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The Fourier Regularization Method for Identifying the Unknown Source for the Helmholtz Equation

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Abstract

In this paper, we discuss the problem of determining an unknown source which depends only on one variable in two-dimensional Helmholtz equation from one supplementary measurement at an internal point. The problem is ill-posed in the sense that the solution (if it exists) does not depend continuously on the data. The regularization solution is obtained by the Fourier regularization method. For the regularization solution, the stability estimate between the regularization solution and the exact solution is given.

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1 Introduction

Consider the following inverse problem: to find a pair of functions $(u(x,y), f(x))$ satisfying

$$\begin{cases} -u_{xx} - u_{yy} - k^2 u = f(x), & -\infty < x < \infty, y > 0, \\ u(x,0) = 0, & -\infty < x < \infty, \\ u(x,y)_{y\to\infty}\text{bounded}, & -\infty < x < \infty, \\ u(x,1) = g(x), & -\infty < x < \infty, \end{cases} \tag{1}$$

where $f(x)$ is the unknown source depending only on one spatial variable, $u(x,1) = g(x)$ is the supplementary condition, $k > 0$ is the wave number. We want to determine the solution $f(x)$ from the data $g(x)$. In applications, input data $g(x)$ can only be measured, there will be measured data function $g_\delta(x)$ which is merely in $L^2(\mathbb{R})$, and satisfies

$$\|g - g_\delta\|_{L^2(\mathbb{R})} \le \delta, \tag{2}$$

where the constant $\delta > 0$ represents a noise level of input data.

This problem is called the inverse problem of identification unknown source. For the heat equation, there has been a large number of research results for different forms of heat source[1−5].

Let $\hat{f}(\xi)$ denote the Fourier transform of $f(x) \in L^2(\mathbb{R})$ which defined by

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} f(x)dx.$$

So the solution of problem (1) is given in frequency space as follows:

$$\hat{f}(\xi) := \begin{cases} \frac{\xi^2 - k^2}{1 - e^{-\sqrt{\xi^2 - k^2}}}\hat{g}(\xi), \ \xi > k \\ \frac{k^2 - \xi^2}{\cos(\sqrt{k^2 - \xi^2}) - 1}\hat{g}(\xi), \ \xi \le k. \end{cases} \tag{3}$$

Thus

Case 1. If $\xi > k$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} \frac{\xi^2 - k^2}{1 - e^{-\sqrt{\xi^2 - k^2}}}\hat{g}(\xi). \tag{4}$$

Case 2. If $\xi \le k$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} \frac{k^2 - \xi^2}{\cos(\sqrt{k^2 - \xi^2}) - 1}\hat{g}(\xi). \tag{5}$$

The unbounded function $\frac{\xi^2 - k^2}{1 - e^{-\sqrt{\xi^2 - k^2}}}$ in (4) can be seen as an amplification factor of $\hat{g}(\xi)$ when $\xi \to \infty$. Therefore when we consider our problem in $L^2(\mathbb{R})$, the exact data function $\hat{g}(\xi)$ must decay as $\xi \to \infty$. But in the applications, the input data $g(x)$ can only be measured and never be exact. We assume the measured data function $g_\delta(x) \in L^2(\mathbb{R})$. Thus if we try to obtain the unknown source $f(x)$, high frequency components in the error are magnified and can destroy the solution. So it is impossible to solve the problem (1) by using classical methods. In the following section, we will use the Fourier regularization method to deal with the ill-posed problem. Before doing that, we impose an a-priori bound on the input data, i.e.,

$$\|f(\cdot)\|_{H^p} \leq E, \qquad p > 0, \tag{6}$$

where $E > 0$ is a constant, $\|\cdot\|_{H^p}$ denotes the norm in Sobolev space $H^p(\mathbb{R})$ defined by

$$\|f(\cdot)\|_{H^p} := \left(\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2(1+\xi^2)^p d\xi\right)^{\frac{1}{2}}. \tag{7}$$

2 A Fourier truncation regularization method

It is obvious that the ill-posedness of problem (1) is caused by disturb of the high frequencies. A natural way to stabilize the problem (1) is to eliminate all high frequencies from the solution $f(x)$. This idea has appeared earlier in [6]. The authors who considered the IHCP, called this method the Fourier regularization. Recently, Fourier regularization method has been effectively applied to solving various types of inverse problems. Yang [7] used it to identify the unknown source on Poisson equation, Fu [8] used it to solve the BHCP, Qian [9] used it to consider the numerical differentiation, Regińska [10] used it to consider a Cauchy problem for the Helmholtz equation. From [6-10], it seems the Fourier regularization method is rather simple and convenience for dealing with some ill-posed problems.

We define a regularization approximation solution of problem (1) for noisy data $g_\delta(x)$ as follows:

$$f_{\delta,\xi_{\max}}(x) := \begin{cases} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} \frac{\xi^2-k^2}{1-e^{-\sqrt{\xi^2-k^2}}} \hat{g}_\delta(\xi)\chi_{\max}\, d\xi, \ \xi > k, \\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} \frac{k^2-\xi^2}{\cos(\sqrt{k^2-\xi^2})-1} \hat{g}_\delta(\xi)\chi_{\max}\, d\xi, \ \xi \leq k, \end{cases} \tag{8}$$

which is called the Fourier regularized solution of problem (1), where $\chi_{\max}$ is the characteristic function of the interval $[-\xi_{\max}, \xi_{\max}]$, i.e.,

$$\chi_{\max}(\xi) = \begin{cases} 1, & |\xi| \leq \xi_{\max}, \\ 0, & |\xi| > \xi_{\max}, \end{cases} \tag{9}$$

and $\xi_{\max}$ is a constant which will be selected appropriately as regularization parameter.

Before giving the main conclusion of this paper, we first introduce two important lemmas.

Lemma 2.1 *For $\xi > k$, there holds*

$$\frac{\xi^2 - k^2}{1 - e^{-\sqrt{\xi^2-k^2}}} \le \max\{2, 2(\xi^2 - k^2)\}. \tag{10}$$

Lemma 2.2 *For $x \ge 0$, there holds*

$$1 - \cos x \ge \frac{x^2}{2} - \frac{x^4}{4}. \tag{11}$$

The main conclusion of this section is:

Theorem 2.3 *Let $f_{\delta,\xi_{\max}}(x)$ be the regularized solution and $f(x)$ be the exact solution with its Fourier transform given by (3). Let $g_\delta(x)$ be the measured data at $y = 1$ satisfying (2) and priori condition (6) holds for $p > 0$. If we select*

$$\xi_{\max} = \sqrt{(\frac{E}{\delta})^{\frac{2}{p+2}} + k^2}, \tag{12}$$

we obtain the following estimate:

$$\|f(\cdot) - f_{\delta,\xi_{\max}}(\cdot)\| \le 2\delta^{\frac{p}{p+2}} E^{\frac{2}{p+2}} (1 + \max\{1, (\frac{\delta}{E})^{\frac{2}{p+2}}\}) + \frac{4}{2-k^2}\delta. \tag{13}$$

Proof. Due to Parseval formula, (3), (8), (2), (10) and (11), we obtain

$$
\begin{aligned}
&\|f(\cdot) - f_{\delta,\xi_{\max}}(\cdot)\| = \|\hat{f}(\cdot) - \hat{f}_{\delta,\xi_{\max}}(\cdot)\| \\
&= (\int\limits_{|\xi|>\xi_{\max}>k} |\frac{\xi^2-k^2}{1-e^{-\sqrt{\xi^2-k^2}}}\hat{g}(\xi)|^2\,d\xi)^{\frac{1}{2}} + (\int\limits_{k<|\xi|\le\xi_{\max}} |\frac{\xi^2-k^2}{1-e^{-\sqrt{\xi^2-k^2}}}(\hat{g}(\xi)-\hat{g}_\delta(\xi))|^2\,d\xi)^{\frac{1}{2}} \\
&\quad + (\int\limits_{|\xi|<k} |\frac{k^2-\xi^2}{1-\cos(\sqrt{k^2-\xi^2})}(\hat{g}(\xi)-\hat{g}_\delta(\xi))|^2\,d\xi)^{\frac{1}{2}} \\
&\le (\int\limits_{|\xi|>\xi_{\max}} |\hat{f}(\xi)(1+\xi^2)^{\frac{p}{2}}(1+\xi^2)^{-\frac{p}{2}}|^2\,d\xi)^{\frac{1}{2}} + \sup_{k<|\xi|\le\xi_{\max}} |\frac{\xi^2-k^2}{1-e^{-\sqrt{\xi^2-k^2}}}|\delta \\
&\quad + \sup_{|\xi|\le k} |\frac{k^2-\xi^2}{1-\cos(\sqrt{k^2-\xi^2})}|\delta \\
&\le \sup_{|\xi|>\xi_{\max}} (1+\xi^2)^{-\frac{p}{2}} (\int\limits_{|\xi|>\xi_{\max}} (\hat{f}(\xi)(1+\xi^2)^{\frac{p}{2}})^2 d\xi)^{\frac{1}{2}} + \max\{2, 2(\xi_{\max}^2-k^2)\}\delta + \frac{4}{2-k^2}\delta \\
&\le (\frac{1}{\xi_{\max}^2-k^2})^{\frac{p}{2}} E + \max\{2, 2(\xi_{\max}^2-k^2)\}\delta + \frac{4}{2-k^2}\delta \\
&= 2\delta^{\frac{p}{p+2}} E^{\frac{2}{p+2}} (1+\max\{1, (\frac{\delta}{E})^{\frac{2}{p+2}}\}) + \frac{4}{2-k^2}\delta.
\end{aligned}
$$

Remark 2.4 *From* (13), *we know when* $k \to \sqrt{2}$, $\frac{4}{2-k^2} \to \infty$. *So we assume* $k \neq \sqrt{2}$.

3 Conclusions

The identification of unknown source on Helmholtz equation is mildly ill-posed and the degree of the ill-posedness is equivalent to the second-order numerical differentiation. The Fourier regularization method which is based on Fourier truncation in the frequency space is applied to determining the unknown source term and the stability error estimate is obtained between the regularization solution and the exact solution.

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