
\[
R_{CW_{X_{a_1}}CW_{X_{a_2}}}(\tau) = (a_1 a_2)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{i\tau \lambda} \frac{\hat{\varphi}(a_2\lambda) \hat{\varphi}(a_1\lambda)}{\lambda} (1 + \lambda^2) d\mu(\lambda).
\]
In particular, the autocorrelation has the power spectral representation as
\[
R_{CW_{X}}(\tau) = a \int_{-\infty}^{\infty} e^{i\tau \lambda} \frac{|\hat{\varphi}(a\lambda)|^2}{\lambda} (1 + \lambda^2) d\mu(\lambda).
\]
Let \( \{X_t\}_{t \in \mathbb{R}^d} \) be a weakly stationary increments with zero mean random field. That is its increments \( \Delta X(t, \tau) = X_{t+\tau} - X_t \) satisfy \( E[\Delta X(t, \tau)] = 0 \) and \( R_{\Delta X(t, \tau_1)\Delta X(s, \tau_2)} = E[(X_{t+\tau_1} - X_t)(X_{s+\tau_2} - X_s)] = D(t - s; \tau_1, \tau_2) \). For a mean-square continuous random field with weakly stationary increment, the autocorrelation function \( D(t - s; \tau_1, \tau_2) \) admits the spectral representation

\[
D(t - s; \tau_1, \tau_2) = \int_{\mathbb{R}^d/\{0\}} e^{it\lambda} (1 - e^{i\tau_1 \cdot \lambda})(1 - e^{-i\tau_2 \cdot \lambda}) \, dF_X(\lambda) + (A\tau_1) \cdot \tau_2
\]

where \( dF_X(\lambda) \) is a measure on \( \mathbb{R}^d/\{0\} \) satisfying

\[
\int_{\mathbb{R}^d/\{0\}} \frac{|\lambda|^2}{1 + |\lambda|^2} \, dF_X(\lambda)
\]

is a nonnegative definite Hermitian matrix. Then the term \((A\tau_1) \cdot \tau_2\) represents the contribution of the integral at \( \lambda = 0 \).

Assumption (*) :

(A1) \( \varphi \in (L^1 \cap L^2)(\mathbb{R}^d) \) and \( \hat{\varphi}(0) = 0 \).

(A2) \( \varphi \) has zero first moments, that is \( \int_{\mathbb{R}^d} u_i \varphi(u) \, du = 0 \) for all \( i = 1, 2, \ldots, d \).

**Theorem 3.5** Let \( H \) be matrix group and \( a, a_1, a_2 \in H \). Let the assumption (*) hold. Then the random fields \( \{\text{CW}^{a_1}_X(t)\}_{t \in \mathbb{R}^d} \) and \( \{\text{CW}^{a_2}_X(t)\}_{t \in \mathbb{R}^d} \) are jointly weakly stationary with zero means, and the autocorrelation and cross-correlation have the power spectral representations and cross power spectral representations

\[
R_{\text{CW}^{a_1}_X,\text{CW}^{a_2}_X}(t, t + \tau) = |\det a_1| \int_{\mathbb{R}^d/\{0\}} e^{i\tau \cdot \lambda} |\hat{\varphi}(a_1 \lambda)|^2 \, dF_X(\lambda)
\]

\[
R_{\text{CW}^{a_1}_X,\text{CW}^{a_2}_X}(t, t + \tau) = |\det a_1| \det a_2 |\frac{1}{2} \int_{\mathbb{R}^d/\{0\}} e^{i\tau \cdot \lambda} \overline{\hat{\varphi}(a_1 \lambda)} \hat{\varphi}(a_2 \lambda) \, dF_X(\lambda)
\]

respectively. Moreover, the power spectral density function is

\[
S_{\text{CW}^{a_1}_X}(\lambda) = |\det a||\hat{\varphi}(a_1 \lambda)|^2 \, dF_X(\lambda)
\]

and the cross power spectral density function is

\[
S_{\text{CW}^{a_1}_X,\text{CW}^{a_2}_X}(\lambda) = |\det a_1| \det a_2 |\frac{1}{2} \overline{\hat{\varphi}(a_1 \lambda)} \hat{\varphi}(a_2 \lambda) \, dF_X(\lambda).
\]

**Proof.** Since \( E[X_u] = 0 \) for all \( u \in \mathbb{R}^d \) we have by Fubini’s theorem that \( E[\text{CW}^{a_1}_X(t)] = 0 \). Let \( a_1, a_2 \in H \), and consider the cross-correlation as follows. Introducing

\[
R_{\text{CW}^{a_1}_X,\text{CW}^{a_2}_X}(t, t + \tau) = E[\text{CW}^{a_1}_X(t) \overline{\text{CW}^{a_2}_X(t + \tau)}]
\]
and using Fubini’s theorem we obtain that
\[
R_{CW^a_1, CW^a_2} (t, t + \tau) = \left| \det a_1 \det a_2 \right|^\frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} E \left[ X_\xi X_\eta \right] \varphi (a_1^{-1} (\xi - t)) \varphi (a_2^{-1} (\eta - t - \tau)) \, d\xi \, d\eta.
\]

Changing variables and adding zero value terms as \( \hat{\varphi}(0) = 0 \), we have
\[
R_{CW^a_1, CW^a_2} (t, t + \tau) = \left| \det a_1 \det a_2 \right|^\frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} E \left[ \left( X_{\xi+t} X_{a_2 \eta+t+\tau} - X_{a_2 \eta+t+\tau} \right) \right] \varphi (\eta) \, d\xi \, d\eta
\]
\[
= \left| \det a_1 \det a_2 \right|^\frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} D (-\tau, a_1 \xi, a_2 \eta) \varphi (\eta) \, d\xi \, d\eta.
\]

Applying the power spectral representation of \( D (-\tau, a_1 \xi, a_2 \eta) \) it follows that
\[
R_{CW^a_1, CW^a_2} (t, t + \tau) = \left| \det a_1 \det a_2 \right|^\frac{1}{2} \int_{\mathbb{R}^d/\{0\}} e^{-i\tau \cdot \lambda} \left( \int_{\mathbb{R}^d} (1 - e^{i a_1 \xi \cdot \lambda}) \overline{\varphi (\xi)} \, d\xi \right) d\lambda
\]
\[
+ \left( a_1 a_2 \right)^\frac{d}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \sum_{i=1}^{d} \sum_{j=1}^{d} a_{ij} \xi_i \eta_j \overline{\varphi (\xi)} \varphi (\eta) \, d\xi \, d\eta.
\]

Since \( \hat{\varphi}(0) = 0 \) and using assumptions (*), we can reduce the above equation to
\[
R_{CW^a_1, CW^a_2} (t, t + \tau) = \left| \det a_1 \det a_2 \right|^\frac{1}{2} \int_{\mathbb{R}^d/\{0\}} e^{-i\tau \cdot \lambda} \varphi (a_1 \lambda) \hat{\varphi} (a_2 \lambda) \, dF_X (\lambda).
\]

If \( a_1 = a_2 = a \), we have
\[
R_{CW^a} (\tau) = \left| \det a \right| \int_{\mathbb{R}^d} e^{i\tau \cdot \lambda} |\hat{\varphi}(a\lambda)|^2 \, dF_X (\lambda).
\]

We can see that the cross-correlation function \( R_{CW^a_1, CW^a_2} (t, t + \tau) \) and the autocorrelation function \( R_{CW^a} (t, t + \tau) \) depend on position translation \( \tau \) only. It follows that the zero mean random fields \( \{ CW^a_1 (t) \}_{t \in \mathbb{R}^d} \) and \( \{ CW^a_2 (t) \}_{t \in \mathbb{R}^d} \) are jointly weakly stationary.

By Bochner’s theorem we have the cross spectral density function
\[
S_{CW^a_1, CW^a_2} (\lambda) = \left| \det a_1 \det a_2 \right|^\frac{1}{2} \varphi (a_1 \lambda) \hat{\varphi} (a_2 \lambda) \, dF_X (\lambda),
\]
Continuous wavelet transform of some classes of random field

and the power spectral density function is

$$S_{CW_X}(\lambda) = \left| \det a |\hat{\varphi}(a\lambda)|^2 dF_X(\lambda).$$

\[ \square \]

4 Ergodicity Properties

4.1 Mean Ergodic Random Fields

Ergodicity in the mean or in short mean ergodicity is saying that the estimate for the mean converges to the true mean in the mean square sense as in the following definition.

**Definition 4.1** A random field \( \{X_t\}_{t \in \mathbb{R}^d} \) with finite mean is said to be mean ergodic if

$$\lim_{T \to \infty} \frac{1}{(2T)^d} \int_{[-T,T]^d} X_t \, dt = E[X_t]$$

(45)

in the mean square sense.

The following theorems provide conditions for mean ergodicity.

**Theorem 4.2** Let \( \{X_t\}_{t \in \mathbb{R}^d} \) be a weakly stationary random field with finite mean and covariance function \( C_X(\tau) \). A necessary and sufficient condition for \( \{X_t\}_{t \in \mathbb{R}^d} \) to be mean ergodic is

$$\lim_{T \to \infty} \frac{1}{(2T)^d} \int_{[-2T,2T]^d} \prod_{i=1}^d (1 - |\tau_i|^2) C_X(\tau) \, d\tau = 0.$$

(46)

Identity (46) is often given as the definition of mean ergodicity, in case of weakly stationary random fields.

**Remark 1.** Let \( \{X_t\}_{t \in \mathbb{R}^d} \) be a weakly stationary random field with zero mean. Then \( C_X(\tau) = R_X(\tau) \) and hence \( \{X_t\}_{t \in \mathbb{R}^d} \) is mean ergodic iff

$$\lim_{T \to \infty} \frac{1}{(2T)^d} \int_{[-2T,2T]^d} \prod_{i=1}^d (1 - |\tau_i|^2) R_X(\tau) \, d\tau = 0.$$

(47)

**Theorem 4.3** Let \( H \) be a matrix group, \( a \in H \) and \( \varphi \in L^1(\mathbb{R}^d) \) be such that \( \hat{\varphi}(0) = 0 \). Let \( \{X_t\}_{t \in \mathbb{R}^d} \) be a weakly stationary random field with zero mean. If \( \{X_t\}_{t \in \mathbb{R}^d} \) is ergodic in mean then \( \{CW_X(t,a)\}_{t \in \mathbb{R}^d} \) is also ergodic in mean.
Proof. Since \( \{X_t\} \) is a weakly stationary, its autocorrelation function \( R_X(\tau) \) is a positive definite function and hence by Bochner’s theorem has the power spectral representation
\[
R_X(\tau) = \int_{\mathbb{R}^d} e^{i\tau \cdot \lambda} dF_X(\lambda)
\]
where \( F_X(\lambda) \) is a finite Borel measure on \( \mathbb{R}^d \).
Since \( \{X_t\} \) is weakly stationary, we have by theorem 3.1 that \( \{CW^a_X(t)\} \) is weakly stationary with zero mean, the autocorrelation function \( R_{CW^a_X}(\tau) \) has the power spectral representation
\[
R_{CW^a_X}(\tau) = |\text{det} a|^2 \int_{\mathbb{R}^d} e^{i\tau \cdot \lambda} |\tilde{\varphi}(a\lambda)|^2 dF_X(\lambda).
\]
Assuming that \( \{X_t\} \) is ergodic in the mean we may use have the alternative definition (47)
\[
\lim_{T \to \infty} \frac{1}{(2T)^d} \int_{[-2T,2T]^d} \prod_{i=1}^{d} \left( 1 - \frac{\tau^i}{2T} \right) R_X(\tau) d\tau = 0.
\]
Then
\[
\frac{1}{(2T)^d} \int_{[-2T,2T]^d} \prod_{i=1}^{d} \left( 1 - \frac{\tau^i}{2T} \right) R_{CW^a_X}(\tau) d\tau
\]
\[
= \frac{1}{(2T)^d} \int_{[-2T,2T]^d} \prod_{i=1}^{d} \left( 1 - \frac{\tau^i}{2T} \right) |\text{det} a|^2 \int_{\mathbb{R}^d} e^{i\tau \cdot \lambda} |\tilde{\varphi}(a\lambda)|^2 dF_X(\lambda) d\tau
\]
\[
\leq |\text{det} a||\varphi||^2 L_1^2 \int_{[-2T,2T]^d} \prod_{i=1}^{d} \left( 1 - \frac{\tau^i}{2T} \right) \int_{\mathbb{R}^d} e^{i\tau \cdot \lambda} dF_X(\lambda) d\tau
\]
\[
= |\text{det} a||\varphi||^2 L_1^2 \int_{[-2T,2T]^d} \prod_{i=1}^{d} \left( 1 - \frac{\tau^i}{2T} \right) R_X(\tau) d\tau
\]
\[
\to 0 \quad \text{as} \quad T \to \infty, \quad \text{by equation (48)}.
\]
Thus
\[
\lim_{T \to \infty} \frac{1}{(2T)^d} \int_{[-2T,2T]^d} \prod_{i=1}^{d} \left( 1 - \frac{\tau^i}{2T} \right) R_{CW^a_X}(\tau) d\tau = 0
\]
so that \( \{CW^a_X(t)\} \) is mean ergodic. \( \square \)
4.2 Ergodic Theorem

The ergodic theorem relates functionals calculated along individual sample paths, say the time average or the maximum attained value, to functionals calculated over the whole distribution, say the expectation or the expected maximum. The basic idea is that the two should be close and they should get closer the longer the trajectory we use, because in some sense any one sample path, carried far enough, is representative of the whole distribution.

The $L^2$ or mean square ergodic theorem, attributed to Von Neumann already holds for weakly stationary random fields.

**Theorem 4.4** *(Ergodic theorem for weakly stationary random fields)*

If $\{X_t\}_{t \in \mathbb{R}^d}$ is a weakly stationary random field, then there exists a random variable $Y$ such that $E[Y] = E[X_0]$ and

$$
\lim_{T \to \infty} \frac{1}{(2T)^d} \int_{[-T,T]^d} X_t \, dt = Y.
$$

(49)

**Corrollary 1.** If $\{X_t\}_{t \in \mathbb{R}^d}$ is a stationary random field with zero mean and autocorrelation function $R_X(\tau)$ then the limit variable

$$
Y = \lim_{T \to \infty} \frac{1}{(2T)^d} \int_{[-T,T]^d} X_t \, dt
$$

(50)

satisfies

$$
E[Y] = 0, \quad E[|Y|^2] = \lim_{T \to \infty} \frac{1}{(2T)^d} \int_{[-2T,2T]^d} R_X(\tau) \, d\tau.
$$

(51)

If $\{X_t\}_{t \in \mathbb{R}^d}$ has a stationary increments or a weakly stationary increments then these ergodic theorem need not apply. Hoever, they apply to their wavelet transform $\{CW_X(t)\}_{t \in \mathbb{R}^d}$, by the following theorem.

**Theorem 4.5**

Let $H$ be a matrix group, $a \in H$ and $\varphi \in L^1(\mathbb{R}^d)$ such that $\hat{\varphi}(0) = 0$. If $\{X_t\}_{t \in \mathbb{R}^d}$ is a weakly stationary increment random field, then there exists a random variable $Y$ such that

$$
\lim_{n \to \infty} \frac{1}{(2T)^d} \int_{[-T,T]^d} CW_X^a(t) \, dt = Y
$$

(52)

in the mean square sense, with $E[Y] = 0$ and

$$
E[|Y|^2] = \lim_{T \to \infty} \frac{1}{(2T)^d} \int_{[-2T,2T]^d} R_{CW_X^a}(\tau) \, d\tau.
$$
Proof. By theorems 3.1, 3.2 and 3.4, \( \{CW_{X}^{a}(t)\}_{t \in \mathbb{R}^d} \) is a weakly stationary random field with zero mean and autocorrelation \( R_{CW_{X}^{a}}(\tau) \). Thus there exists a random variable \( Y \) such that

\[
\lim_{n \to \infty} \frac{1}{(2T)^d} \int_{[-T,T]^d} CW_{X}^{a}(t) \, dt = Y
\]

(53)

with \( E[Y] = 0 \) and

\[
E[|Y|^2] = \lim_{T \to \infty} \frac{1}{(2T)^d} \int_{[-2T,2T]^d} R_{CW_{X}^{a}}(\tau) \, d\tau.
\]

(54)

\[ \square \]

5 Conclusion

In this paper, the continuous wavelet transform of finite dimensional random field with weakly stationary, stationary increment and weakly stationary increment via arbitrary dilation matrix has been proposed. They are joint weakly stationary random fields for different dilation matrices whose cross correlation functions and cross power spectral density functions are determined. The cross power spectral density function of the wavelet transform of a weakly stationary random field is related to the cross power spectral density function of the random field itself. The cross power spectral density function of the wavelet transform of stationary increment random fields and weakly stationary increment random fields are expressed by multiplication of the Fourier transform of the mother wavelet and a finite Borel measure. Moreover, the ergodicities are discussed. For a weakly stationary random field, if it is mean ergodic then its continuous wavelet transform is also mean ergodic. For a weakly stationary increment (also stationary increment and weakly stationary) random field, its continuous wavelet transform satisfies the ergodic theorem.

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