Positive Solutions for Fractional Boundary Value Problem with $p$-Laplacian

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Abstract

In this article, the author investigates the existence and multiplicity of positive solutions for a boundary value problem of fractional differential equations. The differential operator is taken in the Riemann-Liouville sense. By means of the Leggett-Williams fixed point theorem, we obtain the existence of least three solutions.

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1 Introduction

In this paper, is concerned with the existence of multiple positive solutions to nonlinear mixed-order three point boundary value problem for $p$-Laplacian with advanced argument

\[(\varphi_p(D_0^\alpha u(t)))' + a(t)f(u(\theta(t))) = 0, \quad 0 < t < 1, \quad (1)\]

\[D_0^\alpha u(0) = u(0) = u'(0) = 0, \quad D_0^\beta u(1) = \gamma D_0^\beta u(\eta), \quad (2)\]

where $\eta \in (0, 1)$, $\gamma \in (0, \frac{1}{\eta^{\alpha-\beta}-1})$, $D_0^\alpha$, $D_0^\beta$ are the standard Riemann-Liouville fractional derivatives with $\alpha \in (2, 3)$, $\beta \in (1, 2)$ such that $\alpha \geq \beta + 1$. We assume the following conditions throughout:
H1) \( f \in C([0, \infty), [0, \infty)) \),

H2) \( a \in L^1[0, 1] \) is nonnegative and \( a(t) \neq 0 \) on any subinterval of \([0, 1] \),

H3) The advanced argument \( \theta \in C((0, 1), (0, 1]) \) and \( t \leq \theta(t) \leq 1, \forall t \in (0, 1) \).

The equation with a p-Laplacian operator arises in the modeling of different physical and natural phenomena, non-Newtonian mechanics, nonlinear elasticity and glaciology, combustion theory, population biology, nonlinear flow laws and so on. Liang, Peng and Shen [8], used the fixed point theorem of Avery and Henderson to show the existence of at least two positive solutions. Zhao, Wang and Ge [16], studied the existence of at least three positive solutions by using Leggett-Williams fixed point theorem. Chai [4], obtain results for the existence of at least one nonnegative solution and two positive solutions by using fixed point theorem on cone. Su, Wei and Wang [10], studied the existence of one and two positive solution by using the fixed point index theory. Su [9], studied the existence of one and two positive solution by using the method of defining operator by the reverse function of Green function and the fixed-point index theory. Tang, Yan and Q. Liu [12], studied the existence of positive solutions of fractional differential equation with p-laplacian by using the coincidence degree theory. Torres [13], studied the existence and multiplicity for the initial value problem of fractional differential equation envolving Caputo’s differential operator and the boundary conditions with integer order derivatives.

Differential equations with deviated arguments are found to be important mathematical tools for the better understanding of several real world problems in physics, mechanics, engineering, economics, etc. [1],[3]. Integer and fractional order differential equations with deviated argument are found [6],[14],[15] and [11].

Motivated by the above works, we obtain some sufficient conditions for the existence of at least one, two and three positive solutions for (1) and (2).

The organization of this paper is as follows. In Section 2, we present some necessary definitions and preliminary results that will be used to prove our results. In Section 3, we discuss the existence of at least one positive solution for (1) and (2). In Section 4, we discuss the existence of multiple positive solutions for (1) and (2). Finally, we give some examples to illustrate our results in Section 5.

2 Preliminaries

Definition 2.1. Let \( E \) be a real Banach space. A nonempty closed convex set \( K \subset E \) is called cone if
1. $x \in K, \lambda > 0$ then $\lambda x \in K$,
2. $x \in K, -x \in K$ then $x = 0$.

**Definition 2.2.** An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

**Remark 2.3.** By the positive solution of (1), (2) we understand a function $u(t)$ which is positive on $[0, 1]$ and satisfies the differential equation (1) and the boundary conditions (2).

**Definition 2.4.** The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by
\[
I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,
\]
provided that the right side is pointwise defined on $(0, \infty)$.

**Definition 2.5.** The Riemann-Louiville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is given by
\[
D_0^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds,
\]
where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0, \infty)$.

**Remark 2.6.** [2]
1. If $\lambda > -1$,
\[
D_0^\alpha t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha},
\]
and $D_0^\alpha t^\alpha = 0, m = 1, 2, \ldots, n$, where $n = [\alpha] + 1$.
2. $D_0^\alpha I_0^\alpha u(t) = u(t)$ for all $u \in C(0, 1) \cap L^1(0, 1)$.
3. If $u \in L^1(0, 1), \alpha > \beta > 0$, then
\[
D_0^\beta I_0^\alpha u(t) = I_0^{\alpha-\beta} u(t).
\]

**Lemma 2.7.** [2] Let $\alpha > 0$. If we assume that $u \in C(0, 1) \cap L^1(0, 1)$, then the fractional differential equation
\[
D_0^\alpha u(t) = 0,
\]
has $u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \ldots + C_n t^{\alpha-n}, C_i \in \mathbb{R}, i = 1, 2, \ldots, n$, as unique solution, where $n = [\alpha] + 1$. 
Lemma 2.8. [2] Suppose that \( u \in C(0,1) \cap L^1(0,1) \) is such that \( D_{0+}^\alpha u \in C(0,1) \cap L^1(0,1) \). Then
\[
P_0^\alpha D_{0+}^\alpha u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \ldots + C_n t^{\alpha-n},
\]
for some \( C_i \in \mathbb{R}, i = 1, 2, \ldots, n \), where \( n = \lfloor \alpha \rfloor + 1 \).

We shall consider the Banach space \( E = C[0,1] \) equipped with standard norm
\[
\|u\| = \max_{0 \leq t \leq 1} |u(t)|.
\]
The proof of existence of solution is based upon an application of the following theorem.

Theorem 2.9. [5] Let \( E \) be a Banach space and \( K \) be a cone of \( E \). For \( r > 0 \), define \( \Omega_r = \{ u \in K : \|u\| < r \} \) and assume that \( T : \overline{\Omega}_r \rightarrow K \) is a completely continuous operator such that \( Tu \neq u \) for all \( u \in \partial \Omega_r \).

1. If \( \|Tu\| < \|u\| \) for all \( u \in \partial \Omega_r \) then \( i(T, \Omega_r, K) = 1 \),
2. If \( \|Tu\| > \|u\| \) for all \( u \in \partial \Omega_r \) then \( i(T, \Omega_r, K) = 0 \).

Consider the boundary value problem
\[
(\varphi_p(D_{0+}^\alpha u(t)))' + h(t) = 0, \quad 0 < t < 1,
\]
\[
D_{0+}^\alpha u(0) = u(0) = u'(0) = 0, \quad D_{0+}^\beta u(1) = \gamma D_{0+}^\beta u(\eta),
\]
where \( \eta \in (0,1) \) and \( \gamma \in (0, \frac{1}{\eta^{\alpha-\beta-1}}) \).

Lemma 2.10. Suppose that \( h \in C^+[0,1] \), \( \gamma \in (0, \frac{1}{\eta^{\alpha-\beta-1}}) \) and \( \alpha \geq \beta + 1 \). Then the boundary value problem (3) and (4) has a unique solution
\[
u(t) = \int_0^1 G_1(t,s)\varphi_q \left( \int_0^s h(\tau)d\tau \right) d\tau + \frac{\gamma t^{\alpha-1}}{1 - \gamma \eta^{\alpha-\beta-1}} \int_0^1 G_2(\eta,s) d\tau,
\]
where
\[
G_1(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\
\frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1,
\end{cases}
\]
\[
G_2(\eta,s) = \begin{cases} \frac{[\eta(1-s)]^{\alpha-\beta-1}-(\eta-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0 \leq s \leq \eta \leq 1, \\
\frac{[\eta(1-s)]^{\alpha-\beta-1}}{\Gamma(\alpha)}, & 0 \leq \eta \leq s \leq 1.
\end{cases}
\]
Proof. Integrating the equation (3) from 0 to \( t \), we have

\[
\varphi_p(D_0^\alpha u(t)) - \varphi_p(D_0^\alpha u(0)) = -\int_0^t h(s)ds,
\]

and so,

\[
D_0^\alpha u(t) = -\varphi_q\left(\int_0^t h(s)ds\right).
\]

From Lemma 2.8,

\[
\begin{align*}
u(t) &= -I_0^\alpha \varphi_q\left(\int_0^t h(s)ds\right) + At^{\alpha-1} + Bt^{\alpha-2} + Ct^{\alpha-3} \\
&= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi_q\left(\int_0^s h(\tau)d\tau\right)ds + At^{\alpha-1} \\
&+ Bt^{\alpha-2} + Ct^{\alpha-3}.
\end{align*}
\]

From (4), \( B = C = 0 \), and so

\[
u(t) = -I_0^\alpha \varphi_q\left(\int_0^t h(s)ds\right) + At^{\alpha-1}.
\tag{5}
\]

Now, from Remark 2.6

\[
D_0^\beta u(t) = -I_0^{\alpha-\beta} \varphi_q\left(\int_0^t h(s)ds\right) + A\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1} \\
= -\frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} \varphi_q\left(\int_0^s h(\tau)d\tau\right)ds + A\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1}.
\]

Therefore

\[
D_0^\beta u(1) = -\frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} \varphi_q\left(\int_0^s h(\tau)d\tau\right)ds + A\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)},
\]

\[
\gamma D_0^\beta u(\eta) = -\frac{\gamma}{\Gamma(\alpha-\beta)} \int_0^{\eta} (\eta-s)^{\alpha-\beta-1} \varphi_q\left(\int_0^s h(\tau)d\tau\right)ds + A\frac{\Gamma(\alpha)\gamma}{\Gamma(\alpha-\beta)} \eta^{\alpha-\beta-1},
\]

by boundary condition (4), we have

\[
A = \frac{1}{1 - \gamma \eta^{\alpha-\beta-1}} \int_0^1 (1-s)^{\alpha-\beta-1} \varphi_q\left(\int_0^s h(\tau)d\tau\right)ds \\
- \frac{\gamma}{1 - \gamma \eta^{\alpha-\beta-1}} \int_0^{\eta} (\eta-s)^{\alpha-\beta-1} \varphi_q\left(\int_0^s h(\tau)d\tau\right)ds,
\]

\[
\gamma = \frac{1 - \gamma \eta^{\alpha-\beta-1}}{1 - \gamma \eta^{\alpha-\beta-1}} \int_0^1 (1-s)^{\alpha-\beta-1} \varphi_q\left(\int_0^s h(\tau)d\tau\right)ds \\
- \frac{\gamma}{1 - \gamma \eta^{\alpha-\beta-1}} \int_0^{\eta} (\eta-s)^{\alpha-\beta-1} \varphi_q\left(\int_0^s h(\tau)d\tau\right)ds.
\]

\[
\gamma = \frac{1 - \gamma \eta^{\alpha-\beta-1}}{1 - \gamma \eta^{\alpha-\beta-1}} \int_0^1 (1-s)^{\alpha-\beta-1} \varphi_q\left(\int_0^s h(\tau)d\tau\right)ds \\
- \frac{\gamma}{1 - \gamma \eta^{\alpha-\beta-1}} \int_0^{\eta} (\eta-s)^{\alpha-\beta-1} \varphi_q\left(\int_0^s h(\tau)d\tau\right)ds.
\]
and replacing in (5), we obtain
\[
\begin{align*}
u(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi_q \left( \int_0^s h(\tau) d\tau \right) ds \\
&\quad + \frac{t^{\alpha-1}}{1 - \gamma \eta^{\alpha-\beta-1}} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} \varphi_q \left( \int_0^s h(\tau) d\tau \right) ds \\
&\quad - \frac{\gamma t^{\alpha-1}}{1 - \gamma \eta^{\alpha-\beta-1}} \int_0^\eta \frac{(\eta-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} \varphi_q \left( \int_0^s h(\tau) d\tau \right) ds.
\end{align*}
\]

Splitting the second integral in two parts of the form
\[
t^{\alpha-1} + \frac{k}{1 - \gamma \eta^{\alpha-\beta-1}} = \frac{t^{\alpha-1}}{1 - \gamma \eta^{\alpha-\beta-1}},
\]
we have \(k = \gamma \eta^{\alpha-\beta-1} t^{\alpha-1}\), and thus,
\[
\begin{align*}
u(t) &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi_q \left( \int_0^s h(\tau) d\tau \right) ds \\
&\quad + t^{\alpha-1} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} \varphi_q \left( \int_0^s h(\tau) d\tau \right) ds \\
&\quad + \frac{\gamma t^{\alpha-1}}{1 - \gamma \eta^{\alpha-\beta-1}} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} \varphi_q \left( \int_0^s h(\tau) d\tau \right) ds \\
&\quad - \frac{\gamma t^{\alpha-1} t^{\alpha-1}}{(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-1}} \frac{1}{\Gamma(\alpha)} \varphi_q \left( \int_0^s h(\tau) d\tau \right) ds \\
&\quad + \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} \varphi_q \left( \int_0^s h(\tau) d\tau \right) ds \\
&\quad + \frac{\gamma t^{\alpha-1}}{1 - \gamma \eta^{\alpha-\beta-1}} \int_0^\eta \frac{[\eta(1-s)]^{\alpha-\beta-1} - (\eta-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} \varphi_q \left( \int_0^s h(\tau) d\tau \right) ds \\
&\quad \times \varphi_q \left( \int_0^s h(\tau) d\tau \right) ds \\
&\quad + \frac{\gamma t^{\alpha-1}}{1 - \gamma \eta^{\alpha-\beta-1}} \int_\eta^1 \frac{[\eta(1-s)]^{\alpha-\beta-1}}{\Gamma(\alpha)} \varphi_q \left( \int_0^s h(\tau) d\tau \right) ds \\
&\quad = \int_0^1 G_1(t, s) \varphi_q \left( \int_0^s h(\tau) d\tau \right) ds + \frac{\gamma t^{\alpha-1}}{1 - \gamma \eta^{\alpha-\beta-1}} \\
&\quad \times \int_0^1 G_2(\eta, s) \varphi_q \left( \int_0^s h(\tau) d\tau \right) ds.
\end{align*}
\]

This completes the proof. \(\square\)
Lemma 2.11. Let $\rho \in (0, 1)$ be fixed. The kernel $G_1(t, s)$ satisfies the following properties:

1. $G_1(t, s) \in C([0, 1] \times [0, 1])$ and $G_1(t, s) > 0$ for all $t, s \in (0, 1)$,
2. $G_1(t, s) \leq G_1(1, s)$ for all $s \in (0, 1)$,
3. $\min_{\rho \leq t \leq 1} G_1(t, s) \geq \rho^{\alpha - 1} G_1(1, s)$ for all $s \in [0, 1]$.

Proof. 1. It is easy to check that (1) holds.

2. Let us put

$$g_1(t, s) = t^{\alpha - 1}(1 - s)^{\alpha - 1} - (t - s^{\alpha - 1}), \quad g_2 = t^{\alpha - 1}(1 - s)^{\alpha - 1}.$$

Clearly $\frac{\partial g_2(t, s)}{\partial t} \geq 0$.

Now,

$$\frac{\partial g_1(t, s)}{\partial t} = (\alpha - 1)t^{\alpha - 2}(1 - s)^{\alpha - 1} - (\alpha - 1)(t - s)^{\alpha - 2} = (\alpha - 1) \left[ t^{\alpha - 2}(1 - s)^{\alpha - 1} - (t - s)^{\alpha - 2} \right].$$

Notice that,

$$\left(1 - \frac{s}{t}\right)^r \leq (1 - s)^r, \quad \forall t, s \in (0, 1), r > 0. \quad (6)$$

Thus,

$$t^{\alpha - 2}(1 - s)^{\alpha - 1} - (t - s)^{\alpha - 2} = t^{\alpha - 2} \left[ (1 - s)^{\alpha - 1} - (1 - \frac{s}{t})^{\alpha - 2} \right] \geq t^{\alpha - 2} \left[ (1 - s)^{\alpha - 1} - (1 - s)^{\alpha - 2} \right] \geq 0.$$

To get the final inequality we use the fact that $\alpha - 1 \leq \alpha - 2$. Then, $G_1$ is increasing on its domain. Consequently, (2) holds.

3. For $\rho \leq t \leq 1$, we have

$$\min_{\rho \leq t \leq 1} G_1(t, s) = G_1(\rho, s),$$

where

$$G_1(\rho, s) = \begin{cases} \frac{\rho^{\alpha - 1}(1 - s)^{\alpha - 1} - (\rho - s)^{\alpha - 1}}{\Gamma(\alpha)}, & 0 \leq s \leq \rho, \\ \frac{\rho^{\alpha - 1}(1 - s)^{\alpha - 1}}{\Gamma(\alpha)}, & \rho \leq s \leq 1. \end{cases}$$
(a) If \( 0 < s \leq \rho \),
\[
\min_{\rho \leq t \leq 1} G_1(t, s) = \frac{\rho^{\alpha-1}(1 - s)^{\alpha-\beta-1}}{\Gamma(\alpha)} - \frac{(\rho - s)^{\alpha-1}}{\Gamma(\alpha)},
\]
and
\[
\rho^{\alpha-1}G_1(1, s) = \frac{\rho^{\alpha-1}(1 - s)^{\alpha-\beta-1}}{\Gamma(\alpha)} - \frac{\rho^{\alpha-1}(1 - s)^{\alpha-1}}{\Gamma(\alpha)}. \tag{7}
\]
From (6), we have
\[
(\rho - s)^{\alpha-1} = \rho^{\alpha-1}\left(1 - s/\rho\right)^{\alpha-1} \leq \rho^{\alpha-1}(1 - s)^{\alpha-1}. \tag{9}
\]
It follows from (7), (8) and (9) that item (3) in the proof holds.

(b) If \( \rho \leq s \leq 1 \),
\[
\min_{\rho \leq t \leq 1} G_1(t, s) = \frac{\rho^{\alpha-1}(1 - s)^{\alpha-\beta-1}}{\Gamma(\alpha)}, \tag{10}
\]
and
\[
\rho^{\alpha-1}G_1(1, s) = \frac{\rho^{\alpha-1}(1 - s)^{\alpha-\beta-1}}{\Gamma(\alpha)} - \frac{\rho^{\alpha-1}(1 - s)^{\alpha-1}}{\Gamma(\alpha)}. \tag{11}
\]
It follows from (10) and (11) that item (3) in the proof holds.

This completes the proof.

\[\Box\]

**Lemma 2.12.** Let \( h(t) \in C^+[0, 1] \). The unique solution \( u(t) \) of (3), (4) is nonnegative and satisfies
\[
\min_{\rho \leq t \leq 1} u(t) \geq \rho^{\alpha-1}\|u\|.
\]

**Proof.** From the definition, \( u(t) \) is nonnegative. From Lemmas 2.10 and 2.11, we have
\[
u(t) = \int_0^1 G_1(t, s)\varphi_q \left( \int_0^s h(\tau)d\tau \right) ds + \frac{\gamma t^{\alpha-1}}{1 - \gamma \eta^{\alpha-\beta-1}} \int_0^1 G_2(\eta, s) \times \varphi_q \left( \int_0^s h(\tau)d\tau \right) ds \leq \int_0^1 G_1(1, s)\varphi_q \left( \int_0^s h(\tau)d\tau \right) ds + \frac{\gamma}{1 - \gamma \eta^{\alpha-\beta-1}} \int_0^1 G_2(\eta, s) \times \varphi_q \left( \int_0^s h(\tau)d\tau \right) ds.\]
Then
\[ \| u \| \leq \int_0^1 G_1(1, s) \varphi_q \left( \int_0^s h(\tau) d\tau \right) ds + \frac{\gamma}{1 - \gamma \eta^{\alpha-\beta-1}} \int_0^1 G_2(\eta, s) \times \varphi_q \left( \int_0^s h(\tau) d\tau \right) ds. \]

On the other hand, Lemmas 2.10 and 2.11 imply that, for any \( t \in [\rho, 1] \),
\[
\begin{align*}
u(t) &= \int_0^1 G_1(t, s) \varphi_q \left( \int_0^s h(\tau) d\tau \right) ds + \frac{\gamma \tau^{\alpha-1}}{1 - \gamma \eta^{\alpha-\beta-1}} \int_0^1 G_2(\eta, s) \times \varphi_q \left( \int_0^s h(\tau) d\tau \right) ds \\
&\geq \int_0^1 \rho^{-1} G_1(1, s) \varphi_q \left( \int_0^s h(\tau) d\tau \right) ds + \frac{\gamma \rho^{-1}}{1 - \gamma \eta^{\alpha-\beta-1}} \int_0^1 G_2(\eta, s) \times \varphi_q \left( \int_0^s h(\tau) d\tau \right) ds \\
&= \rho^{-1} \left[ \int_0^1 G_1(1, s) \varphi_q \left( \int_0^s h(\tau) d\tau \right) ds + \frac{\gamma \rho^{-1}}{1 - \gamma \eta^{\alpha-\beta-1}} \int_0^1 G_2(\eta, s) \times \varphi_q \left( \int_0^s h(\tau) d\tau \right) ds \right] \\
&\geq \rho^{-1} \| u \|.
\end{align*}
\]

Therefore,
\[ \min_{\rho \leq t \leq 1} u(t) \geq \rho^{-1} \| u \|. \]

This completes the proof. \( \square \)

Define the cone \( K \) by
\[ K = \{ u \in E : u(t) \geq 0 \text{ and } \min_{\rho \leq t \leq 1} u(t) \geq \rho^{-1} \| u \| \} \]

and the operator \( T : E \to E \) by
\[
T u(t) = \int_0^1 G_1(t, s) \varphi_q \left( \int_0^s a(\tau) f(u(\theta(\tau))) d\tau \right) ds + \frac{\gamma \tau^{\alpha-1}}{1 - \gamma \eta^{\alpha-\beta-1}} \int_0^1 G_2(\eta, s) \varphi_q \left( \int_0^s a(\tau) f(u(\theta(\tau))) d\tau \right) ds.
\] (12)
Remark 2.13. Since \( t \leq \theta(t) \leq 1 \) and \( u \in K \)

\[
\min_{\rho \leq t \leq 1} u(\theta(t)) \geq \min_{\rho \leq t \leq 1} u(t) \geq \rho^{\alpha-1}\|u\|.
\]  

Thus, Lemma 2.12 and (13) show that \( T : K \to K \).

Lemma 2.14. \([4]\) Let \( c > 0, a > 0 \). For any \( x, y \in [0, c] \), we have that

1. if \( a > 1 \), then \( |x^a - y^a| \leq ac^{a-1}|x - y| \),
2. if \( 0 < a \leq 1 \), then \( |x^a - y^a| \leq |x - y|^a \).

Lemma 2.15. \( T \) is completely continuous and \( T(K) \subseteq K \).

Proof. By Remark 2.13, \( T(K) \subseteq K \). In view of the assumption of nonnegativity and continuity of functions \( G_i(x, y) \) with \( i = 1, 2 \) and \( a(t)f(u(\theta(t))) \), we conclude that \( T : K \to K \) is continuous.

Let \( \Omega \subset K \) be bounded, then there exists \( L > 0 \) such that

\[
|f(u(\theta(t)))| \leq L, \quad \forall u \in \Omega.
\]

Then from \( u \in \Omega \) and from Lemmas 2.10 and 2.11, we have

\[
|Tu(t)| = \left| \int_0^1 G_1(t, s)\varphi_q \left( \int_0^s a(\tau)f(u(\theta(\tau)))d\tau \right)ds \right|
\]

\[
+ \frac{\gamma \eta^{\alpha-1}}{1 - \gamma \eta^{\alpha-\beta-1}} \int_0^1 G_2(\eta, s)\varphi_q \left( \int_0^s a(\tau)f(u(\theta(\tau)))d\tau \right)ds
\]

\[
\leq \int_0^1 G_1(1, s)\varphi_q \left( \int_0^1 a(\tau)Ld\tau \right)ds
\]

\[
+ \frac{\gamma}{1 - \gamma \eta^{\alpha-\beta-1}} \int_0^1 G_2(\eta, s)\varphi_q \left( \int_0^1 a(\tau)Ld\tau \right)ds
\]

\[
\leq L^{q-1}\varphi_q \left( \int_0^1 a(\tau)d\tau \right) \left[ \int_0^1 (1 - s)^{\alpha-\beta-1} \frac{1}{\Gamma(\alpha)}ds + \frac{\gamma}{1 - \gamma \eta^{\alpha-\beta-1}} \right]
\]

\[
\times \int_0^1 \eta^{\alpha-\beta-1}(1 - s)^{\alpha-\beta-1} \frac{1}{\Gamma(\alpha)}ds
\]

\[
= \frac{L^{q-1}}{(1 - \gamma \eta^{\alpha-\beta-1})\Gamma(\alpha)}\varphi_q \left( \int_0^1 a(\tau)d\tau \right) \int_0^1 (1 - s)^{\alpha-\beta-1} \]

\[
= \frac{L^{q-1}}{(\alpha - \beta)(1 - \gamma \eta^{\alpha-\beta-1})\Gamma(\alpha)}\varphi_q \left( \int_0^1 a(\tau)d\tau \right) \leq l.
\]

Hence, \( T(\Omega) \) is bounded.

On the other hand, let \( u \in \Omega, t_1, t_2 \in [0, 1] \) with \( t_1 < t_2 \), from Lemmas 2.10
and 2.14 we have

\[
|Tu(t_2) - Tu(t_1)| = \int_0^1 G_1(t_2, s)\varphi_q \left( \int_0^s a(\tau)f(u(\theta(\tau)))d\tau \right) ds \\
+ \frac{\gamma t_2^{\alpha-1}}{1 - \gamma t_2^{\alpha-\beta-1}} \int_0^1 G_2(\eta, s)\varphi_q \left( \int_0^s a(\tau)f(u(\theta(\tau)))d\tau \right) ds \\
- \int_0^1 G_1(t_1, s)\varphi_q \left( \int_0^s a(\tau)f(u(\theta(\tau)))d\tau \right) ds \\
- \frac{\gamma t_1^{\alpha-1}}{1 - \gamma t_1^{\alpha-\beta-1}} \int_0^1 G_2(\eta, s)\varphi_q \left( \int_0^s a(\tau)f(u(\theta(\tau)))d\tau \right) ds \\
\leq L^q - 1 \int_0^1 |G_1(t_2, s) - G_1(t_1, s)|\varphi_q \left( \int_0^s a(\tau)d\tau \right) ds \\
+ \frac{L^q - 1}{1 - \gamma t_2^{\alpha-\beta-1}} |t_2^{\alpha-1} - t_1^{\alpha-1}| \int_0^1 G_2(\eta, s)\varphi_q \left( \int_0^s a(\tau)d\tau \right) ds \\
\leq L^q - 1 \int_0^1 |G_1(t_2, s) - G_1(t_1, s)|\varphi_q \left( \int_0^s a(\tau)d\tau \right) ds \\
+ \frac{L^q - 1}{1 - \gamma t_2^{\alpha-\beta-1}} |t_2^{\alpha-1} - t_1^{\alpha-1}| \int_0^1 G_2(\eta, s) \\
\times \varphi_q \left( \int_0^s a(\tau)d\tau \right) ds.
\]

The continuity of \( G_1 \) implies that the right-side of the above inequality tends to zero if \( t_2 \to t_1 \). Therefore, \( T \) is completely continuous by Arzela-Ascoli Theorem. □

Consider the following:

\[
\Lambda_1 = \left[ \rho^{2(\alpha-1)} \left( \int_0^1 G_1(1, s)\varphi_q \left( \int_0^s a(\tau)d\tau \right) ds \right. \right. \\
+ \frac{\gamma}{1 - \gamma t_2^{\alpha-\beta-1}} \int_0^1 G_2(\eta, s)\varphi_q \left( \int_0^s a(\tau)d\tau \right) ds \left. \right) \right]^{-1},
\]

\[
\Lambda_2 = \left[ \frac{1}{(\alpha - \beta)(1 - \gamma t_2^{\alpha-\beta-1})\Gamma(\alpha)}\varphi_q \left( \int_0^1 a(\tau)d\tau \right) \right]^{-1}.
\]

Then \( 0 < \Lambda_2 < \Lambda_1 \).
In fact,

\[
\Lambda_1^{-1} \leq \varphi_q \left( \int_0^1 a(\tau) d\tau \right) \left[ \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} ds + \frac{\gamma}{1-\gamma} \right] \\
\times \int_0^1 \frac{\eta^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)} ds \\
= \frac{1}{(1-\gamma)\eta^{\alpha-\beta-1}\Gamma(\alpha)} \varphi_q \left( \int_0^1 a(\tau) d\tau \right) \int_0^1 (1-s)^{\alpha-\beta-1} ds \\
= \frac{1}{(\alpha-\beta)(1-\gamma)\eta^{\alpha-\beta-1}\Gamma(\alpha)} \varphi_q \left( \int_0^1 a(\tau) d\tau \right) \\
= \Lambda_2^{-1}.
\]

3 Triple Solutions

To show the existence of multiple solutions we will use the Leggett-Williams fixed point theorem [7]. To this end define the following subsets of a cone \( K \).

\[
K_c = \{ u \in K : \|u\| < c \}, \quad K(\psi,b,d) = \{ u \in K : b \leq \psi(u), \|u\| \leq d \}.
\]

**Definition 3.1.** A map \( \alpha : K \to [0, +\infty) \) is said to be a nonnegative continuous concave functional on a cone \( K \) of a real Banach space \( E \) if \( \alpha \) is continuous and

\[
\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)
\]

for all \( x,y \in K \) and \( t \in [0,1] \).

**Theorem 3.2.** [7] Let \( T : K_c \to K_c \) be a completely continuous operator and \( \psi \) a nonnegative continuous concave functional on \( K \) such that \( \psi(u) \leq \|u\| \) for all \( u \in K_c \). Suppose that there exist constants \( 0 < a < b < d \leq c \) such that

\[
\begin{align*}
(B1) \quad & \{ u \in K(\psi,b,d) : \psi(u) > b \} \neq \emptyset \text{ and } \psi(Tu) > b \text{ if } u \in K(\psi,b,d), \\
(B2) \quad & \|Tu\| < a \text{ if } u \in K_a, \\
(B3) \quad & \psi(Tu) > b \text{ for } u \in K(\psi,b,c) \text{ with } \|Tu\| > d.
\end{align*}
\]

Then, \( T \) has at least three fixed points \( u_1, u_2 \) and \( u_3 \) such that \( \|u_1\| < a, b < \psi(u_2) \) and \( \|u_3\| > a \) with \( \psi(u_3) < b \).

**Theorem 3.3.** Suppose that there exist \( a, b, c \) with \( 0 < a < \rho^{\alpha-1}b < b \leq c \) such that

\[
\begin{align*}
(C1) \quad & f(u(\theta(t))) < (a\Lambda_2)^{p-1}, \quad (t,u) \in [0,1] \times [0,a],
\end{align*}
\]
(C2) \( f(u(\theta(t))) \geq (\rho^{a-1}b\Lambda_1)^{p-1}, \quad (t, u) \in [\rho, 1] \times [\rho^{a-1}b, b], \)

(C3) \( f(u(\theta(t))) \leq (c\Lambda_2)^{p-1}, \quad (t, u) \in [0, 1] \times [0, c]. \)

Then (1), (2) has at least three positive solution \( u_1, u_2 \) and \( u_3 \) satisfying
\[
\|u_1\| < a, \quad \rho^{a-1}b < \psi(u_2), \quad \|u_3\| > a \quad \text{with} \quad \psi(u_3) < \rho^{a-1}b.
\]

Proof. By Lemma 2.15, \( T : K \to K \) is completely continuous.

Let \( \psi(u) = \min_{\rho \leq t \leq 1} u(t) \), it is obvious that \( \psi \) is a nonnegative continuous concave functional on \( K \) with \( \psi(u) \leq \|u\|, \) for \( u \in \overline{K}_c \). Now we will show that the conditions of Theorem 3.2 are satisfied. Suppose that \( u \in \overline{K}_c, \) this is, \( \|u\| \leq c. \) For \( t \in [0, 1] \) by (12), Lemma 2.15 and (C3), we have
\[
Tu(t) = \int_0^1 G_1(t, s)\varphi_q\left( \int_0^s a(\tau)f(u(\theta(\tau)))d\tau \right)ds
+ \frac{\gamma t^{\alpha-1}}{1 - \gamma \eta^{\alpha-1}} \int_0^1 G_2(\eta, s)\varphi_q\left( \int_0^s a(\tau)f(u(\theta(\tau)))d\tau \right)ds
\leq \int_0^1 G_1(1, s)\varphi_q\left( \int_0^1 a(\tau)(c\Lambda_2)^{p-1}d\tau \right)ds
+ \frac{\gamma}{1 - \gamma \eta^{\alpha-1}} \int_0^1 G_2(\eta, s)\varphi_q\left( \int_0^1 a(\tau)(c\Lambda_2)^{p-1}d\tau \right)ds
= \frac{\Lambda_2c}{(\alpha - \beta)(1 - \gamma \eta^{\alpha-1})\Gamma(\alpha)}\varphi_q\left( \int_0^1 a(\tau)d\tau \right) = c.
\]

This implies \( T : \overline{K}_c \to \overline{K}_c \). By the same method, if \( u \in \overline{K}_a, \) then we can get \( \|Tu\| < a \) and therefore (B2) is satisfied.

Next, we assert that \( \{u \in K(\psi, \rho^{a-1}b, b) : \psi(u) > \rho^{a-1}b \} \neq \emptyset \) and \( \psi(Tu) > \rho^{a-1}b \) for all \( u \in K(\psi, \rho^{a-1}b, b). \) In fact, the constant function
\[
\frac{\rho^{a-1}b + b}{2} \in \{u \in K(\psi, \rho^{a-1}b, b) : \psi(u) > \rho^{a-1}b \}.
\]

On the other hand, for \( u \in K(\psi, \rho^{a-1}b, b), \) we have
\[
\rho^{a-1}b \leq \psi(u) = \min_{\rho \leq t \leq 1} u(t) \leq \|u\| \leq b, \quad t \in [\rho, 1].
\]
Thus, in view of (12), Lemma 2.11 and (C2), we have

\[
\psi(Tu) = \min_{\rho \leq t \leq 1} \left[ \int_0^1 G_1(t, s) \varphi_q \left( \int_0^s a(\tau)f(u(\theta(\tau)))d\tau \right) ds \\
+ \frac{\gamma t^{\alpha-1}}{1 - \gamma \eta^{\alpha-1}} \int_0^1 G_2(\eta, s) \varphi_q \left( \int_0^s a(\tau) \left( \rho^{\alpha-1} b \Lambda_1 \right)^{p-1} d\tau \right) ds \right]
\]

\[
\geq \int_0^1 \rho^{\alpha-1} G_1(1, s) \varphi_q \left( \int_0^s a(\tau) \left( \rho^{\alpha-1} b \Lambda_1 \right)^{p-1} d\tau \right) ds
\]

\[
+ \frac{\gamma \rho^{\alpha-1}}{1 - \gamma \eta^{\alpha-1}} \int_0^1 G_2(\eta, s) \varphi_q \left( \int_0^s a(\tau) d\tau \right) ds
\]

\[
= b \Lambda_1 \left[ \rho^{2(\alpha-1)} \left( \int_0^1 G_1(1, s) \varphi_q \left( \int_0^s a(\tau) d\tau \right) \right) d\tau \right]
\]

\[
+ \frac{\gamma}{1 - \gamma \eta^{\alpha-1}} \int_0^1 G_2(\eta, s) \varphi_q \left( \int_0^s a(\tau) d\tau \right) ds
\]

\[
= b > \rho^{\alpha-1} b.
\]

Thus, (B1) is satisfied.

Finally, we assert that if \( u \in K(\psi, \rho^{\alpha-1} b, c) \) with \( \|Tu\| > b \) then \( \psi(Tu) > \rho^{\alpha-1} b \). To see this, suppose that \( u \in K(\psi, \rho^{\alpha-1} b, c) \) with \( \|Tu\| > b \), then by Lemma 2.15, we have

\[
\psi(Tu) = \min_{\rho \leq t \leq 1} (Tu)(t) \geq \rho^{\alpha-1} \|Tu\| > \rho^{\alpha-1} b.
\]

Thus, (B3) is satisfied.

Hence, an application of Lemma 3.2 completes the proof. \(\square\)

### 4 Example

Consider the fractional differential equation with advanced argument for \(p\)-Laplacian

\[
(D^{5/2}_{0+} u(t))' + \frac{\sqrt{7}}{2} t^{-1/2} f(u(\theta(t))) = 0, \quad 0 < t < 1,
\]

\[
D^{5/2}_{0+} u(0) = u(0) = u'(0) = 0, \quad D^{7/6}_{0+} u(1)) = \frac{7}{10} D^{7/6}_{0+} u(\frac{1}{2}),
\]

where \( q = 3, \rho = \frac{1}{3}, \theta(t) = t^\nu, 0 < \nu < 1 \) and

\[
f(t, u) = \begin{cases} 14u^2, & u \leq 1, \\ 13 + u^{1/4}, & u > 1. \end{cases}
\]
Through a simple calculation, we have \( \Lambda_1 \approx 10.49539567 \) and \( \Lambda_2 \approx 0.5697619380 \). Choosing \( a = \frac{1}{14}, b = 18 \) and \( c = 1296 \), we get

\[
f(u) < f\left(\frac{1}{14}\right) \approx 0.07... < (a\Lambda_2)^{1/2} \approx 0.201..., \quad u \in \left[0, \frac{1}{14}\right],
\]

\[
f(u) > 13 + (2\sqrt{3})^{1/4} \approx 14.36426160... > (\rho^{\alpha-1} b \Lambda_1)^{1/2} \approx 6.029686319..., \quad u \in [2\sqrt{3}, 18],
\]

\[
f(u) < f(1296) \approx 19.000... < (c\Lambda_2)^{1/2} \approx 27.173..., \quad u \in [0, 1296].
\]

Then the conditions \((C1-C3)\) are satisfied. Therefore, it follows from Theorem 3.3 that (14) and (15) has at least three positive solution \( u_1, u_2 \) and \( u_3 \) such that

\[
\|u_1\| < \frac{1}{14}, \quad 2\sqrt{3} < \psi(u_2), \quad \|u_3\| > \frac{1}{14} \text{ with } \psi(u_3) < 2\sqrt{3}.
\]

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**References**


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