Powers of Balancing Polynomials and Some Consequences for Fibonacci Sums

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Abstract

In this short article, we derive identities for powers of balancing and Lucas-balancing polynomials. These polynomials are a natural extension of balancing and Lucas-balancing numbers. As an application of the new identities we present closed-forms for sums with certain binomial coefficients involving balancing, Lucas-balancing, Fibonacci and Lucas numbers.

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1 Introduction and Preliminaries

Behera and Panda [1] introduced the notion of balancing numbers \( (B_n)_{n \geq 0} \). Balancing numbers are solutions of a certain Diophantine equation. They satisfy the relation \( B_{n+1} = 6B_n - B_{n-1}, n \geq 1 \), with initial values \( B_0 = 0 \) and \( B_1 = 1 \). The sequence \( C_n = \sqrt{8B_n^2 + 1} \) is called a Lucas-balancing number. It satisfies the same recurrence relation as \( B_n \): \( C_{n+1} = 6C_n - C_{n-1}, n \geq 1 \), with initial values \( C_0 = 1 \) and \( C_1 = 3 \). (\( B_n \)) is sequence A001109 in the OEIS [12], whereas (\( C_n \)) has id-number A001541 in OEIS. Ab initio both

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sequences gained some popularity and have been studied extensively. The latest results on the topic can be found in the articles [2]-[4], [6]-[9] and [11].

A natural extension of balancing and Lucas-balancing numbers are the so-called balancing and Lucas-balancing polynomials \((B_n(x))_{n \geq 0}\) and \((C_n(x))_{n \geq 0}\), respectively. See [5], [9] or [10]. Balancing polynomials are defined by the recurrence

\[ B_n(x) = 6xB_{n-1}(x) - B_{n-2}(x), \quad n \geq 2, \]

with \(B_0(x) = 0\) and \(B_1(x) = 1\). Similarly, Lucas-balancing polynomials are defined by

\[ C_n(x) = 6xC_{n-1}(x) - C_{n-2}(x), \quad n \geq 2, \]

with \(C_0(x) = 1\) and \(C_1(x) = 3x\). The argument \(x\) is assumed to be a complex number. These polynomials are connected with Chebyshev and Legendre polynomials (see [5]). The first few polynomials are

\[ B_0(x) = 0, \quad B_1(x) = 1, \quad B_2(x) = 6x, \quad B_3(x) = 36x^2 - 1, \]

\[ B_4(x) = 216x^3 - 12x, \quad B_5(x) = 1296x^4 - 108x^2 + 1, \ldots \]

and

\[ C_0(x) = 1, \quad C_1(x) = 3x, \quad C_2(x) = 18x^2 - 1, \quad C_3(x) = 108x^3 - 9x, \]

\[ C_4(x) = 648x^4 - 72x^2 + 1, \quad C_5(x) = 3888x^5 - 540x^3 + 15x, \ldots \]

Obviously, \(B_n(1) = B_n\) and \(C_n(1) = C_n\), respectively. The Binet forms for the polynomials are

\[ B_n(x) = \frac{\lambda^n(x) - \lambda^{-n}(x)}{\lambda(x) - \lambda^{-1}(x)} \quad \text{and} \quad C_n(x) = \frac{1}{2} \left( \lambda^n(x) + \lambda^{-n}(x) \right), \]

where \(\lambda(x) = 3x + \sqrt{9x^2 - 1}\) and \(\lambda^{-1}(x) = 3x - \sqrt{9x^2 - 1}\). The relations

\[ B_n(-x) = (-1)^{n+1} B_n(x) \quad \text{and} \quad C_n(-x) = (-1)^n C_n(x), \]

follow from \(\lambda(-x) = -\lambda^{-1}(x)\) and \(\lambda^{-1}(-x) = -\lambda(x)\). Also, it is not difficult to see that \(B_{-n}(x) = -B_n(x)\) and \(C_{-n}(x) = C_n(x)\).

In this paper, we present new identities for powers of balancing and Lucas-balancing polynomials. We apply these results to state closed-forms for sums of balancing, Lucas-balancing, Fibonacci and Lucas numbers with certain binomial coefficients.
2 Powers of balancing and Lucas-balancing polynomials

The following lemma will play an essential role to prove our results.

**Lemma 2.1.** Let \( z \in \mathbb{C} \) and \( m \in \mathbb{N} \). Then it holds that

\[
\left( z + \frac{1}{z} \right)^{2m} = \frac{(2m)!}{(m!)^2} + \sum_{k=1}^{m} \left( \begin{array}{c} 2m \\ m-k \\ \end{array} \right) z^{2k} + z^{-2k}, \tag{5}
\]

\[
\left( z + \frac{1}{z} \right)^{2m+1} = \sum_{k=0}^{m} \left( \begin{array}{c} 2m + 1 \\ m-k \\ \end{array} \right) z^{2k+1} + z^{-(2k+1)}, \tag{6}
\]

\[
\left( z - \frac{1}{z} \right)^{2m} = (-1)^m \frac{(2m)!}{(m!)^2} + \sum_{k=1}^{m} \left( \begin{array}{c} 2m \\ m-k \\ \end{array} \right) (-1)^{m-k} z^{2k} + z^{-2k}, \tag{7}
\]

and

\[
\left( z - \frac{1}{z} \right)^{2m+1} = \sum_{k=0}^{m} \left( \begin{array}{c} 2m + 1 \\ m-k \\ \end{array} \right) (-1)^{m-k} z^{2k+1} - z^{-(2k+1)}. \tag{8}
\]

**Proof:** We prove (5) and (6). The proofs of the other two identities use similar arguments and are omitted. From the binomial theorem we get

\[
\left( z + \frac{1}{z} \right)^m = \sum_{k=0}^{m} \left( \begin{array}{c} m \\ k \\ \end{array} \right) z^{m-2k}.
\]

Therefore,

\[
\left( z + \frac{1}{z} \right)^{2m} = \sum_{k=0}^{2m} \left( \begin{array}{c} 2m \\ k \\ \end{array} \right) z^{2m-2k} = \sum_{k=0}^{m} \left( \begin{array}{c} 2m \\ k \\ \end{array} \right) z^{2m-2k} + \sum_{k=m+1}^{2m} \left( \begin{array}{c} 2m \\ k \\ \end{array} \right) z^{2m-2k} = \sum_{k=0}^{m} \left( \begin{array}{c} 2m \\ m-k \\ \end{array} \right) z^{2k} + \sum_{k=1}^{m} \left( \begin{array}{c} 2m \\ m+k \\ \end{array} \right) z^{-2k} = \frac{(2m)!}{(m!)^2} + \sum_{k=1}^{m} \left( \begin{array}{c} 2m \\ m-k \\ \end{array} \right) z^{2k} + \sum_{k=1}^{m} \left( \begin{array}{c} 2m \\ m+k \\ \end{array} \right) z^{-2k}.
\]
and (5) follows. Similarly,

\[
\left(z + \frac{1}{z}\right)^{2m+1} = \sum_{k=0}^{m} \binom{2m + 1}{k} z^{2m-2k+1} + \sum_{k=m+1}^{2m+1} \binom{2m + 1}{k} z^{2m-2k+1}
\]

\[
= \sum_{k=0}^{m} \binom{2m + 1}{m-k} z^{2k+1} + \sum_{k=1}^{m} \binom{2m + 1}{m+k} z^{-2k+1}
\]

\[
= \sum_{k=0}^{m} \binom{2m + 1}{m-k} z^{2k+1} + \sum_{k=0}^{m} \binom{2m + 1}{m+k+1} z^{-2k-1}.
\]

□

Now, we can state the main results of this article.

**Theorem 2.2.** Let \(m, n \in \mathbb{N}\). The following identities hold for powers of balancing and Lucas-balancing polynomials:

\[
B_{2m}^n(x) = 2^{-2m}(9x^2 - 1)^m (-1)^m \frac{(2m)!}{(m!)^2} + 2 \sum_{k=1}^{m} \binom{2m}{m-k} (-1)^{m-k} C_{2kn}(x),
\]

(9)

\[
B_{2m+1}^n(x) = 2^{-2m}(9x^2 - 1)^m \sum_{k=0}^{m} \binom{2m + 1}{m-k} (-1)^{m-k} B_{(2k+1)n}(x),
\]

(10)

\[
C_{2m}^n(x) = 2^{-2m} \left( \frac{(2m)!}{(m!)^2} + 2 \sum_{k=1}^{m} \binom{2m}{m-k} C_{2kn}(x) \right),
\]

(11)

and

\[
C_{2m+1}^n(x) = 2^{-2m} \sum_{k=0}^{m} \binom{2m + 1}{m-k} C_{(2k+1)n}(x).
\]

(12)

Proof: Use the results from Lemma 2.1 with \(z = \lambda^n(x)\). □

**Corollary 2.3.** Let \(m, n \in \mathbb{N}\). Then,

\[
\sum_{k=1}^{m} \binom{2m}{m-k} (-1)^{m-k} C_{2kn} = 2^{5m-1} B_{2m}^n - \frac{1}{2} (-1)^m \frac{(2m)!}{(m!)^2},
\]

(13)

\[
\sum_{k=0}^{m} \binom{2m + 1}{m-k} (-1)^{m-k} B_{(2k+1)n} = 2^{5m} B_{2m+1}^n,
\]

(14)

\[
\sum_{k=1}^{m} \binom{2m}{m-k} C_{2kn} = 2^{2m-1} C_{2n}^m - \frac{1}{2} (2m)! \frac{(2m)!}{(m!)^2},
\]

(15)

and

\[
\sum_{k=0}^{m} \binom{2m + 1}{m-k} C_{(2k+1)n} = 2^{2m} C_{2n}^{2m+1}.
\]

(16)
Proof: Evaluate Theorem 2.2 at \( x = 1 \) and rearrange. \( \square \)

**Corollary 2.4.** Let \( F_n \) and \( L_n \) denote the Fibonacci and Lucas numbers, respectively. Then, for integers \( m, n \) and \( q \) with \( n \geq 0 \) and \( m, q \geq 1 \) (or \( m, q \geq 0 \)) we have

\[
\sum_{k=1}^{m} \binom{2m}{m-k} (-1)^{m-k} L_{4q+1} = 5^m F_{2q}^{2m} - (-1)^m \frac{(2m)!}{(m!)^2}, \tag{17}
\]

\[
\sum_{k=0}^{m} \binom{2m+1}{m-k} (-1)^{m-k} F_{2q(2k+1)} = 5^m F_{2q}^{2m+1}, \tag{18}
\]

\[
\sum_{k=1}^{m} \binom{2m}{m-k} (-1)^{k(n-1)} L_{(2q+1)2k} = (-1)^{m(n-1)} 5^m F_{(2q+1)n}^{2m} - \frac{(2m)!}{(m!)^2}, \tag{19}
\]

and

\[
\sum_{k=0}^{m} \binom{2m+1}{m-k} (-1)^{k(n-1)} F_{(2q+1)(2k+1)n} = (-1)^{m(n-1)} 5^m F_{(2q+1)n}^{2m+1}. \tag{20}
\]

In addition, we have the following closed-form evaluations:

\[
\sum_{k=1}^{m} \binom{2m}{m-k} L_{4q+1} = L_{2q}^{2m} - \frac{(2m)!}{(m!)^2}, \tag{21}
\]

\[
\sum_{k=0}^{m} \binom{2m+1}{m-k} L_{2q(2k+1)} = L_{2q}^{2m+1}, \tag{22}
\]

\[
\sum_{k=0}^{m} \binom{2m+1}{m-k} (-1)^k L_{(2q+1)(2k+1)n} = (-1)^{mn} L_{(2q+1)n}^{2m+1}, \tag{23}
\]

and

\[
\sum_{k=1}^{m} \binom{2m}{m-k} (-1)^{kn} L_{(2q+1)2kn} = (-1)^{mn} L_{(2q+1)n}^{2m} - \frac{(2m)!}{(m!)^2}. \tag{24}
\]

Proof: To prove the first four sums evaluate (9) and (10) from Theorem 2.2 at the points \( x = L_{2q}/6 \) and \( x = iL_{2q+1}/6 \) with \( i = \sqrt{-1} \) being the imaginary unit, and use the following results from [5]:

\[
B_n\left(\frac{L_{2q}}{6}\right) = \frac{F_{2q}}{F_{2q}}, \quad C_n\left(\frac{L_{2q}}{6}\right) = \frac{L_{2q}}{2},
\]

\[
B_n\left(\frac{i}{6}L_{2q+1}\right) = i^{n-1} \frac{F_{(2q+1)n}}{F_{2q+1}}, \quad C_n\left(\frac{i}{6}L_{2q+1}\right) = i^n \frac{L_{(2q+1)n}}{2}.
\]

To simplify the term \( 9x^2 - 1 \) use the known relation \( L_n^2 = 5F_n^2 + (-1)^n4 \). The remaining sums are evaluations of (11) and (12) from Theorem 2.2 at the same points. \( \square \)
3 Further Comments

With little effort we can modify the main findings stated in Theorem 2.2 to produce more results of similar nature. The obvious way how one can proceed is to differentiate both sides of the identities. To keep the article readable but to give the reader an appreciation of the extensions that are possible, we focus on just two of the identities from Theorem 2.2, equations (11) and (12).

**Theorem 3.1.** Let $m, n \in \mathbb{N}$. Then, for each $x \in \mathbb{C}$ it holds that

$$\sum_{k=1}^{m} \left( \frac{2m}{m-k} \right) kB_{2kn}(x) = m2^{2m-1}B_n(x)C_n^{2m-1}(x),$$

and

$$\sum_{k=0}^{m} \left( \frac{2n+1}{m-k} \right) (2k+1)B_{(2k+1)n}(x) = (2m+1)2^{2m}B_n(x)C_n^{2m}(x).$$

Proof: Differentiate equations (11) and (12) w.r.t. $x$ and use the relation (see [5], for instance)

$$C'_n(x) = 3nB_n(x).$$

Evaluating the equations at $x = 1$, $x = L_{2q}/6$ and $x = iL_{2q+1}/6$, respectively, we immediately get the next corollary.

**Corollary 3.2.** Let $m, n, q \in \mathbb{N}$. Then,

$$\sum_{k=1}^{m} \left( \frac{2m}{m-k} \right) kB_{2kn} = m2^{2m-1}B_nC_n^{2m-1},$$

and

$$\sum_{k=0}^{m} \left( \frac{2n+1}{m-k} \right) (2k+1)B_{(2k+1)n} = (2m+1)2^{2m}B_nC_n^{2m},$$

$$\sum_{k=1}^{m} \left( \frac{2m}{m-k} \right) kF_{4kn} = mF_{2qn}L_{2qn}^{2m-1},$$

$$\sum_{k=0}^{m} \left( \frac{2m+1}{m-k} \right) (2k+1)F_{(2k+1)2qn} = (2m+1)F_{2qn}L_{2qn}^{2m},$$

$$\sum_{k=1}^{m} \left( \frac{2m}{m-k} \right) (-1)^{kn}kF_{2bn(2q+1)} = m(-1)^{mn}F_{(2q+1)n}L_{(2q+1)n}^{2m-1},$$

and

$$\sum_{k=0}^{m} \left( \frac{2m+1}{m-k} \right) (-1)^{kn}(2k+1)F_{(2k+1)(2q+1)n} = (2m+1)(-1)^{mn}F_{(2q+1)n}L_{(2q+1)n}^{2m}.$$
References


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