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On Tri α-Separation Axioms in Fuzzifying Tri-Topological Spaces

Barah M. Sulaiman and Tahir H. Ismail

Mathematics Department
College of Computer Science and Mathematics
University of Mosul, Iraq

Abstract

The present article introduce $\alpha T_0^{(1,2,3)}$(Kolmogorov), $\alpha T_1^{(1,2,3)}$(Fréchet), $\alpha T_2^{(1,2,3)}$(Hausdorff), $\alpha \mathcal{R}^{(1,2,3)}$($\alpha$-regular), $\alpha \mathcal{N}^{(1,2,3)}$($\alpha$-normal),$\alpha R_0^{(1,2,3)}$, $\alpha R_1^{(1,2,3)}$ and $\alpha R_2^{(1,2,3)}$ separation axioms in fuzzifying tri-topological spaces and studying the relation among them and also some of their properties.

Keywords: Fuzzifying Tri topology; Fuzzifying tri α-separation axioms

1 Introduction

Ying (1991-1993) introduced the concept of the term "fuzzifying topology" [7-9]. Wuyts and Lowen (1983) studied "separation axioms in fuzzy topological spaces" [6]. Shen (1993) introduced and studied T_0, T_1, T_2 (Hausdorff), T_3 (regularity), T_4 (normality)-separation axioms in fuzzifying topology [3]. Khedr et al. (2001) studied "separation axioms in fuzzifying topology" [2]. Sayed (2014) presented "α-separation axioms based on Łukasiewicz logic" [4]. Allam et al. (2015) studied "semi separation axioms in fuzzifying bitopological spaces" [1]. We use the fundamentals of fuzzy logic with consonant set theoretical notations which are introduced by Ying (1991-1993) [7-9] throughout this paper.

Definition 1.1 [5]
If $(X, \tau_1, \tau_2, \tau_3)$ is a fuzzifying tri-topological space (FTTS),

(i) The family of fuzzifying (1,2,3) α-open sets in X, symbolized as $\alpha\tau_{(1,2,3)} \in \Im(P(X))$, and defined as
$E \in \alpha\tau_{(1,2,3)} \coloneqq \forall\, x\, (x \in E \to x \in int_1(cl_2(int_3(E))))$,
i.e., $\alpha\tau_{(1,2,3)}(E) = \inf_{x\in E}(int_1(cl_2(int_3(E))))(x)$.

(ii) The family of fuzzifying (1,2,3) α-closed sets in X, symbolized as $\alpha\mathcal{F}_{(1,2,3)}$, and defined by $F \in \alpha\mathcal{F}_{(1,2,3)} \coloneqq X{\sim}F \in \alpha\tau_{(1,2,3)}$.

(iii) The (1,2,3) α-neighborhood system of x, denoted by $\alpha N_x^{(1,2,3)}$ and defined as
$E \in \alpha N_x^{(1,2,3)} \coloneqq \exists\, F\, (F \in \alpha\tau_{(1,2,3)} \wedge x \in F \subseteq E)$;
i.e. $\alpha N_x^{(1,2,3)}(E) = \sup_{x\in F\subseteq E} \alpha\tau_{(1,2,3)}(F)$.

(iv) The (1,2,3) α-derived set of E ⊆ X, denoted by $\alpha d_{(1,2,3)}(E)$ and defined as
$x \in \alpha d_{(1,2,3)}(E) \coloneqq \forall\, F\, (F \in \alpha N_x^{(1,2,3)} \to F \cap (E - \{x\}) \neq \emptyset)$,
i.e., $\alpha d_{(1,2,3)}(E)(x) = \inf_{F\cap(E-\{x\})\neq\emptyset} (1 - \alpha N_x^{(1,2,3)}(F))$.

(v) The (1,2,3) α-closure set of $E \subseteq X$, denoted by $\alpha cl_{(1,2,3)}(E)$ and defined as
$x \in \alpha cl_{(1,2,3)}(E) \coloneqq \forall\, F\, (F \supseteq E) \cap (F \in \alpha\mathcal{F}_{(1,2,3)}) \to x \in F)$,
i.e., $\alpha cl_{(1,2,3)}(E)(x) = \inf_{x\notin F\supseteq E} (1 - \alpha\mathcal{F}_{(1,2,3)}(F))$.

(vi) The (1,2,3) α-interior set of $E \subseteq X$, denoted by $\alpha int_{(1,2,3)}(E)$ and defined as
$\alpha int_{(1,2,3)}(E)(x) = \alpha N_x^{(1,2,3)}(E)$.

(vii) The (1,2,3) α-exterior set of $E \subseteq X$, denoted by $\alpha ext_{(1,2,3)}(E)$ and defined as
$x \in \alpha ext_{(1,2,3)}(E) \coloneqq x \in \alpha int_{(1,2,3)}(X{\sim}E)(x)$,
i.e. $\alpha ext_{(1,2,3)}(E)(x) = \alpha int_{(1,2,3)}(X{\sim}E)(x)$.

(viii) The (1,2,3) α-boundary set of $E \subseteq X$, denoted by $\alpha b_{(1,2,3)}(E)$ and defined as
$x \in \alpha b_{(1,2,3)}(E) \coloneqq (x \notin \alpha int_{(1,2,3)}(E)) \wedge (x \notin \alpha int_{(1,2,3)}(X{\sim}E))$,
i.e. $\alpha b_{(1,2,3)}(E)(x) \coloneqq min(1 - \alpha int_{(1,2,3)}(E)(x)) \wedge (1 - \alpha int_{(1,2,3)}(X{\sim}E)(x))$.

2 Tri α-Separation axioms in fuzzifying tri-topological spaces

Remark 2.2 We consider the following notations:
$\alpha\mathcal{K}_{x,y}^{(1,2,3)} \coloneqq \exists\, G\, ((G \in \alpha N_x^{(1,2,3)} \wedge y \notin G) \vee (G \in \alpha N_y^{(1,2,3)} \wedge x \notin G))$;
$\alpha\mathcal{H}_{x,y}^{(1,2,3)} \coloneqq \exists\, H\, \exists\, E\, (H \in \alpha N_x^{(1,2,3)} \wedge E \in \alpha N_y^{(1,2,3)} \wedge y \notin H \wedge x \notin E)$;
$\alpha\mathcal{M}_{x,y}^{(1,2,3)} \coloneqq \exists\, H\, \exists\, E\, (H \in \alpha N_x^{(1,2,3)} \wedge E \in \alpha N_y^{(1,2,3)} \wedge H \cap E = \emptyset)$.

Definition 2.3 If Ω is the class of all FTTSs. The predicates $\alpha T_i^{(1,2,3)}, \alpha R_i^{(1,2,3)} \in \Im(\Omega)$, $i = 0,1,2$, are defined as follow
$(X, \tau_1, \tau_2, \tau_3) \in \alpha T_0^{(1,2,3)} \coloneqq \forall\, x\, \forall\, y\, (x \in X \wedge y \in X \wedge x \neq y \to \alpha\mathcal{K}_{x,y}^{(1,2,3)})$;
$(X, \tau_1, \tau_2, \tau_3) \in \alpha T_1^{(1,2,3)} \coloneqq \forall\, x\, \forall\, y\, (x \in X \wedge y \in X \wedge x \neq y \to \alpha\mathcal{H}_{x,y}^{(1,2,3)})$;
$(X, \tau_1, \tau_2, \tau_3) \in \alpha T_2^{(1,2,3)} \coloneqq \forall\, x\, \forall\, y\, (x \in X \wedge y \in X \wedge x \neq y \to \alpha\mathcal{M}_{x,y}^{(1,2,3)})$;

$(X, \tau_1, \tau_2, \tau_3) \in \alpha R_0^{(1,2,3)} \coloneqq \forall\, x\, \forall\, y\, (x \in X \wedge y \in X \wedge x \neq y\ \to (\alpha\mathcal{K}_{x,y}^{(1,2,3)} \to \alpha\mathcal{H}_{x,y}^{(1,2,3)})$;
$(X, \tau_1, \tau_2, \tau_3) \in \alpha R_1^{(1,2,3)} \coloneqq \forall\, x\, \forall\, y\, (x \in X \wedge y \in X \wedge x \neq y\ \to (\alpha\mathcal{K}_{x,y}^{(1,2,3)} \to \alpha\mathcal{M}_{x,y}^{(1,2,3)})$;
$(X, \tau_1, \tau_2, \tau_3) \in \alpha R_2^{(1,2,3)} \coloneqq \forall\, x\, \forall\, y\, (x \in X \wedge y \in X \wedge x \neq y\ \to (\alpha\mathcal{H}_{x,y}^{(1,2,3)} \to \alpha\mathcal{M}_{x,y}^{(1,2,3)})$.

Definition 2.4 If Ω is the class of all FTTSs. The predicates $\alpha\mathcal{R}^{(1,2,3)}, \alpha\mathcal{N}^{(1,2,3)} \in \Im(\Omega)$, are defined as follow
(1)$(X, \tau_1, \tau_2, \tau_3) \in \alpha\mathcal{R}^{(1,2,3)} \coloneqq \forall\, x\, \forall\, U\, (x \in X \wedge U \in \alpha\mathcal{F}_{(1,2,3)} \wedge x \notin U \to \exists\, G\, \exists\, H\, (G \in \alpha N_x^{(1,2,3)} \wedge H \in \alpha\tau_{(1,2,3)} \wedge U \subseteq H \wedge G \cap H = \emptyset))$;
(2)$(X, \tau_1, \tau_2, \tau_3) \in \alpha\mathcal{N}^{(1,2,3)} \coloneqq \forall\, G\, \forall\, H\, (G \in \alpha\mathcal{F}_{(1,2,3)} \wedge H \in \alpha\mathcal{F}_{(1,2,3)} \wedge G \cap H = \emptyset) \to \exists\, U\, \exists\, V\, (U \in \alpha\tau_{(1,2,3)} \wedge V \in \alpha\tau_{(1,2,3)} \wedge G \subseteq V \wedge H \subseteq U \wedge U \cap V = \emptyset)$.

Definition 2.5 If Ω is the class of all FTTSs. The predicates $\alpha T_3^{(1,2,3)}$, $\alpha T_4^{(1,2,3)} \in \Im(\Omega)$ are defined as follow
(1)$\alpha T_3^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3) \coloneqq \alpha\mathcal{R}^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3) \wedge \alpha T_1^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)$;
(2)$\alpha T_4^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3) \coloneqq \alpha\mathcal{N}^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3) \wedge \alpha T_1^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)$.

Remark 2.6 If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS. Note that
(1) $\alpha T_i^{(1,2,3)} = \alpha T_i^{(3,2,1)}$, $i = 0,1,2,3,4$;
(2) $\alpha R_i^{(1,2,3)} = \alpha R_i^{(3,2,1)}$, $i = 0,1,2$.

Lemma 2.7 If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS. Then
(1)$\models\ \alpha\mathcal{M}_{x,y}^{(1,2,3)} \to \alpha\mathcal{H}_{x,y}^{(1,2,3)}$;
(2)$\models\ \alpha\mathcal{H}_{x,y}^{(1,2,3)} \to \alpha\mathcal{K}_{x,y}^{(1,2,3)}$;
(3)$\models\ \alpha\mathcal{M}_{x,y}^{(1,2,3)} \to \alpha\mathcal{K}_{x,y}^{(1,2,3)}$.
Proof.
(1) $[\alpha M_{x,y}^{(1,2,3)}] = \sup_{B \cap C = \emptyset} \min(\alpha N_x^{(1,2,3)}(B), \alpha N_y^{(1,2,3)}(C)) \leq \sup_{y \notin B, x \notin C} \min(\alpha N_x^{(1,2,3)}(B), \alpha N_y^{(1,2,3)}(C)) = [\alpha\mathcal{H}_{x,y}^{(1,2,3)}]$.
(2) $[\alpha\mathcal{K}_{x,y}^{(1,2,3)}] = \max(\sup_{y \notin A} \alpha N_x^{(1,2,3)}(A), \sup_{x \notin A} \alpha N_y^{(1,2,3)}(A)) \geq \sup_{y \notin A} \alpha N_x^{(1,2,3)}(A) \geq \sup_{y \notin A, x \notin B} \min(\alpha N_x^{(1,2,3)}(A), \alpha N_y^{(1,2,3)}(B)) = [\alpha\mathcal{H}_{x,y}^{(1,2,3)}]$.
(3) is concluded from (1) and (2) above.

Theorem 2.8 If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS. Then

$\vDash (X, \tau_1, \tau_2, \tau_3) \in \alpha T_0^{(1,2,3)} \leftrightarrow \forall\, x\, \forall\, y\, (x \in X \wedge y \in X \wedge x \neq y\; \rightarrow x \notin \alpha cl_{(1,2,3)}(\{y\}) \vee y \notin \alpha cl_{(1,2,3)}(\{x\}))$.

Proof.

$$\alpha T_0^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)$$
$$= \inf_{x \neq y} max(\sup_{y \notin A} \alpha N_x^{(1,2,3)}(A), \sup_{x \notin A} \alpha N_y^{(1,2,3)}(A))$$
$$= \inf_{x \neq y} max(\alpha N_x^{(1,2,3)}(X \sim \{y\}), \alpha N_y^{(1,2,3)}(X \sim \{x\}))$$
$$= \inf_{x \neq y} max(1 - \alpha cl_{(1,2,3)}(\{y\})(x), 1 - \alpha cl_{(1,2,3)}(\{x\})(y))$$
$$= [\forall\, x\, \forall\, y\, (x \in X \wedge y \in X \wedge x \neq y\; \rightarrow x \notin \alpha cl_{(1,2,3)}(\{y\}) \vee y \notin \alpha cl_{(1,2,3)}(\{x\}))].$$

Theorem 2.9 If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS. Then

$$\vDash \forall\, x\, (\{x\} \in \alpha \mathcal{F}_{(1,2,3)}) \leftrightarrow (X, \tau_1, \tau_2, \tau_3) \in \alpha T_1^{(1,2,3)}.$$

Proof.

$$\alpha T_1^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)$$
$$= \inf_{x_1 \neq x_2} min(\sup_{x_2 \notin A} \alpha N_{x_1}^{(1,2,3)}(A), \sup_{x_1 \notin B} \alpha N_{x_2}^{(1,2,3)}(B)) =$$
$$\inf_{x_1 \neq x_2} min(\alpha N_{x_1}^{(1,2,3)}(X \sim \{x_2\}), \alpha N_{x_2}^{(1,2,3)}(X \sim \{x_1\})) \leq$$
$$\inf_{x_1 \neq x_2} \alpha N_{x_1}^{(1,2,3)}(X \sim \{x_2\}) = \inf_{x_2 \in X} \inf_{x_1 \in X \sim \{x_2\}} \alpha N_{x_1}^{(1,2,3)}(X \sim \{x_2\})$$
$$= \inf_{x_2 \in X} \alpha \tau_{(1,2,3)}(X \sim \{x_2\}) = \inf_{x \in X} \alpha \tau_{(1,2,3)}(X \sim \{x\}) = \inf_{x \in X} \alpha \mathcal{F}_{(1,2,3)}(\{x\}).$$

Now, for any $x_1, x_2 \in X$ with $x_1 \neq x_2$.

$$[\forall\, x\, (\{x\} \in \alpha \mathcal{F}_{(1,2,3)})]$$
$$= \inf_{x \in X} [\{x\} \in \alpha \mathcal{F}_{(1,2,3)}] = \inf_{x \in X} \alpha \tau_{(1,2,3)}(X \sim \{x\}) = \inf_{x \in X} \inf_{y \in X \sim \{x\}} \alpha N_y^{(1,2,3)}(X \sim \{x\})$$
$$\leq \inf_{y \in X \sim \{x_2\}} \alpha N_y^{(1,2,3)}(X \sim \{x_2\}) \leq \alpha N_{x_2}^{(1,2,3)}(X \sim \{x_2\}) = \sup_{x_2 \notin A} \alpha N_{x_1}^{(1,2,3)}(A).$$

By the same way, we have

$[\forall\, x\, (\{x\} \in \alpha \mathcal{F}_{(1,2,3)})] \leq \sup_{x_1 \notin A} \alpha N_{x_2}^{(1,2,3)}(B)$. *So*

$$[\forall\, x\, (\{x\} \in \alpha \mathcal{F}_{(1,2,3)})] \leq \inf_{x_1 \neq x_2} min(\sup_{x_2 \notin A} \alpha N_{x_1}^{(1,2,3)}(A), \sup_{x_1 \notin B} \alpha N_{x_2}^{(1,2,3)}(B))$$
$$= \alpha T_1^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3).$$

Therefore $\alpha T_1^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3) = [\forall\, x\, (\{x\} \in \alpha \mathcal{F}_{(1,2,3)})]$.

Definition 2.10 If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS, we define

(1)$^{(1)}\alpha \mathcal{R}^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3) \coloneqq \forall\, x\, \forall\, U\, (x \in X \wedge U \in \alpha \mathcal{F}_{(1,2,3)} \wedge x \notin U \rightarrow \exists\, G\, (G \in \alpha N_x^{(1,2,3)} \wedge \alpha cl_{(1,2,3)}(G) \cap U = \emptyset))$;

(2)$^{(2)}\alpha \mathcal{R}^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3) \coloneqq \forall\, x\, \forall\, U\, (x \in X \wedge U \in \alpha \tau_{(1,2,3)} \wedge x \in U \rightarrow \exists\, G\, \exists\, H\, (G \in \alpha N_x^{(1,2,3)} \wedge H \in \alpha \tau_{(1,2,3)} \wedge G \subseteq U \wedge G \cap H = \emptyset))$.

Theorem 2.11 If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS. Then
$\vDash \alpha\mathcal{R}^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3) \leftrightarrow {}^{(i)}\alpha\mathcal{R}^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)$, $i = 1,2$.

Proof.

(a) $[{}^{(1)}\alpha\mathcal{R}^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)]$

$$= \inf_{x\notin U} \min(1, 1 - \alpha\mathcal{F}_{(1,2,3)}(U) + \sup_{G\in P(X)} \min(\alpha N_x^{(1,2,3)}(G), \inf_{y\in U} (1 - \alpha cl_{(1,2,3)}(G)(y))))$$

$$= \inf_{x\notin U} \min(1, 1 - \alpha\mathcal{F}_{(1,2,3)}(U) + \sup_{G\in P(X)} \min(\alpha N_x^{(1,2,3)}(G), \inf_{y\in U} \alpha N_y^{(1,2,3)}(X\sim G)))$$

$$= \inf_{x\notin U} \min(1, 1 - \alpha\mathcal{F}_{(1,2,3)}(U) + \sup_{G\cap U=\emptyset, G\in P(X)} \min(\alpha N_x^{(1,2,3)}(G), \inf_{y\in U} \alpha N_y^{(1,2,3)}(X\sim G)))$$

$$= \inf_{x\notin U} \min(1, 1 - \alpha\mathcal{F}_{(1,2,3)}(U) + \sup_{G\cap H=\emptyset, G\in P(X)} \min(\alpha N_x^{(1,2,3)}(G), \inf_{y\in U} \sup_{y\in H\subseteq X\sim G} \alpha\tau_{(1,2,3)}(H)))$$

$$= \inf_{x\notin U} \min(1, 1 - \alpha\mathcal{F}_{(1,2,3)}(U) + \sup_{G\cap H=\emptyset, G\in P(X)} \min(\alpha N_x^{(1,2,3)}(G), \sup_{G\cap H=\emptyset, U\subseteq H} \alpha\tau_{(1,2,3)}(H)))$$

$$= \inf_{x\notin U} \min(1, 1 - \alpha\mathcal{F}_{(1,2,3)}(U) + \sup_{G\cap H=\emptyset, G\in P(X)} \sup_{G\cap H=\emptyset, U\subseteq H} \min(\alpha N_x^{(1,2,3)}(G), \alpha\tau_{(1,2,3)}(H)))$$

$$= \inf_{x\notin U} \min(1, 1 - \alpha\mathcal{F}_{(1,2,3)}(U) + \sup_{G\cap H=\emptyset, U\subseteq H} \min(\alpha N_x^{(1,2,3)}(G), \alpha\tau_{(1,2,3)}(H)))$$

$= [\alpha\mathcal{R}^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)]$.

(b) $[{}^{(2)}\alpha\mathcal{R}^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)]$

$$= \inf_{x\in U} \min(1, 1 - \alpha\tau_{(1,2,3)}(U) + \sup_{G\cap H=\emptyset, G\subseteq U} \min(\alpha N_x^{(1,2,3)}(G), \alpha\tau_{(1,2,3)}(H)))$$

$$= \inf_{x\notin X\sim U} \min(1, 1 - \alpha\mathcal{F}_{(1,2,3)}(X\sim U) + \sup_{G\cap X\sim U=\emptyset, H\subseteq X\sim U} \min(\alpha N_x^{(1,2,3)}(G), \alpha\tau_{(1,2,3)}(H)))$$

$= [\alpha\mathcal{R}^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)]$.

Definition 2.12 If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS, we define

(1)$^{(1)}\alpha\mathcal{N}^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3) \coloneqq \forall G\, \forall H\, (G \in \alpha\mathcal{F}_{(1,2,3)} \wedge H \in \alpha\tau_{(1,2,3)} \wedge G \subseteq H \rightarrow \exists U\, \exists V\, (U \in \alpha\mathcal{F}_{(1,2,3)} \wedge V \in \alpha\tau_{(1,2,3)} \wedge U \subseteq V \wedge V\cap H = \emptyset))$;

(2)$^{(2)}\alpha\mathcal{N}^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3) \coloneqq \forall G\, \forall H\, (G \in \alpha\mathcal{F}_{(1,2,3)} \wedge H \in \alpha\mathcal{F}_{(1,2,3)} \wedge G\cap H = \emptyset \rightarrow \exists U\, (U \in \alpha\tau_{(1,2,3)} \wedge G \subseteq U \wedge \alpha cl_{(1,2,3)}(U)\cap H = \emptyset))$.

Theorem 2.13 If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS. Then

$\vDash \alpha\mathcal{N}^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3) \leftrightarrow {}^{(i)}\alpha\mathcal{N}^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3)$, $i = 1,2$.

Proof.

(a) $[{}^{(1)}\alpha\mathcal{N}^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3)]$

$$= \inf_{G\subseteq H} min(1, 1 - min(\alpha\mathcal{F}_{(1,2,3)}(G), \alpha\tau_{(1,2,3)}(H)) + \sup_{E\subseteq F, F\cap H=\emptyset} min(\alpha\mathcal{F}_{(1,2,3)}(E), \alpha\tau_{(1,2,3)}(F)))$$

$$= \inf_{G\cap X\sim H=\emptyset} min(1, 1 - min(\alpha\mathcal{F}_{(1,2,3)}(G), \alpha\mathcal{F}_{(1,2,3)}(X\sim H)) + \sup_{X\sim E\cap F=\emptyset, G\subseteq X\sim E, F\subseteq X\sim H} min(\alpha\tau_{(1,2,3)}(X\sim E), \alpha\tau_{(1,2,3)}(F)))$$

$$= [\alpha\mathcal{N}^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3)].$$

(b) is analogous to the proof of (a) of Theorem (2.11).

3 Relations among α-separation axioms in fuzzifying tri-topological spaces

Theorem 3.1 If (X,τ_1,τ_2,τ_3) is a FTTS. Then

(1)$\vDash (X,\tau_1,\tau_2,\tau_3) \in \alpha T_1^{(1,2,3)} \to (X,\tau_1,\tau_2,\tau_3) \in \alpha T_0^{(1,2,3)}$;

(2)$\vDash (X,\tau_1,\tau_2,\tau_3) \in \alpha T_2^{(1,2,3)} \to (X,\tau_1,\tau_2,\tau_3) \in \alpha T_i^{(1,2,3)}$, $i = 0,1$.

Proof. From Lemma (2.7), it is clear.

Theorem 3.2 If (X,τ_1,τ_2,τ_3) is a FTTS. Then

(1) $\vDash (X,\tau_1,\tau_2,\tau_3) \in \alpha R_1^{(1,2,3)} \to (X,\tau_1,\tau_2,\tau_3) \in \alpha R_i^{(1,2,3)}$, $i = 0,2$;

(2) $\vDash (X,\tau_1,\tau_2,\tau_3) \in \alpha T_1^{(1,2,3)} \to (X,\tau_1,\tau_2,\tau_3) \in \alpha R_0^{(1,2,3)}$;

(3) $\vDash (X,\tau_1,\tau_2,\tau_3) \in \alpha T_2^{(1,2,3)} \to (X,\tau_1,\tau_2,\tau_3) \in \alpha R_i^{(1,2,3)}$, $i = 0,1,2$.

Proof.

(1) (a) From (1) of Lemma (2.7), we have

$$\alpha R_1^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3) = \inf_{x\neq y} min(1, 1 - [\alpha\mathcal{K}_{x,y}^{(1,2,3)}] + [\alpha\mathcal{M}_{x,y}^{(1,2,3)}])$$
$$\leq \inf_{x\neq y} min(1, 1 - [\alpha\mathcal{K}_{x,y}^{(1,2,3)}] + [\alpha\mathcal{H}_{x,y}^{(1,2,3)}])$$
$$= \alpha R_0^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3)$$

(b) From (2) of Lemma (2.7), we have

$$\alpha R_1^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3) = \inf_{x\neq y} min(1, 1 - [\alpha\mathcal{K}_{x,y}^{(1,2,3)}] + [\alpha\mathcal{M}_{x,y}^{(1,2,3)}])$$
$$\leq \inf_{x\neq y} min(1, 1 - [\alpha\mathcal{H}_{x,y}^{(1,2,3)}] + [\alpha\mathcal{M}_{x,y}^{(1,2,3)}])$$
$$= \alpha R_2^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3)$$

(2) Using Lemma 2.2 in [2], we have

$$\alpha T_1^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3) = \inf_{x\neq y} [\alpha\mathcal{H}_{x,y}^{(1,2,3)}]$$
$$\leq \inf_{x\neq y} [\alpha\mathcal{K}_{x,y}^{(1,2,3)} \to \alpha\mathcal{H}_{x,y}^{(1,2,3)}]$$

$$= \alpha R_0^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3).$$

(3) (a) From (2) above and (2) of Theorem (3.1), we have

$$\alpha T_2^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3) = \inf_{x\neq y} [\alpha \mathcal{M}_{x,y}^{(1,2,3)}] \leq \inf_{x\neq y} [\alpha \mathcal{K}_{x,y}^{(1,2,3)} \to \alpha \mathcal{M}_{x,y}^{(1,2,3)}]$$
$$= \alpha R_1^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3)$$
$$= \inf_{x\neq y} \min(1, 1-[\alpha \mathcal{K}_{x,y}^{(1,2,3)}] + [\alpha \mathcal{M}_{x,y}^{(1,2,3)}])$$
$$\leq \inf_{x\neq y} \min(1, 1-[\alpha \mathcal{K}_{x,y}^{(1,2,3)}] + [\alpha \mathcal{H}_{x,y}^{(1,2,3)}])$$
$$= \alpha R_0^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3).$$

(b) Using Lemma 2.2 in [2], we have

$$\alpha T_2^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3) = \inf_{x\neq y} [\alpha \mathcal{M}_{x,y}^{(1,2,3)}]$$
$$\leq \inf_{x\neq y} [\alpha \mathcal{K}_{x,y}^{(1,2,3)} \to \alpha \mathcal{M}_{x,y}^{(1,2,3)}]$$
$$= \alpha R_1^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3).$$

(c) Using Lemma 2.2 in [2], we have

$$\alpha T_2^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3) = \inf_{x\neq y} [\alpha \mathcal{M}_{x,y}^{(1,2,3)}]$$
$$\leq \inf_{x\neq y} [\alpha \mathcal{H}_{x,y}^{(1,2,3)} \to \alpha \mathcal{M}_{x,y}^{(1,2,3)}]$$
$$= \alpha R_2^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3).$$

Theorem 3.3 If (X,τ_1,τ_2,τ_3) is a FTTS. Then
$\models \alpha \mathcal{R}^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3) \wedge \alpha T_1^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3) \to \alpha T_2^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3).$

Proof.

It suffices to show that
$[\alpha T_2^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3)] \geq max(0, \alpha \mathcal{R}^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3))] +$
$[\alpha T_1^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3)] - 1).$
Since $[\alpha T_2^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3)] \geq 0$.
Then from Theorem (3.2), we have

$$[\alpha T_1^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3)] = \inf_{x\in X} \alpha \mathcal{F}_{(1,2,3)}(\{x\}) = \inf_{x\in X} \alpha\tau_{(1,2,3)}(X\sim\{x\})$$

So $[\alpha \mathcal{R}^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3)] + [\alpha T_1^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3)]$

$$= \inf_{x\notin U} \min(1, 1-\alpha\tau_{(1,2,3)}(X\sim U)$$
$$+ \sup_{G\cap H=\emptyset, U\subseteq H} \min(\alpha N_x^{(1,2,3)}(G), \alpha\tau_{(1,2,3)}(H))) + \inf_{z\in X} \alpha\tau_{(1,2,3)}(X\sim\{z\})$$
$$\leq \inf_{x\in X, x\neq y}\ \inf_{y\in X} \min(1, 1-\alpha\tau_{(1,2,3)}(X\sim\{y\}) +$$
$$\sup_{G\cap H=\emptyset, y\in H} \min(\alpha N_x^{(1,2,3)}(G), \alpha\tau_{(1,2,3)}(H))) + \inf_{z\in X} \alpha\tau_{(1,2,3)}(X\sim\{z\}))$$
$$\leq \inf_{x\in X, x\neq y}\ \inf_{y\in X} \min(1, 1-\alpha\tau_{(1,2,3)}(X\sim\{y\}) +$$

$$\sup_{G\cap H=\emptyset, y\in H} min(\alpha N_x^{(1,2,3)}(G), \alpha N_y^{(1,2,3)}(H))) + \alpha\tau_{(1,2,3)}(X\sim\{y\}))$$

$= \inf_{x\in X, x\neq y} \inf_{y\in X} (min(1,1 + \sup_{G\cap H=\emptyset} min(\alpha N_x^{(1,2,3)}(G), \alpha N_y^{(1,2,3)}(H))))$

$= \inf_{x\in X, x\neq y} \inf_{y\in X} (1 + \sup_{G\cap H=\emptyset} min(\alpha N_x^{(1,2,3)}(G), \alpha N_y^{(1,2,3)}(H)))$

$= 1 + \inf_{x\neq y} \sup_{G\cap H=\emptyset} min(\alpha N_x^{(1,2,3)}(G), \alpha N_y^{(1,2,3)}(H))$

$= 1 + [\alpha T_2^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)]$.

Thus

$[\alpha T_2^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)] \geq max(0, \alpha\mathcal{R}^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3))] + [\alpha T_1^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)] - 1)$.

Corollary 3.4 If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS. Then

(1) $\vDash \alpha T_3^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3) \rightarrow \alpha T_2^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)$.

(2) $\vDash \alpha T_3^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3) \rightarrow \alpha R_i^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)$, $i = 0,1,2$.

Theorem 3.5 If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS. Then

$\vDash \alpha T_4^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3) \rightarrow \alpha\mathcal{R}^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)$.

Proof.

$\alpha T_4^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3) = max(0, [\alpha\mathcal{N}^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3))] + [\alpha T_1^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)] - 1)$,

now we prove that

$[\alpha\mathcal{R}^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)] \geq [\alpha\mathcal{N}^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)] + [\alpha T_1^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)] - 1$.

In fact

$[\alpha\mathcal{N}^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)] + [\alpha T_1^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)]$

$= \inf_{U\cap V=\emptyset} min(1,1 - min(\alpha\mathcal{F}_{(1,2,3)}(U), \alpha\mathcal{F}_{(1,2,3)}(V))$

$+ \sup_{G\cap H=\emptyset, U\subseteq H, V\subseteq G} min(\alpha\tau_{(1,2,3)}(G), \alpha\tau_{(1,2,3)}(H))) + \inf_{z\in X} \alpha\tau_{(1,2,3)}(X\sim\{z\})$

$= \inf_{U\cap V=\emptyset} min(1,1 - min(\alpha\tau_{(1,2,3)}(X\sim U), \alpha\tau_{(1,2,3)}(X\sim V))$

$+ \sup_{G\cap H=\emptyset, U\subseteq H, V\subseteq G} min(\alpha\tau_{(1,2,3)}(G), \alpha\tau_{(1,2,3)}(H))) + \inf_{z\in X} \alpha\tau_{(1,2,3)}(X\sim\{z\})$

$\leq \inf_{x\notin U} min(1,1 - min(\alpha\tau_{(1,2,3)}(X\sim U), \alpha\tau_{(1,2,3)}(X\sim\{x\}))$

$+ \sup_{G\cap H=\emptyset, U\subseteq H} min(\alpha N_x^{(1,2,3)}(G), \alpha\tau_{(1,2,3)}(H)) + \inf_{z\in X} \alpha\tau_{(1,2,3)}(X\sim\{z\})$

$= \inf_{x\notin U} min(1, max(1 - \alpha\tau_{(1,2,3)}(X\sim U)$

$+ \sup_{G\cap H=\emptyset, U\subseteq H} min(\alpha N_x^{(1,2,3)}(G), \alpha\tau_{(1,2,3)}(H)), 1 - \alpha\tau_{(1,2,3)}(X\sim\{x\})$

$+ \sup_{G\cap H=\emptyset, U\subseteq H} min(\alpha N_x^{(1,2,3)}(G), \alpha\tau_{(1,2,3)}(H))) + \inf_{z\in X} \alpha\tau_{(1,2,3)}(X\sim\{z\})$

$$= \inf_{x \notin U} max(min(1, 1 - \alpha\tau_{(1,2,3)}(X \sim U) + \sup_{G \cap H = \emptyset, U \subseteq H} min(\alpha N_x^{(1,2,3)}(G), \alpha\tau_{(1,2,3)}(H))), min(1, 1 - \alpha\tau_{(1,2,3)}(X \sim \{x\}) + \sup_{G \cap H = \emptyset, U \subseteq H} min(\alpha N_x^{(1,2,3)}(G), \alpha\tau_{(1,2,3)}(H))) + \inf_{z \in X} \alpha\tau_{(1,2,3)}(X \sim \{z\})$$

$$\leq \inf_{x \notin U} max(min(1, 1 - \alpha\tau_{(1,2,3)}(X \sim U) + \sup_{G \cap H = \emptyset, U \subseteq H} min(\alpha N_x^{(1,2,3)}(G), \alpha\tau_{(1,2,3)}(H))) + \alpha\tau_{(1,2,3)}(X \sim \{x\}), min(1, 1 - \alpha\tau_{(1,2,3)}(X \sim \{x\})) + \sup_{G \cap H = \emptyset, U \subseteq H} min(\alpha N_x^{(1,2,3)}(G), \alpha\tau_{(1,2,3)}(H))) + \alpha\tau_{(1,2,3)}(X \sim \{x\}))$$

$$\leq \inf_{x \notin U} max(min(1, 1 - \alpha\tau_{(1,2,3)}(X \sim U) + \sup_{G \cap H = \emptyset, U \subseteq H} min(\alpha N_x^{(1,2,3)}(G), \alpha\tau_{(1,2,3)}(H))) + \alpha\tau_{(1,2,3)}(X \sim \{x\}), 1 + \sup_{G \cap H = \emptyset, U \subseteq H} min(\alpha N_x^{(1,2,3)}(G), \alpha\tau_{(1,2,3)}(H)))$$

$$\leq \inf_{x \notin U} min(1, 1 - \alpha\tau_{(1,2,3)}(X \sim U) + \sup_{G \cap H = \emptyset, U \subseteq H} min(\alpha N_x^{(1,2,3)}(G), \alpha\tau_{(1,2,3)}(H))) + 1$$

$$= [\alpha\mathcal{R}^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)] + 1.$$

Theorem 3.6 If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS. Then

(1) $\models (X, \tau_1, \tau_2, \tau_3) \in \alpha T_1^{(1,2,3)} \to (X, \tau_1, \tau_2, \tau_3) \in \alpha R_0^{(1,2,3)} \wedge (X, \tau_1, \tau_2, \tau_3) \in \alpha T_0^{(1,2,3)}$;

(2) If $\alpha T_0^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3) = 1$, then $\models (X, \tau_1, \tau_2, \tau_3) \in \alpha T_1^{(1,2,3)} \leftrightarrow (X, \tau_1, \tau_2, \tau_3) \in \alpha R_0^{(1,2,3)} \wedge (X, \tau_1, \tau_2, \tau_3) \in \alpha T_0^{(1,2,3)}$.

Proof.

(1) Follows from (1) of Theorem (3.1) and (2) of Theorem (3.2).

(2) Since $\alpha T_0^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3) = 1$, then for every $x, y \in X$ such that $x \neq y$, we have $[\alpha\mathcal{K}_{x,y}^{(1,2,3)}] = 1$. So

$$\alpha R_0^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3) \wedge \alpha T_0^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)$$
$$= \alpha R_0^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3)$$
$$= \inf_{x \neq y} min(1, 1 - [\alpha\mathcal{K}_{x,y}^{(1,2,3)}] + [\alpha\mathcal{H}_{x,y}^{(1,2,3)}])$$
$$= \inf_{x \neq y} \left[\alpha\mathcal{H}_{x,y}^{(1,2,3)}\right] = \alpha T_1^{(1,2,3)}(X, \tau_1, \tau_2, \tau_3).$$

Theorem 3.7 If $(X, \tau_1, \tau_2, \tau_3)$ is a FTTS. Then

(1) $\vDash (X,\tau_1,\tau_2,\tau_3) \in \alpha T_2^{(1,2,3)} \to (X,\tau_1,\tau_2,\tau_3) \in \alpha R_1^{(1,2,3)} \wedge (X,\tau_1,\tau_2,\tau_3) \in \alpha T_0^{(1,2,3)}$;

(2) If $\alpha T_0^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3) = 1$, then
$\vDash (X,\tau_1,\tau_2,\tau_3) \in \alpha T_2^{(1,2,3)} \leftrightarrow (X,\tau_1,\tau_2,\tau_3) \in \alpha R_1^{(1,2,3)} \wedge (X,\tau_1,\tau_2,\tau_3) \in \alpha T_0^{(1,2,3)}$.

Proof.
(1) Follows from (3) and (4) of Theorems (3.1) and (3.2) respectively.
(2) Likewise from (2) theorem 3.6.

Theorem 3.8 If (X,τ_1,τ_2,τ_3) is a FTTS. Then
(1) $\vDash (X,\tau_1,\tau_2,\tau_3) \in \alpha T_2^{(1,2,3)} \to (X,\tau_1,\tau_2,\tau_3) \in \alpha R_0^{(1,2,3)} \wedge (X,\tau_1,\tau_2,\tau_3) \in \alpha T_1^{(1,2,3)}$;

(2) If $\alpha T_1^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3) = 1$, then
$\vDash (X,\tau_1,\tau_2,\tau_3) \in \alpha T_2^{(1,2,3)} \leftrightarrow (X,\tau_1,\tau_2,\tau_3) \in \alpha R_0^{(1,2,3)} \wedge (X,\tau_1,\tau_2,\tau_3) \in \alpha T_1^{(1,2,3)}$.

Proof.
(1) Follows from (2) and (3) of Theorems (3.1) and (3.2) respectively.
(2) Likewise from (3) Theorem 3.6.

Remark 3.9 If (X,τ_1,τ_2,τ_3) is a FTTS. Then we have
(1) $\vDash (X,\tau_1,\tau_2,\tau_3) \in \alpha T_1^{(1,2,3)} \to (X,\tau_1,\tau_2,\tau_3) \in \alpha R_0^{(1,2,3)} \wedge (X,\tau_1,\tau_2,\tau_3) \in \alpha T_0^{(1,2,3)}$;

(2) $\vDash (X,\tau_1,\tau_2,\tau_3) \in \alpha T_2^{(1,2,3)} \to (X,\tau_1,\tau_2,\tau_3) \in \alpha R_1^{(1,2,3)} \wedge (X,\tau_1,\tau_2,\tau_3) \in \alpha T_0^{(1,2,3)}$.

(3) $\vDash (X,\tau_1,\tau_2,\tau_3) \in \alpha T_2^{(1,2,3)} \to (X,\tau_1,\tau_2,\tau_3) \in \alpha R_0^{(1,2,3)} \wedge (X,\tau_1,\tau_2,\tau_3) \in \alpha T_1^{(1,2,3)}$.

Theorem 3.10 If (X,τ_1,τ_2,τ_3) is a FTTS. Then
(1) $(X,\tau_1,\tau_2,\tau_3) \in \alpha R_0^{(1,2,3)} \wedge (X,\tau_1,\tau_2,\tau_3) \in \alpha T_0^{(1,2,3)} \to (X,\tau_1,\tau_2,\tau_3) \in \alpha T_1^{(1,2,3)}$;

(2) If $\alpha T_0^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3) = 1$, then
$\vDash (X,\tau_1,\tau_2,\tau_3) \in \alpha R_0^{(1,2,3)} \wedge (X,\tau_1,\tau_2,\tau_3) \in \alpha T_0^{(1,2,3)} \leftrightarrow (X,\tau_1,\tau_2,\tau_3) \in \alpha T_1^{(1,2,3)}$.

Proof.
(1) $[(X,\tau_1,\tau_2,\tau_3) \in \alpha R_0^{(1,2,3)} \wedge (X,\tau_1,\tau_2,\tau_3) \in \alpha T_0^{(1,2,3)}]$

$$= max(0, \alpha R_0^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3) + \alpha T_0^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3) - 1)$$

$$= max(0, \inf_{x\neq y} min(1,1 - [\alpha\mathcal{K}_{x,y}^{(1,2,3)}] + [\alpha\mathcal{H}_{x,y}^{(1,2,3)}]) + \inf_{x\neq y} [\alpha\mathcal{K}_{x,y}^{(1,2,3)}] - 1)$$

$\leq max(0, \inf_{x\neq y} (min(1, 1-[\alpha\mathcal{K}_{x,y}^{(1,2,3)}]+[\alpha\mathcal{H}_{x,y}^{(1,2,3)}])+[\alpha\mathcal{K}_{x,y}^{(1,2,3)}]-1)$

$\leq max(0, \inf_{x\neq y} (1-[\alpha\mathcal{K}_{x,y}^{(1,2,3)}]+[\alpha\mathcal{H}_{x,y}^{(1,2,3)}]+[\alpha\mathcal{K}_{x,y}^{(1,2,3)}]-1)$

$= \inf_{x\neq y} [\alpha\mathcal{H}_{x,y}^{(1,2,3)}] = \alpha T_1^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3)$.

(2) Follows from (2) Theorem (3.6).

Theorem 3.11 If (X,τ_1,τ_2,τ_3) is a FTTS. Then

(1) $(X,\tau_1,\tau_2,\tau_3) \in \alpha R_1^{(1,2,3)} \wedge (X,\tau_1,\tau_2,\tau_3) \in \alpha T_0^{(1,2,3)} \to (X,\tau_1,\tau_2,\tau_3) \in \alpha T_2^{(1,2,3)}$;

(2) If $\alpha T_0^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3) = 1$, then
$\models (X,\tau_1,\tau_2,\tau_3) \in \alpha R_1^{(1,2,3)} \wedge (X,\tau_1,\tau_2,\tau_3) \in \alpha T_0^{(1,2,3)} \leftrightarrow (X,\tau_1,\tau_2,\tau_3) \in \alpha T_2^{(1,2,3)}$.

Proof.

(1) $[(X,\tau_1,\tau_2,\tau_3) \in \alpha R_1^{(1,2,3)} \wedge (X,\tau_1,\tau_2,\tau_3) \in \alpha T_0^{(1,2,3)}]$

$= max(0, \alpha R_1^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3) + \alpha T_0^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3) - 1)$

$= max(0, \inf_{x\neq y} min(1, 1-[\alpha\mathcal{K}_{x,y}^{(1,2,3)}]+[\alpha\mathcal{M}_{x,y}^{(1,2,3)}]) + \inf_{x\neq y} [\alpha\mathcal{K}_{x,y}^{(1,2,3)}]-1)$

$\leq max(0, \inf_{x\neq y} (min(1, 1-[\alpha\mathcal{K}_{x,y}^{(1,2,3)}]+[\alpha\mathcal{M}_{x,y}^{(1,2,3)}])+[\alpha\mathcal{K}_{x,y}^{(1,2,3)}]-1)$

$\leq max(0, \inf_{x\neq y} (1-[\alpha\mathcal{K}_{x,y}^{(1,2,3)}]+[\alpha\mathcal{M}_{x,y}^{(1,2,3)}]+[\alpha\mathcal{K}_{x,y}^{(1,2,3)}]-1)$

$= \inf_{x\neq y} [\alpha\mathcal{M}_{x,y}^{(1,2,3)}] = \alpha T_2^{(1,2,3)}(X,\tau_1,\tau_2,\tau_3)$.

(2) Follows from (2) Theorem (3.6).

Theorem 3.12 If (X,τ_1,τ_2,τ_3) is a FTTS. Then

(1) $\models (X,\tau_1,\tau_2,\tau_3) \in \alpha T_0^{(1,2,3)} \to ((X,\tau_1,\tau_2,\tau_3) \in \alpha R_0^{(1,2,3)} \to (X,\tau_1,\tau_2,\tau_3) \in \alpha T_1^{(1,2,3)}$;

(2) $\models (X,\tau_1,\tau_2,\tau_3) \in \alpha R_0^{(1,2,3)} \to ((X,\tau_1,\tau_2,\tau_3) \in \alpha T_0^{(1,2,3)} \to (X,\tau_1,\tau_2,\tau_3) \in \alpha T_1^{(1,2,3)}$;

(3) $\models (X,\tau_1,\tau_2,\tau_3) \in \alpha T_0^{(1,2,3)} \to ((X,\tau_1,\tau_2,\tau_3) \in \alpha R_1^{(1,2,3)} \to (X,\tau_1,\tau_2,\tau_3) \in \alpha T_2^{(1,2,3)}$;

(4) $\models (X,\tau_1,\tau_2,\tau_3) \in \alpha R_1^{(1,2,3)} \to ((X,\tau_1,\tau_2,\tau_3) \in \alpha T_0^{(1,2,3)} \to (X,\tau_1,\tau_2,\tau_3) \in \alpha T_2^{(1,2,3)}$.

Proof.

(1) From (2) Theorem (3.1) and (3) Theorem (3.2), we have

$[(X,\tau_1,\tau_2,\tau_3) \in \alpha T_0^{(1,2,3)} \to ((X,\tau_1,\tau_2,\tau_3) \in \alpha R_0^{(1,2,3)} \to (X,\tau_1,\tau_2,\tau_3) \in \alpha T_1^{(1,2,3)}]$

$= min(1,1 - [(X, \tau_1, \tau_2, \tau_3) \in \alpha T_0^{(1,2,3)}] + min(1,1 - [(X, \tau_1, \tau_2, \tau_3) \in \alpha R_0^{(1,2,3)}] + [(X, \tau_1, \tau_2, \tau_3) \in \alpha T_1^{(1,2,3)}]))$
$= min(1,1 - [(X, \tau_1, \tau_2, \tau_3) \in \alpha T_0^{(1,2,3)}] + 1 - [(X, \tau_1, \tau_2, \tau_3) \in \alpha R_0^{(1,2,3)}] + [(X, \tau_1, \tau_2, \tau_3) \in \alpha T_1^{(1,2,3)}]))$
$= min(1,1 - ([(X, \tau_1, \tau_2, \tau_3) \in \alpha R_0^{(1,2,3)}] + [(X, \tau_1, \tau_2, \tau_3) \in \alpha T_0^{(1,2,3)}] - 1) + [(X, \tau_1, \tau_2, \tau_3) \in \alpha T_1^{(1,2,3)}]) = 1.$
(2) From (1) Theorem (3.1) and (3) Theorem (3.6), we have
$[(X, \tau_1, \tau_2, \tau_3) \in \alpha R_0^{(1,2,3)} \rightarrow ((X, \tau_1, \tau_2, \tau_3) \in \alpha T_0^{(1,2,3)} \rightarrow (X, \tau_1, \tau_2, \tau_3) \in \alpha T_1^{(1,2,3)}]$
$= min(1,1 - [(X, \tau_1, \tau_2, \tau_3) \in \alpha R_0^{(1,2,3)}] + min(1,1 - [(X, \tau_1, \tau_2, \tau_3) \in \alpha T_0^{(1,2,3)}] + [(X, \tau_1, \tau_2, \tau_3) \in \alpha T_1^{(1,2,3)}]))$
$= min(1,1 - [(X, \tau_1, \tau_2, \tau_3) \in \alpha R_0^{(1,2,3)}] + 1 - [(X, \tau_1, \tau_2, \tau_3) \in \alpha T_0^{(1,2,3)}] + [(X, \tau_1, \tau_2, \tau_3) \in \alpha T_1^{(1,2,3)}]))$
$= min(1,1 - ([(X, \tau_1, \tau_2, \tau_3) \in \alpha R_0^{(1,2,3)}] + [(X, \tau_1, \tau_2, \tau_3) \in \alpha T_0^{(1,2,3)}] - 1) + [(X, \tau_1, \tau_2, \tau_3) \in \alpha T_1^{(1,2,3)}]) = 1.$
(3) and (4) are likewise (2) and (3) above

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