International Journal of Mathematical Analysis
Vol. 13, 2019, no. 2, 71 - 79
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ijma.2019.913

Some Extreme Points

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Abstract

In this paper we will obtain using a method with principal minors, the local extreme points for two special functions which can be used for a Young-type inequality.

Mathematics Subject Classification: 26D20

Keywords: Young-type inequalities, arithmetic mean, geometric mean

1. Introduction

The classical inequality of Young is

$$a^{\nu}b^{1-\nu} < \nu a + (1-\nu)b,$$

where a and b are distinct positive real numbers and $0 < \nu < 1$, see [9].

In the paper of [1] are proven new inequalities which extend many generalizations of Young 's inequality given before. The following inequality is a refinement of the left-hand side of a refinement of the inequality of Young proved in 2010 and 2011 by Kittaneh and Manasrah [8] and [7]. Many generalizations and refinements of Young's inequality are given also in [2], [3], [4], [5] and references therein.

Theorem 1. *([1]) Let λ, ν and τ be real numbers with $\lambda \geq 1$ and $0 < \nu < \tau < 1$.*

Then

$$\left(\frac{\nu}{\tau}\right)^{\lambda} < \frac{A_{\nu}(a,b)^{\lambda} - G_{\nu}(a,b)^{\lambda}}{A_{\tau}(a,b)^{\lambda} - G_{\tau}(a,b)^{\lambda}} < \left(\frac{1-\nu}{1-\tau}\right)^{\lambda},$$

for all positive and distinct real numbers a and b. Moreover, both bounds are sharp.

2. Local extreme points using principal minors of Hessian matrix for a special two variable function

We soppose that $a,\ b > 0$ are two distinct numbers. In [1]), the authors proved in Theorem 1 a generalization of Young inequality considering a function where they used the variable ν instead of $\frac{1}{p}$ and variablse a and b.We shall consider instead the two variables function

$$f(a,b) = \frac{1}{p}a^p + \frac{1}{q}b^q - ab - \frac{p_1}{p}\left(\frac{1}{p_1}a^p + \frac{1}{q_1}b^q - a^{\frac{p}{p_1}}b^{\frac{q}{q_1}}\right),$$

where $a,\ b > 0,\ a \neq 1,\ b \neq 1$, $p,\ p_1,\ q,\ q_1 > 0$ and we study some properties of this function.

We will find below the stationary points of this function.

Proposition 1. *(a) The stationary points of previous function are $A(b^{q-1}, b)$, for every $b > 0$ and $b \neq 1$, if $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{p_1} + \frac{1}{q_1} = 1$.*

(b) The local extreme points of previous function are $A(b^{q-1}, b)$, if in adittion, $p > p_1 > 1$. In this case previous local extreme points are local minimum points.

Proof. (a) In order to find the stationary points it is necessary to solve the sistem:

$$\begin{cases} \frac{\partial f}{\partial a} = -b + a^{\frac{p}{p_1}-1}b^{\frac{q}{q_1}} = 0 \\ \frac{\partial f}{\partial b} = b^{q-1}\left(1 - \frac{p_1 q}{q_1 p}\right) - a + \frac{p_1 q}{q_1 p}a^{\frac{p}{p_1}}b^{\frac{q}{q_1}-1} = 0 \end{cases}$$

, which, by calculus becomes,

$$\begin{cases} a^{\frac{p}{p_1}-1}b^{\frac{q}{q_1}-1} = 1 \\ a = b^{q-1} \end{cases},$$

because we can replace $b^{\frac{q}{q_1}-1}$ by $\frac{1}{a^{\frac{p}{p_1}-1}}$ in the second equation and take into account that $\frac{p_1 q}{q_1 p} \neq 1$ in our hypothesis, when $p_1 \neq p$.

We can see that the second equation of our last system, $a = b^{q-1}$ verifies the first equation of the system, $a^{\frac{p}{p_1}-1}b^{\frac{q}{q_1}-1} = 1$, if $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{p_1} + \frac{1}{q_1} = 1$.

Therefore the stationary points are the points, $A(b^{q-1}, b)$. We see also that $f(b^{q-1}, b) = 0$.

(b) Now we will compute the Hessian matrix of the function f in these points. The partial second derivatives of the function f will be:

$$\frac{\partial^2 f}{\partial a^2} = \left(\frac{p}{p_1} - 1\right) a^{\frac{p}{p_1}-2} b^{\frac{q}{q_1}},$$

$$\frac{\partial^2 f}{\partial a \partial b} = \frac{\partial^2 f}{\partial b \partial a} = -1 + \frac{q}{q_1} a^{\frac{p}{p_1}-1} b^{\frac{q}{q_1}-1},$$

$$\frac{\partial^2 f}{\partial b^2} = (q-1)\left(1 - \frac{p_1 q}{q_1 p}\right) b^{q-2} + \frac{p_1 q}{q_1 p}\left(\frac{q}{q_1} - 1\right) a^{\frac{p}{p_1}} b^{\frac{q}{q_1}-2}.$$

Then $\Delta_1 = \left(\frac{p}{p_1} - 1\right) a^{-1} b > 0$ if $\frac{p}{p_1} > 1$ and $\Delta_1 < 0$ if $\frac{p}{p_1} < 1$. By calculus we have,

$$\Delta_2 = \left(\frac{p}{p_1} - 1\right)\left[(q-1)\left(1 - \frac{p_1 q}{q_1 p}\right) + \frac{p_1 q}{q_1 p}\left(\frac{q}{q_1} - 1\right)\right] - \left(\frac{q}{q_1} - 1\right)^2$$

or

$$\Delta_2 = 0,$$

by calculus, taking into account that $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{p_1} + \frac{1}{q_1} = 1$.

Because $\Delta_1 > 0,\ \Delta_2 = 0$. that means they are greather or equal than zero and the principal minors of Hessian of f are positive , i.e. $d_1 = \Delta_1 > 0$ and

$$d_2 = \frac{a}{b}\left[(q-1)\left(1 - \frac{p_1 q}{q_1 p}\right) + \frac{p_1 q}{q_1 p}\left(\frac{q}{q_1} - 1\right)\right] = \frac{a}{b}\frac{p - p_1}{p_1 (p-1)^2} > 0,$$

then the Hessian matrix of the function f is semi-positive definite.The used criterion is like a generalization of Sylvester criterion and can be found also on [10], [11] and many books and papers. □

Here the function

$$g(x, y) = \frac{3}{7} x^{\frac{7}{3}} + \frac{4}{7} y^{\frac{7}{4}} - xy$$

from Figure 1 is used in the classical Young's inequality for particular values of p and q.

3. Local extreme points using principal minors of Hessian matrix for a special three variable function

We suppose that $a,\ b,\ c > 0$ are three distinct numbers and $p_1,\ p_2,\ p_3 > 0$, $p_1^{'},\ p_2^{'},\ p_3^{'} > 0$ with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ and $\frac{1}{p_1^{'}} + \frac{1}{p_2^{'}} + \frac{1}{p_3^{'}} = 1$.

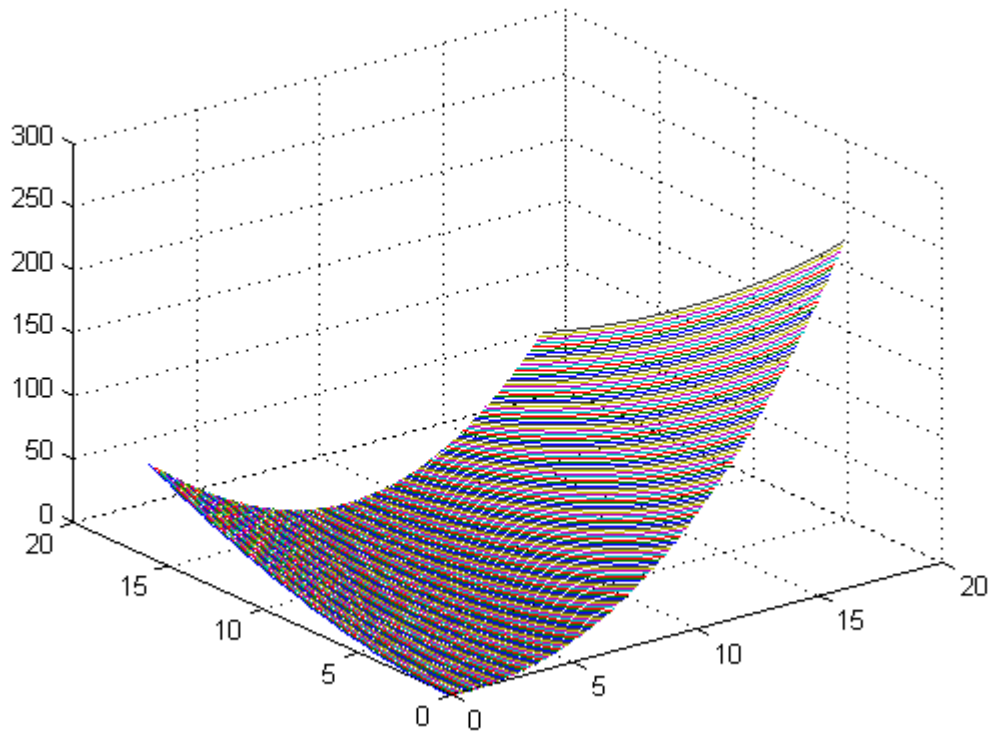

FIGURE 1. The function g(x,y) on $[0,16] \times [0,16]$ when $p = \frac{7}{3}, q = \frac{7}{4}$.

Theorem 2. *The stationary points for the three variables function*

$$f(a,b,c) = \frac{1}{p_1}a^{p_1} + \frac{1}{p_2}b^{p_2} + \frac{1}{p_3}c^{p_3} - abc - \frac{p_1'}{p_1}\left(\frac{1}{p_1'}a^{p_1} + \frac{1}{p_2'}b^{p_2} + \frac{1}{p_3'}c^{p_3} - a^{\frac{p_1}{p_1'}}b^{\frac{p_2}{p_2'}}c^{\frac{p_3}{p_3'}}\right)$$

are $A(c^{\frac{p_3}{p_1}}, c^{\frac{p_3}{p_2}}, c)$ *with* $c > 0$, $c \neq 1$.

Proof. Like before first we will find the stationary points from the following system:

$$\begin{cases} \frac{\partial f}{\partial a} = -bc + a^{\frac{p_1}{p_1'}-1} b^{\frac{p_2}{p_2'}} c^{\frac{p_3}{p_3'}} = 0 \\ \frac{\partial f}{\partial b} = b^{p_2-1}\left(1 - \frac{p_1'}{p_1}\frac{p_2}{p_2'}\right) - ac\left(1 - \frac{p_1'}{p_1}\frac{p_2}{p_2'}a^{\frac{p_1}{p_1'}-1} b^{\frac{p_2}{p_2'}-1} c^{\frac{p_3}{p_3'}-1}\right) = 0 \\ \frac{\partial f}{\partial c} = c^{p_3-1}\left(1 - \frac{p_1'}{p_1}\frac{p_3}{p_3'}\right) - ab\left(1 - \frac{p_1'}{p_1}\frac{p_3}{p_3'}a^{\frac{p_1}{p_1'}-1} b^{\frac{p_2}{p_2'}-1} c^{\frac{p_3}{p_3'}-1}\right) = 0 \end{cases}$$

, which, by calculus becomes,

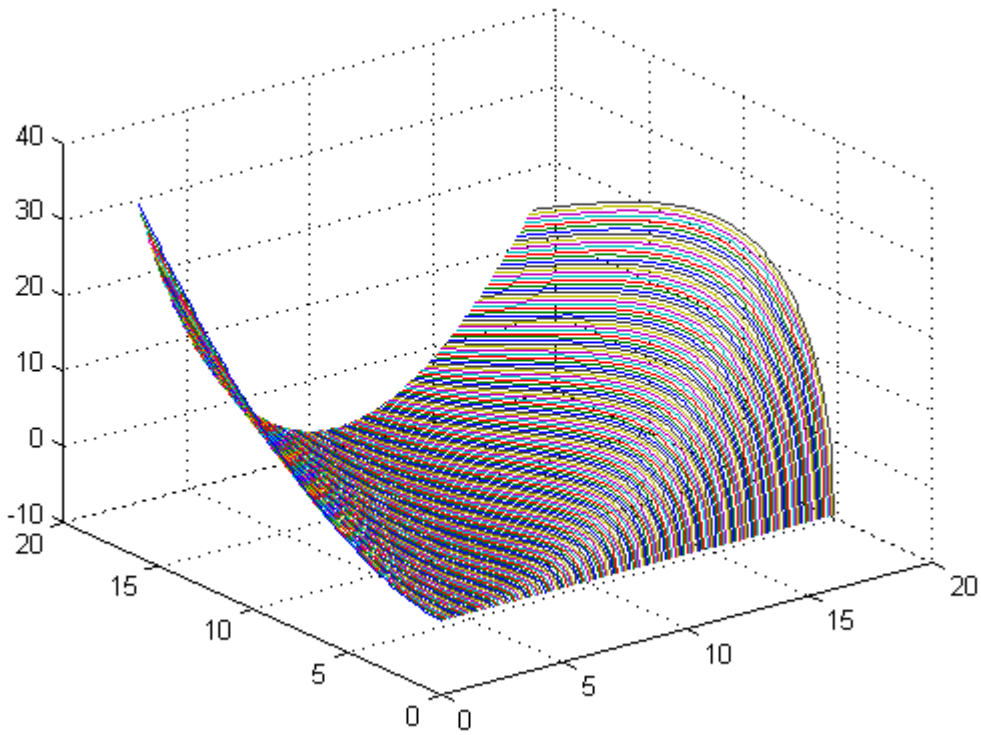

FIGURE 2. The function f(x,y) on $[0,16] \times [0,16]$ when $p = \frac{7}{3}, q = \frac{7}{4}, p_1 = \frac{5}{3}, q_1 = \frac{5}{2}$.

$$\begin{cases} a^{\frac{p_1}{p_1'}} b^{\frac{p_2}{p_2'}} c^{\frac{p_3}{p_3'}} = abc \\ b^{p_2}\left(1 - \frac{p_1'}{p_1}\frac{p_2}{p_2'}\right) = abc\left(1 - \frac{p_1'}{p_1}\frac{p_2}{p_2'} a^{\frac{p_1}{p_1'}-1} b^{\frac{p_2}{p_2'}-1} c^{\frac{p_3}{p_3'}-1}\right) \\ c^{p_3}\left(1 - \frac{p_1'}{p_1}\frac{p_3}{p_3'}\right) = abc\left(1 - \frac{p_1'}{p_1}\frac{p_3}{p_3'} a^{\frac{p_1}{p_1'}-1} b^{\frac{p_2}{p_2'}-1} c^{\frac{p_3}{p_3'}-1}\right) \end{cases}.$$

When $c \neq 1$, by calculus using the hypothesis that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ and $\frac{1}{p_1'} + \frac{1}{p_2'} + \frac{1}{p_3'} = 1$ we obtain $a = c^{\frac{p_3}{p_1}}$ and $b = c^{\frac{p_3}{p_2}}$. If $c = 1$ then we have, the stationary point $A(1,1,1)$. but then $a,\ b,\ c$ are not distinct numbers. We find by an easy calculus that,

$$f(c^{\frac{p_3}{p_1}}, c^{\frac{p_3}{p_2}}, c) = 0.$$

□

The function

$$f(x,y)=\frac{3}{7}x^{\frac{7}{3}}+\frac{4}{7}y^{\frac{7}{4}}-xy-\frac{5}{7}(\frac{3}{5}x^{\frac{7}{3}}+\frac{2}{5}y^{\frac{7}{4}}-x^{\frac{7}{5}}y^{\frac{7}{10}})$$

in the Figure 2 is a particular case from the generalized Young's inequality for $\lambda=1$, given by [1].

Theorem 3. *The local extreme points of the above function are* $A(c^{\frac{p_3}{p_1}},c^{\frac{p_3}{p_2}},c)$. *If the following conditions are satisfied*

$$\frac{p_1}{p_1^{'}}-1\geq\max\{\frac{1}{p_2}|\frac{p_1}{p_1^{'}}-\frac{p_2}{p_2^{'}}|,\frac{1}{p_3}|\frac{p_1}{p_1^{'}}-\frac{p_3}{p_3^{'}}|\},$$

$$1\geq\frac{1}{p_2}\frac{\frac{p_1}{p_1^{'}}-\frac{p_2^2}{(p_2^{'})^2}}{\frac{p_1}{p_1^{'}}-\frac{p_2}{p_2^{'}}}+\frac{1}{p_3}\frac{\frac{p_1}{p_1^{'}}-\frac{p_3^2}{(p_3^{'})^2}}{\frac{p_1}{p_1^{'}}-\frac{p_3}{p_3^{'}}}+\frac{p_1}{p_1^{'}}\frac{1}{p_2p_3}\frac{\left(\frac{p_2}{p_2^{'}}-\frac{p_3}{p_3^{'}}\right)^2}{\left(\frac{p_1}{p_1^{'}}-\frac{p_2}{p_2^{'}}\right)\left(\frac{p_1}{p_1^{'}}-\frac{p_3}{p_3^{'}}\right)}.$$

then these points are local minimum points for the function f.

Proof. We use the criterion with principal minors because we will see that the third determinant , Δ_3 from Sylvester criterion will be zero.

Bu,t for the begining it is necessary to compute the second derivatives of the function f and then the Hessian matrix in our points $A(c^{\frac{p_3}{p_1}},c^{\frac{p_3}{p_2}},c)$.

We have,

$$\frac{\partial^2 f}{\partial a^2}=\left(\frac{p_1}{p_1^{'}}-1\right)a^{\frac{p_1}{p_1^{'}}-2}b^{\frac{p_2}{p_2^{'}}}c^{\frac{p_3}{p_3^{'}}}$$

$$\frac{\partial^2 f}{\partial b^2}=(p_2-1)b^{p_2-2}\left(1-\frac{p_1^{'}}{p_1}\frac{p_2}{p_2^{'}}\right)+\frac{p_1^{'}}{p_1}\frac{p_2}{p_2^{'}}\left(\frac{p_2}{p_2^{'}}-1\right)a^{\frac{p_1}{p_1^{'}}}b^{\frac{p_2}{p_2^{'}}-2}c^{\frac{p_3}{p_3^{'}}},$$

$$\frac{\partial^2 f}{\partial c^2}=(p_3-1)c^{p_3-2}\left(1-\frac{p_1^{'}}{p_1}\frac{p_3}{p_3^{'}}\right)+\frac{p_1^{'}}{p_1}\frac{p_3}{p_3^{'}}\left(\frac{p_3}{p_3^{'}}-1\right)a^{\frac{p_1}{p_1^{'}}}b^{\frac{p_2}{p_2^{'}}}c^{\frac{p_3}{p_3^{'}}-2},$$

$$\frac{\partial^2 f}{\partial a\partial b}=\frac{\partial^2 f}{\partial b\partial a}=-c+\frac{p_2}{p_2^{'}}a^{\frac{p_1}{p_1^{'}}-1}b^{\frac{p_2}{p_2^{'}}-1}c^{\frac{p_3}{p_3^{'}}},$$

$$\frac{\partial^2 f}{\partial a\partial c}=\frac{\partial^2 f}{\partial c\partial a}=-b+\frac{p_3}{p_3^{'}}a^{\frac{p_1}{p_1^{'}}-1}b^{\frac{p_2}{p_2^{'}}}c^{\frac{p_3}{p_3^{'}}-1},$$

$$\frac{\partial^2 f}{\partial b\partial c}=\frac{\partial^2 f}{\partial c\partial b}=-a+\frac{p_1^{'}}{p_1}\frac{p_2}{p_2^{'}}\frac{p_3}{p_3^{'}}a^{\frac{p_1}{p_1^{'}}}b^{\frac{p_2}{p_2^{'}}-1}c^{\frac{p_3}{p_3^{'}}-1}.$$

In our points, these derivatives, by calculus, become:

$$\frac{\partial^2 f}{\partial a^2}(c^{\frac{p_3}{p_1}},c^{\frac{p_3}{p_2}},c)=\left(\frac{p_1}{p_1^{'}}-1\right)c^{p_3(1-\frac{2}{p_1})},$$

$$\frac{\partial^2 f}{\partial b^2}(c^{\frac{p_3}{p_1}},c^{\frac{p_3}{p_2}},c)=\left(p_2-1-\frac{p_1^{'}}{p_1}\frac{p_2^2}{p_2^{'}}+\frac{p_1^{'}}{p_1}\frac{p_2^2}{(p_2^{'})^2}\right)c^{p_3(1-\frac{2}{p_2})},$$

$$\frac{\partial^2 f}{\partial c^2}(c^{\frac{p_3}{p_1}}, c^{\frac{p_3}{p_2}}, c) = \left(p_3 - 1 - \frac{p_1^{'}}{p_1}\frac{p_3^2}{p_3^{'}} + \frac{p_1^{'}}{p_1}\frac{p_3^2}{(p_3^{'})^2}\right) c^{p_3-2},$$

$$\frac{\partial^2 f}{\partial a \partial b}(c^{\frac{p_3}{p_1}}, c^{\frac{p_3}{p_2}}, c) = \frac{\partial^2 f}{\partial b \partial a}(c^{\frac{p_3}{p_1}}, c^{\frac{p_3}{p_2}}, c) = c\left(\frac{p_2}{p_2^{'}} - 1\right),$$

$$\frac{\partial^2 f}{\partial a \partial c}(c^{\frac{p_3}{p_1}}, c^{\frac{p_3}{p_2}}, c) = \frac{\partial^2 f}{\partial c \partial a}(c^{\frac{p_3}{p_1}}, c^{\frac{p_3}{p_2}}, c) = c^{\frac{p_3}{p_2}}\left(\frac{p_3}{p_3^{'}} - 1\right),$$

$$\frac{\partial^2 f}{\partial b \partial c}(c^{\frac{p_3}{p_1}}, c^{\frac{p_3}{p_2}}, c) = \frac{\partial^2 f}{\partial c \partial b}(c^{\frac{p_3}{p_2}}, c^{\frac{p_3}{p_2}}, c) = c^{\frac{p_3}{p_1}}\left(\frac{p_1^{'}}{p_1}\frac{p_2}{p_2^{'}}\frac{p_3}{p_3^{'}} - 1\right).$$

Then we replace thiese values in the Hessian matrix,

$$H_f(A) = \begin{pmatrix} \frac{\partial^2 f}{\partial a^2}(c^{\frac{p_3}{p_1}}, c^{\frac{p_3}{p_2}}, c) & \frac{\partial^2 f}{\partial a \partial b}(c^{\frac{p_3}{p_1}}, c^{\frac{p_3}{p_2}}, c) & \frac{\partial^2 f}{\partial a \partial c}(c^{\frac{p_3}{p_1}}, c^{\frac{p_3}{p_2}}, c) \\ \frac{\partial^2 f}{\partial b \partial a}(c^{\frac{p_3}{p_1}}, c^{\frac{p_3}{p_2}}, c) & \frac{\partial^2 f}{\partial b^2}(c^{\frac{p_3}{p_1}}, c^{\frac{p_3}{p_2}}, c) & \frac{\partial^2 f}{\partial b \partial c}(c^{\frac{p_3}{p_1}}, c^{\frac{p_3}{p_2}}, c) \\ \frac{\partial^2 f}{\partial c \partial a}(c^{\frac{p_3}{p_1}}, c^{\frac{p_3}{p_2}}, c) & \frac{\partial^2 f}{\partial c \partial b}(c^{\frac{p_3}{p_1}}, c^{\frac{p_3}{p_2}}, c) & \frac{\partial^2 f}{\partial c^2}(c^{\frac{p_3}{p_1}}, c^{\frac{p_3}{p_2}}, c) \end{pmatrix}$$

and we compute the three determinant from the Sylvester's criterion, Δ_1, Δ_2 and Δ_3.

We get,

$$\Delta_1 = \left(\frac{p_1}{p_1^{'}} - 1\right) c^{p_3(1-\frac{2}{p_1})},$$

$$\Delta_2 = c^2\{\left(\frac{p_1}{p_1^{'}} - 1\right)\left[p_2 - 1 - \frac{p_1^{'}}{p_1}\frac{p_2^2}{p_2^{'}} + \frac{p_1^{'}}{p_1}\frac{p_2^2}{(p_2^{'})^2}\right] - \left(\frac{p_2}{p_2^{'}} - 1\right)^2\}$$

and

$$\Delta_3 = detH_f(A) = 0,$$

like in the case of the two variable function f which have two variables.

Therefore we will use the principal minors criterion, see [10], [11] which implies the calculation of all determinants obtained by elimination of the same rows and columns of the Hessian matrix, in this case. These determinants must be greather or equal with zero in order that the Hessian to be semi-positive definite.

In this way we have, in addition, the following principal minors:

$$\Delta_{1212}^{'} = c^{p_3-2}\left[p_3 - 1 - \frac{p_1^{'}}{p_1}\frac{p_3^2}{p_3^{'}} + \frac{p_1^{'}}{p_1}\frac{p_3^2}{(p_3^{'})^2}\right],$$

$$\Delta_{1313}^{'} = c^{p_3(1-\frac{2}{p_2})}\left[p_2 - 1 - \frac{p_1^{'}}{p_1}\frac{p_2^2}{p_2^{'}} + \frac{p_1^{'}}{p_1}\frac{p_2^2}{(p_2^{'})^2}\right],$$

$$\Delta_{2323}^{'} = \Delta_1,$$

$$\Delta_{11}^{'} = a^2\{\left[p_3 - 1 - \frac{p_1^{'}}{p_1}\frac{p_3^2}{p_3^{'}} + \frac{p_1^{'}}{p_1}\frac{p_3^2}{(p_3^{'})^2}\right]\cdot\left[p_2 - 1 - \frac{p_1^{'}}{p_1}\frac{p_2^2}{p_2^{'}} + \frac{p_1^{'}}{p_1}\frac{p_2^2}{(p_2^{'})^2}\right] - \left(1 - \frac{p_1^{'}}{p_1}\frac{p_2}{p_2^{'}}\frac{p_3}{p_3^{'}}\right)^2\},$$

$$\Delta_{22}^{'} = b^2\{\left(\frac{p_1}{p_1^{'}} - 1\right)\left[p_3 - 1 - \frac{p_1^{'}}{p_1}\frac{p_3^2}{p_3^{'}} + \frac{p_1^{'}}{p_1}\frac{p_3^2}{(p_3^{'})^2}\right] - \left(\frac{p_3}{p_3^{'}} - 1\right)^2\}$$

and

$$\Delta_{33}^{'} = \Delta_2.$$

We can notice that $\Delta_{1212}^{'} \geq 0$ and $\Delta_{1313}^{'} \geq 0$ are fulfilled when $p_1 > p_1^{'}$ because

the conditions $\Delta_2 \geq 0$ and $\Delta_{22}^{'} \geq 0$ are stronger. So we have thus only the following three conditions: $\Delta_2 \geq 0,\ \Delta_{22}^{'} \geq 0$ and $\Delta_{11}^{'} \geq 0$.

By calculus these conditions will be: $p_2\left(\frac{p_1}{p_1^{'}} - 1\right) \geq |\frac{p_1}{p_1^{'}} - \frac{p_2}{p_2^{'}}|$,

$p_3\left(\frac{p_1}{p_1^{'}} - 1\right) \geq |\frac{p_1}{p_1^{'}} - \frac{p_3}{p_3^{'}}|$, and

$$1 \geq \frac{1}{p_2}\frac{\frac{p_1}{p_1^{'}} - \frac{p_2^2}{(p_2^{'})^2}}{\frac{p_1}{p_1^{'}} - \frac{p_2}{p_2^{'}}} + \frac{1}{p_3}\frac{\frac{p_1}{p_1^{'}} - \frac{p_3^2}{(p_3^{'})^2}}{\frac{p_1}{p_1^{'}} - \frac{p_3}{p_3^{'}}} + \frac{p_1}{p_1^{'}}\frac{1}{p_2p_3}\frac{\left(\frac{p_2}{p_2^{'}} - \frac{p_3}{p_3^{'}}\right)^2}{\left(\frac{p_1}{p_1^{'}} - \frac{p_2}{p_2^{'}}\right)\left(\frac{p_1}{p_1^{'}} - \frac{p_3}{p_3^{'}}\right)}.$$

□

Example 1. *We choose* $p_1 = 10,\ p_2 = 2,\ p_3 = \frac{10}{4}$ *and* $p_1^{'} = 6,\ p_2^{'} = 3,\ p_3^{'} = 2$ *and the function* f *is,*

$$f(a,b,c) = \frac{1}{10}a^{10} + \frac{1}{2}b^2 + \frac{2}{5}c^{\frac{5}{2}} - abc - \frac{3}{5}\left(\frac{1}{6}a^{10} + \frac{1}{3}b^2 + \frac{1}{2}c^{\frac{5}{2}} - a^{\frac{5}{3}}b^{\frac{2}{3}}c^{\frac{5}{4}}\right).$$

We see that $\Delta_3 = 0$ and the three hypothesis of Theorem 3 are satisfied, where

$$H_f(A) = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{4} \\ -\frac{1}{3} & \frac{7}{15} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & \frac{9}{16} \end{pmatrix}.$$

and the principal minors are positive.

References

[1] H. Alzer, C. M. Fonseca and A. Kovacec, Young-type inequalities and their matrix analogues, *Linear and Multilinear Algebra*, **63**, 3 (2015), 622-635. https://doi.org/10.1080/03081087.2014.891588

[2] S. S. Dragomir, Some asymmetric reverses of Young's scalar and operator inequalities with applications, *RGMIA Res. Rep. Coll.*, **19** (2016), art. 44, 16 pp.

[3] S. S. Dragomir, Some new reverses of Young's operator inequality, *RGMIA Res. Rep. Coll.*, **18** (2015), 12 pp.

[4] S. S. Dragomir, Trace inequalities for positive operators via recent refinements and reverses of Young's inequality, *RGMIA Res. Rep. Coll.*, **19** (2015).

[5] S. S. Dragomir, A note on new refinements and reverses of Young's inequality, *RGMIA Res. Rep. Coll.*, **18** (2015), Art. 131.

[6] J. I. Fujii, E. Kamei, Uhlmann's interpolational method for operators means, *Math. Japon.*, **34** (1989), no. 4, 541-547.

[7] F. Kittaneh, Y. Manasrah, Reverse Young and Heinz inequalities for matrices, *Linear and Multilinear Algebra*, **59**, 9 (2011), 1031–1037. https://doi.org/10.1080/03081087.2010.551661

[8] F. Kittaneh, Y. Manasrah, Improved Young and Heinz inequalities for matrices, *J. Math. Anal. Appl.*, **361** (2010), 262–269. https://doi.org/10.1016/j.jmaa.2009.08.059

[9] W.H. Young, On classes of summable functions and their Fourier series, *Proc. Royal Soc. London, Ser. A.*, **87** (1912), 225-229.

[10] Eivind Eriksen, Lecture 5: Principal minors and Hessian, B1 Norwegian School of Management, Department of Economics, October 1, 2010.

[11] Sudhir R. Ghorpage and Balmohan V. Limaye, Sylvester's minorant criterion, Lagrange-Beltrami identity and nonnegative definiteness, The Mathematics Student, Special Centenary Volume, (2007), 123-130.

Received: January 19, 2019; Published: February 27, 2019