Generalized Some New Sequence Space

\[ H_\infty (G, \| \cdots \| q, s) \text{ by Orlicz Function} \]

on a n-Normed Space

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Abstract

The aim of this paper, we study generalized some the new Hilbert Sequence Space \( H_\infty (G, \| \cdots \| q, s) \) through Orlicz functions. We discussed some topological properties and inclusion relations involving the space.

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Introduction

The German mathematician Hilbert had initially developed the Hilbert matrix in the middle of 1894. Further, many mathematician study the Hilbert matrix, Hilbert operators, Hilbert transforms, Hilbert space concept and obtained
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various results. Recently, Harun Polat introduced the new type of Hilbert sequence space and generalized the various results.

The matrix $B = (b_{uv})$ $(u, v \in N)$ is called an infinite matrix of complex or real numbers $b_{uv}$. Then, the matrix transformation $B:X \rightarrow Y$, $X$ and $Y$ be any two sequence space. We write $B_x = \sum_{i=0}^{\infty} b_{uv} x_i$ and $B_x = (B_u x) \forall n \in N$. All the series $B_u x$ converge. The matrix domain $X_B$ is defined by

$$X_B = \{ x = (x_i) \in \omega : B_x \in X \} \quad (1)$$

Which is a sequence space. Mathematicians Taylor, Euler, Cesaro, Riesz and Noruled using the idea, constructing a new sequence space by means of the matrix domain of a particular limitation method. They introduced the sequence spaces, following this idea recently mathematician Harun Polat [2] introduced the new Hilbert sequence spaces $h_c$, $h_0$ and $h_\infty$.

The Hilbert matrix $(m \times m)$ is defined by

$$H_m = [h_{uv}] = \left[ \frac{1}{u + v - 1} \right]_{u,v=1}^{m} \text{ for each } m \in N$$

The inverse of Hilbert matrix is defined by

$$H_m^{-1} = (-1)^{u+v}(u + v - 1) \left( \begin{array}{c} m + u - 1 \\ m - u \end{array} \right) \left( \begin{array}{c} m + v - 1 \\ m - v \end{array} \right)^2$$

For all $u, v, m \in N$. Polat (2) has defined new spaces by using the Hilbert transform.

Let $h_c$, $h_0$, $h_\infty$ be convergent Hilbert, null convergent and bounded Hilbert spaces respectively. Then we have

$$X = \{ x \in \omega ; H_x \in Y \}$$

Where $X = \{ h_c, h_0, h_\infty \}$ and $Y = \{ C, C_0, C_\infty \}$

There new spaces as the set of all sequences whose $H$-transform of them are in the Hilbert Sequence spaces which are defined by Polat (2).

Let $(S_k)$ represent the sequences of partial sums of the infinite series $\sum_{i=0}^{\infty} a_i$ and $q > 0$. The Hilbert transform $(H, q)$ of the sequences $S = (S_n)$ is defined by

$$H_m^q(s) = \sup_{m} \sum_{v=1}^{m} \frac{1}{u + v - 1} S_k < \infty \quad (A)$$

The series $\sum_{n=0}^{\infty} a_n$ is called to be summable (H, q) to S provided

$$H_m^q(s) = \sum_{v=1}^{m} \frac{1}{u + v - 1} s_k = 0 \text{ as } j \rightarrow \infty$$

Also is absolutely summable (H,q) or summable $|H, q|$ if
\[
\sum_{k} |H_{k}^{q}(s) - H_{k-1}^{q}(s)| < \infty
\]

Let \( x = (x_k) \) be a sequence of scalars we write \( T_k(x) = H_{k}^{q}(x) - H_{k-1}^{q}(x) \), where \( H_{m}^{q}(x) \) is defined by (A).

Note that for any sequence \( x = (x_i) \) and \( y = (y_i) \) and scalar \( \lambda \), we have
\[
T_{i}(x + y) = T_{i}(x) + T_{i}(y) \quad \text{and} \quad T_{i}(\lambda x) = \lambda T_{i}(x)
\]

In the middle of 1960, Mathematician Gahler had initially developed the 2-normed spaces. Later Mathematicians Misiak, Gunman and many others have developed the n-normed space & provided various results in n-normed spaces.

Let \( m \in \mathbb{N} \) and \( X \) is a vector space over the field of \( \mathbb{R} \) real numbers of dimension \( D \). For \( 2 \leq m \leq D \).

The real valued function \( \| \ldots \| \) on \( X^m \) is following the four conditions

(i) \( \| x_1, x_2, \ldots, x_m \| = 0 \) iff \( \| x_1, x_2, \ldots, x_m \| \) are linearly dependent.

(ii) \( \| x_1, x_2, \ldots, x_m \| \) is invariant under permutation.

(iii) \( \| \alpha x_1, x_2, \ldots, x_m \| = |\alpha| \| x_1, x_2, \ldots, x_m \| \) for any \( \alpha \in \mathbb{R} \)

(iv) \( \| x_1 + \bar{x}, x_2, \ldots, x_m \| \leq \| x_1, x_2, \ldots, x_m \| + \| \bar{x}, x_2, \ldots, x_m \| \)

Is said to be n-norm on \( X \) and pair \( (\| \ldots \|, X) \) is called a n-normed space.

Let the symbol \( w \) represent the set of all complex sequences or real. If \( x \in w \), then simply \( x = (x_k), k = 1 \) to \( \infty \). Let \( \ell_{\infty}, c \) and \( c_0 \) denote the bounded sequence space, convergent sequence space and null sequence space respectively. Also we denote \( bs,c,\ell, \) and \( \ell_p \) the space of all convergent, bounded, absolutely convergent and \( p \)-absolutely summable series, respectively; where \( 1 \leq p < \infty \).

The function \( G: [0, \infty) \to [0, \infty) \) is said to be an Orlicz function if it is non-decreasing, convex and continuous with
\[
l_{u \to 0} G(u) = 0, \quad G(u) > 0 \quad \text{with} \quad u > 0 \quad \text{and} \quad l_{u \to \infty} G(u) = \infty .\]

If convexity of \( G \) is replaced by \( G(u + v) \leq G(u) + G(v) \) is said to be modulus function.

The orlicz function \( G \) is satisfied \( \Delta_2 \)-Condition if there exists a constant \( \mu \geq 2 \) and \( u_0 > 0 \) such that \( G(2u) \leq \mu G(u) \) where \( |u| \leq u_0 \).

Lindenstrauss and Tzafriri construct new sequence space through Orlicz function
\[
\ell_{G} = \left\{ (x_i) \in \omega : \sum_{i=1}^{\infty} G\left( \frac{|x_i|}{\rho} \right) < \infty, \quad \rho > 0 \right\}
\]

which is said to be an Orlicz sequence space. The space \( \ell_{G} \) with the norm
\[ \|x\| = \inf \left\{ \sum_{i=1}^{\infty} G\left( \frac{|x_i|}{\rho} \right) \leq 1, \ \rho > 0 \right\} \]

which is connected to \( \ell_p \) space, \( G(x) = x^p \), for \( 1 \leq p < \infty \).

A paranorm \( h: X \rightarrow R \), \( Y \) be a vector space, satisfied the axioms

(i) \( h(\theta) = 0 \) (where \( \theta = 0, 0, 0, \ldots, 0 \ldots \)) is zero of the space

(ii) \( h(-x) = h(x) \), \( \forall x \in X \)

(iii) \( h(x + y) \leq h(x) + h(y) \), \( \forall x, y \in X \)

(iv) If \( (\lambda n) \) is a sequence of scalars with \( \lambda n \rightarrow \lambda \) as \( n \rightarrow \infty \), \( (y_i) \) is a sequence of Vectors with \( (x_n \rightarrow x) \) as \( n \rightarrow \infty \) then \( P(\lambda n x_n \rightarrow \lambda x) \rightarrow 0 \) as \( n \rightarrow \infty \)

A paranorm space \( (X, h) \) is a vector space with a paranorm \( h \)

The following inequality will be used this paper \( |a_i + b_i|^p \leq D(|a_i|^p + |b_i|^p) \) where \( a_i \) and \( b_i \) are complex numbers and \( H = \sup p_i < \infty, h = \inf p_i, D = \max(1, 2^{H-1}) \)

Let \( G \) be an Orlicz function, \( P = (P_k) \) be a Sequence \( (0 < \sup p < \infty) \) of positive real numbers and "\( A = a_{mn} \)" be an infinite matrix and \( (\|a_{i},\ldots,a_{m}\|,X) \) be an \( n \)-normed space. We define the new sequence space

\[
H_x(G,\ldots\|q,s) = \left\{ (x_i) \in \omega : \sup_n \sum_{i=1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \left\| T_n(x) \right\| / \rho, z_i \right) \right]^{p_k} < \infty, \ \rho > 0, s \geq 0, z_i \in X \right\}
\]

\[
H_c(G,\ldots\|q,s) = \left\{ (x_i) \in \omega : \lim_n \sum_{i=1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \left\| T_n(x) \right\| / \rho, z_i \right) - L, z_i \right]^{p_k} \text{ exists}, \ \rho > 0, s \geq 0, L, z_i \in X \right\}
\]

\[
H_o(G,\ldots\|q,s) = \left\{ (x_i) \in \omega : \lim_n \sum_{i=1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \left\| T_n(x) \right\| / \rho, z_i \right) \right]^{p_k} = 0, \ \rho > 0, s \geq 0, z_i \in X \right\}
\]

**Theorem 1:**

Suppose that \( G \) be an Orlicz function, \( P = (P_k) \) is a bounded sequence of positive real number then the space \( H_x(G,\ldots\|q,s) \) is a vector space over the real field

**Proof:**

Let \((x),(y) \in H_x(G,\ldots\|q,s)\) and \( \alpha, \beta \in R, \exists \) a number \( \rho_1 \) and \( \rho_2 \)

S.T
\[ \sum_{i=1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \left\| \frac{T_i(x)}{\rho_1}, z_i \right\| \right) \right]^{p_i} < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \left\| \frac{T_i(x)}{\rho_2}, z_i \right\| \right) \right]^{p_i} < \infty \]

Define \( \rho_3 = \max(2|x|\rho_1, 2|\beta|\rho_2) \)

Since \( G \) is a convex, non-decreasing and by using inequality, we have

\[ \sum_{i=1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \left\| \frac{T_i(\alpha x + \beta y)}{\rho_3}, z_i \right\| \right) \right]^{p_i} \]
\[ \leq \sum_{k=1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \left\| \frac{T_i(\alpha x)}{\rho_3}, z_i \right\| + \left\| \frac{T_i(\beta y)}{\rho_3}, z_i \right\| \right) \right]^{p_i} \]
\[ \leq \sum_{k=1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \left\| \frac{T_i(\alpha x)}{\rho_3}, z_i \right\| \right) \right]^{p_i} + D \sum_{k=1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \left\| \frac{T_i(\beta y)}{\rho_3}, z_i \right\| \right) \right]^{p_i} \]

Therefore, \( \alpha x + \beta y \in H_\infty \) \( (G, \ldots, q, s) \)

Hence \( H_\infty \) \( (G, \ldots, q, s) \) be a vector space

**Theorem 2:**

Let \( G \) be an Orlicz function, bounded sequence \( P = (p_k) \) is a positive real numbers. Then the space \( H_\infty \) \( (G, \ldots, q, s) \) is a paranormed space with the paranorm defined by

\[ h(x) = \inf \left\{ \rho_i^{\frac{p_i}{n}} : \sum_{i=1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \left\| \frac{T_i(x)}{\rho_i}, z_i \right\| \right) \right]^{p_i} \leq 1 \right\} \]

\( n = -1, 2, 3, \ldots \) where \( H = \max(1, \sup_{k, p} \rho) \)

**Proof:**

It is clear that \( h(x) = h(-x) \) and \( h(x + y) \leq h(x) + h(y) \) since \( G(0) = 0 \), we get \( \inf \left\{ p^{\frac{p}{n}} \right\} = 0 \) for \( y = 0 \). Finally, we get prove that multiplication is continuous. Let \( \lambda \neq 0 \) be any complex number, by the definition, we have
\[ h(\lambda x) = \inf \left\{ \frac{p_s}{\rho^n} : \sum_{i=1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \left\| \frac{\lambda T_k(x)}{\rho}, z_i \right\| \right) \right]^{\frac{1}{p_s}} \leq 1, \ n = 1, 2, 3, \ldots \right\} \]

Thus, we have

\[ h(\lambda x) = \inf \left\{ (\lambda |s|)^{\frac{p_s}{\rho_n}} : \sum_{i=1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \left\| \frac{T_k(x)}{\rho}, z_i \right\| \right) \right]^{\frac{1}{p_s}} \leq 1, \ n = 1, 2, 3, \ldots \right\} \]

where \( s = \frac{p}{|\lambda|} \), since \( |\lambda|^{p_k} \leq \max(1, |\lambda|^{p_k}) \) we have

\[ h(\lambda x) = (\max(1, |\lambda|^{p_k}))^{\frac{1}{p_n}} \inf \left\{ (s)^{\frac{p_s}{\rho_n}} : \sum_{i=1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \left\| \frac{T_k(x)}{\rho}, z_i \right\| \right) \right]^{\frac{1}{p_s}} \leq 1, \ n = 1, 2, 3, \ldots \right\} \]

hence. when \( h(y) \) converges to zero in \( H_\infty (G, \ldots, q, s, s) \), \( h(\lambda x) \) also converges to zero. Now, suppose that \( \lambda_n \to 0 \) as \( n \to \infty \) and \( y \) is in \( H_\infty (G, s, q, \ldots, \ldots, \ldots) \).

For arbitrary \( \epsilon > 0 \). Let \( n_0 \) be a +ve integer s.t

\[ \sum_{i=n_0+1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \left\| \frac{T_k(x)}{\rho}, z_i \text{ for some } \rho > 0 \right\| \right) \right]^{p_s} < \frac{\epsilon}{2} \]

\[ \Rightarrow \sum_{i=n_0+1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \left\| \frac{T_k(x)}{\rho}, z_i \right\| \right) \right]^{p_s} < \frac{\epsilon}{2} \]

\[ \left( \sum_{i=n_0+1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \left\| \frac{T_k(x)}{\rho}, z_i \right\| \right) \right]^{p_s} \right)^{\frac{1}{p_s}} \leq \frac{\epsilon}{2} \]

Let \( 0 < |\lambda| < 1 \), then using convexity of \( G \), We get

\[ \sum_{i=n_0+1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \left\| \frac{\lambda T_k(x)}{\rho}, z_i \right\| \right) \right]^{p_s} < |\lambda| \sum_{i=n_0+1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \left\| \frac{T_k(x)}{\rho}, z_i \right\| \right) \right]^{p_s} < \left( \frac{\epsilon}{2} \right)^{p_s} \]

Since \( G \) is Orlicz function, then

\[ h(t) = \sum_{k=1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \left\| \frac{T_k(x)}{\rho}, z_i \right\| \right) \right]^{p_s} \]
Is continuous at 0. So there is $0 < \delta < 1$ such that $|h(t)| < \frac{\epsilon}{2}$ for $0 < t < \delta$.

Let $k$ be such that $|\lambda_n| < \delta \quad \forall \quad n > k$; we have

$$
\sum_{k=1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \frac{\|\lambda_n T_k(x)\|}{\rho}, z_i \right) \right]^{\frac{1}{n}} < \frac{\epsilon}{2}
$$

Thus

$$
\sum_{k=1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \frac{\|\lambda_n T_k(x)\|}{\rho}, z_i \right) \right]^{\frac{1}{n}} \leq \epsilon \quad \text{for} \quad n > k
$$

Hence $h(\lambda x) \to 0$ as $\lambda \to 0$

Hence the theorem.

**Theorem 3**

Let $G$ be an Orlicz function. If $0 < r_k \leq q_k < \infty$ for each $k \in N$.

Then $H_\infty (G, r, s, \|[\ldots]\|) \subseteq H_\infty (G, q, s, \|[\ldots]\|)$

**Proof:**

Let $x \in H_\infty (G, \|[\ldots]\|, r, s)$ then there exists some $\rho > 0$ such that

$$
\sum_{k=1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \frac{\|T_k(x)\|}{\rho}, z_i \right) \right]^{n} < \infty
$$

$$
\Rightarrow \sum_{k=1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \frac{\|T_k(x)\|}{\rho}, z_i \right) \right]^{n} \leq 1
$$

for sufficiently large value of $k$. Say $k \geq k_0$ for some fixed $k_0 \in N$. Since $G$ is Orlicz function (non decreasing), we get

$$
\sum_{k=1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \frac{\|T_k(x)\|}{\rho}, z_i \right) \right]^{q_k} \leq \sum_{k=1}^{\infty} \frac{a_{mn}}{n^s} \left[ G\left( \frac{\|T_k(x)\|}{\rho}, z_i \right) \right]^{q_k} < \infty
$$

Hence $x \in H_\infty (G, \|[\ldots]\|, q, s)$
References


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