Distribution of Zeros of Appell-type Degenerate Twisted $q$-Tangent Numbers and Polynomials

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Abstract

In this paper, we construct the Appell-type degenerate twisted $q$-tangent numbers and polynomials associated with the $p$-adic integral on $\mathbb{Z}_p$. We also give some explicit formulas for Appell-type degenerate twisted $q$-tangent numbers and polynomials. Finally, we investigate the distribution of the zero of Appell-type degenerate twisted $q$-tangent polynomials.

Mathematics Subject Classification: 11B68, 11S40, 11S80

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1 Introduction

L. Carlitz constructed the degenerate Bernoulli polynomials(see [1]). Feng Qi et al.[2] introduced the partially degenerate Bernoulli polynomials of the first kind in $p$-adic field. P.T. Young derived some properties of degenerate Bernoulli polynomials(see [7]). T. Kim introduced the Barnes’ type multiple degenerate Bernoulli and Euler polynomials(see [3]). Recently, Ryoo introduced the Appell-type degenerate twisted tangent numbers and polynomials(see [5, 6]). In this paper, we introduce Appell-type degenerate twisted
\(q\)-tangent numbers \(T_{n,\lambda}(\lambda)\) and \(q\)-tangent polynomials \(T_{n,\lambda}(x, \lambda)\). Throughout this paper we use the following notations. By \(\mathbb{N}\) we denote the set of natural numbers, \(\mathbb{C}\) denotes the complex number field, and \(\mathbb{Z}_+ = \mathbb{N} \cup \{0\}\). Let \(r\) be a positive integer, and let \(\zeta\) be \(r\)th root of 1. We recall that the degenerate twisted \(q\)-tangent polynomials are defined by the generating function

\[
\sum_{n=0}^{\infty} T_{n,\lambda}(x, \lambda) \frac{t^n}{n!} = \frac{2}{\zeta q(1 + \lambda t)^{2/\lambda} + 1} (1 + \lambda t)^{x/\lambda}.
\] (1.1)

For \(x = 0\), formula (1.1) reduces to the generating function of the degenerate twisted \(q\)-tangent numbers

\[
\sum_{n=0}^{\infty} T_{n,\zeta}(\lambda) \frac{t^n}{n!} = \frac{2}{\zeta q(1 + \lambda t)^{2/\lambda} + 1}.
\] (1.2)

2 Appell-type degenerate twisted \(q\)-tangent polynomials

In this section, we introduce Appell-type degenerate twisted \(q\)-tangent numbers and polynomials, and we obtain explicit formulas for them. Let us define the Appell-type degenerate twisted \(q\)-tangent numbers \(T_{n,\lambda}(\lambda)\) and polynomials \(T_{n,\lambda}(x, \lambda)\) as follows:

\[
\left( \frac{2}{\zeta q(1 + \lambda t)^{2/\lambda} + 1} \right) e^t = \sum_{n=0}^{\infty} T_{n,\lambda}(x, \lambda) \frac{t^n}{n!},
\] (2.1)

\[
\frac{2}{\zeta q(1 + \lambda t)^{2/\lambda} + 1} = \sum_{n=0}^{\infty} T_{n,\lambda}(\lambda) \frac{t^n}{n!}.
\] (2.2)

Note that \((1 + \lambda t)^{1/\lambda}\) tends to \(e^t\) as \(\lambda \to 0\). From (2.1), we note that

\[
\sum_{n=0}^{\infty} \lim_{\lambda \to 0} T_{n,\lambda}(x, \lambda) \frac{t^n}{n!} = \lim_{\lambda \to 0} \left( \frac{2}{\zeta q(1 + \lambda t)^{2/\lambda} + 1} \right) e^t = \left( \frac{2}{\zeta q e^{2t} + 1} \right) e^t = \sum_{n=0}^{\infty} T_{n,\lambda}(x) \frac{t^n}{n!}.
\]

Thus, we get

\[
\lim_{\lambda \to 0} T_{n,\lambda}(x, \lambda) = T_{n,\lambda}(x), (n \geq 0),
\]
where, $T_{n,q,\zeta}(x)$ are the usual twisted $q$-tangent polynomials (see [4]). From (2.1), we have
\[
\sum_{n=0}^{\infty} T_{n,q,\zeta}(x,\lambda) \frac{t^n}{n!} = \left( \frac{2}{\zeta q(1 + \lambda t)^{2/\lambda} + 1} \right) e^{xt}
\]
\[
= \left( \sum_{m=0}^{\infty} T_{m,q,\zeta}(\lambda) \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} x^l l! \right)
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} T_{l,q,\zeta}(\lambda) x^{n-l} \right) \frac{t^n}{n!}.
\]

Therefore, by (2.2) and (2.3), we obtain the following theorem.

**Theorem 2.1** For $n \geq 0$, we have
\[
T_{n,q,\zeta}(x,\lambda) = \sum_{l=0}^{n} \binom{n}{l} T_{l,q,\zeta}(\lambda) x^{n-l}.
\]

By (2.1), we see that
\[
\frac{d}{dx} T_{m,q,\zeta}(x,\lambda) = m T_{m-1,q,\zeta}(x,\lambda).
\]

By (2.4) we get
\[
\int_{0}^{x} \frac{d}{dt} \left( \frac{T_{n+1,q,\zeta}(t,\lambda)}{n+1} \right) dt = \int_{0}^{x} T_{n,q,\zeta}(t,\lambda) dt
\]
\[
= \frac{T_{n+1,q,\zeta}(x,\lambda) - T_{n+1,q,\zeta}(\lambda)}{n+1}.
\]

By (2.5), we have the following theorem.

**Theorem 2.2** For $n \in \mathbb{Z}_+$, we have
\[
\frac{T_{n+1,q,\zeta}(x,\lambda) - T_{n+1,q,\zeta}(\lambda)}{n+1} = \int_{0}^{x} T_{n,q,\zeta}(t,\lambda) dt.
\]

From (2.2), we can derive the following recurrence relation:
\[
2 = (\zeta q(1 + \lambda t)^{2/\lambda} + 1) \sum_{n=0}^{\infty} T_{n,q,\zeta}(\lambda) \frac{t^n}{n!}
\]
\[
= \zeta q(1 + \lambda t)^{2/\lambda} \sum_{n=0}^{\infty} T_{n,q,\zeta}(\lambda) \frac{t^n}{n!} + \sum_{n=0}^{\infty} T_{n,q,\zeta}(\lambda) \frac{t^n}{n!}
\]
\[
= \left( \sum_{l=0}^{\infty} \zeta q(2|\lambda|) tl! \sum_{m=0}^{\infty} T_{m,q,\zeta}(\lambda) \frac{t^m}{m!} \right) + \sum_{n=0}^{\infty} T_{n,q,\zeta}(\lambda) \frac{t^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} \zeta q(2|\lambda|) T_{l,q,\zeta}(\lambda) \right) \frac{t^n}{n!}.\]
By comparing of the coefficients \( \frac{n^n}{n!} \) on the both sides of (2.6), we have the following theorem.

**Theorem 2.3** For \( n \in \mathbb{Z}_+ \), we have

\[
\zeta_q \sum_{l=0}^{n} \binom{n}{l} (2|\lambda)_{l} T_{n-l,q,\zeta}(\lambda) + T_{n,q,\zeta}(\lambda) = \begin{cases} 
2, & \text{if } n = 0, \\
0, & \text{if } n \neq 0.
\end{cases}
\]

By (2.1), we get

\[
\sum_{n=0}^{\infty} T_{n,q,\zeta}(1-x,\lambda) \frac{t^n}{n!} = \frac{2}{\zeta_q(1+\lambda t)^{2/\lambda} + 1} e^{(1-x)t} 
= \frac{2}{(1+\lambda t)^{2/\lambda} + 1} e^t e^{-xt} 
= \left( \sum_{n=0}^{\infty} T_{n,q,\zeta}(1,\lambda) \frac{t^n}{n!} \right) \left( \sum_{l=0}^{\infty} (-x)^{l} \frac{t^l}{l!} \right) 
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} T_{n-l,q,\zeta}(1,\lambda)(-x)^{l} \right) \frac{t^n}{n!}.
\]

By comparing of the coefficients \( \frac{n^n}{n!} \) on the both sides of (2.7), we have the following theorem.

**Theorem 2.4** For \( n \in \mathbb{Z}_+ \), we have

\[
T_{m,q,\zeta}(1-x,\lambda) = \sum_{l=0}^{n} \binom{n}{l} T_{n-l,q,\zeta}(1,\lambda)(-x)^{l}.
\]

Again, from (2.2), we have

\[
\sum_{n=0}^{\infty} T_{n,q,\zeta}(x+y,\lambda) \frac{t^n}{n!} = \frac{2}{(1+\lambda t)^{2/\lambda} + 1} e^{(x+y)t} 
= \frac{2}{\zeta_q(1+\lambda t)^{2/\lambda} + 1} e^t e^{yt} 
= \left( \sum_{n=0}^{\infty} T_{m,q,\zeta}(x,\lambda) \frac{t^n}{n!} \right) \left( \sum_{n=0}^{\infty} y^n \frac{t^n}{n!} \right) 
= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} \binom{n}{l} T_{l,q,\zeta}(x,\lambda)y^{n-l} \right) \frac{t^n}{n!}.
\]

Therefore, by (2.8), we have the following theorem.
Theorem 2.5 For \( n \in \mathbb{Z}_+ \), we have
\[
T_{n,q,\zeta}(x + y, \lambda) = \sum_{l=0}^{n} \binom{n}{l} T_{l,q,\zeta}(x, \lambda) y^{n-l}.
\]

Then, it is easy to deduce that \( T_{n,q,\zeta}(x, \lambda) \) are polynomials of degree \( n \). Here is the list of the first Appell-type degenerate twisted \( q \)-tangent’s polynomials.

\[
T_{0,q,\zeta}(x, \lambda) = \frac{2}{1 + \zeta q},
\]

\[
T_{1,q,\zeta}(x, \lambda) = -\frac{4\zeta q}{(1 + \zeta q)^2} + \frac{2x}{(1 + \zeta q)^2} + \frac{2\zeta qx}{(1 + \zeta q)^2},
\]

\[
T_{2,q,\zeta}(x, \lambda) = -\frac{8\zeta q}{(1 + \zeta q)^3} + \frac{4\lambda q}{(1 + \zeta q)^3} + \frac{8\zeta^2 q^2}{(1 + \zeta q)^3} + \frac{4\lambda q^2}{(1 + \zeta q)^3} - \frac{2x^2}{(1 + \zeta q)^3} - \frac{8\zeta qx}{(1 + \zeta q)^3} - \frac{2\lambda q^3}{(1 + \zeta q)^3}.
\]

\[
T_{3,q,\zeta}(x, \lambda) = -\frac{16\zeta q}{(1 + \zeta q)^4} + \frac{24\lambda q}{(1 + \zeta q)^4} - \frac{8\lambda^2 \zeta q}{(1 + \zeta q)^4} + \frac{64\zeta^2 q^2}{(1 + \zeta q)^4} - \frac{16\lambda^2 q^2}{(1 + \zeta q)^4} - \frac{16\zeta^3 q^3}{(1 + \zeta q)^4} - \frac{24\lambda q^2}{(1 + \zeta q)^4} - \frac{12\lambda q^3}{(1 + \zeta q)^4} - \frac{24\zeta q^2 x}{(1 + \zeta q)^4} - \frac{12\lambda q^2 x}{(1 + \zeta q)^4} - \frac{2\lambda q^3 x}{(1 + \zeta q)^4} + \frac{2x^3}{(1 + \zeta q)} - \frac{2x^2}{(1 + \zeta q)}.
\]

3 Zeros of the Appell-type degenerate twisted \( q \)-tangent polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the Appell-type degenerate twisted \( q \)-tangent polynomials \( T_{n,q,\zeta}(x, \lambda) \).

We investigate the zeros of the \( T_{n,q,\zeta}(x, \lambda) \) by using a computer. Let \( \zeta = e^{\frac{2\pi i}{N}} \) in \( \mathbb{C} \). We plot the zeros of the Appell-type degenerate twisted \( q \)-tangent polynomials \( T_{n,q,\zeta}(x, \lambda) \) for \( n = 30, N = 1, 3, 5, 7 \) and \( x \in \mathbb{C} \)(Figure 1). In Figure 1(top-left), we choose \( n = 30, q = 1/2, \lambda = 1/10, \) and \( \zeta = e^{\frac{2\pi i}{5}} \). In Figure 1(top-right), we choose \( n = 30, q = 1/2, \lambda = 1/10, \) and \( \zeta = e^{\frac{2\pi i}{10}} \). In Figure 1(bottom-left), we choose \( n = 30, q = 1/2, \lambda = 1/10, \) and \( \zeta = e^{\frac{2\pi i}{15}} \). In Figure 1(bottom-right), we choose \( n = 30, q = 1/2, \lambda = 1/10, \) and \( \zeta = e^{\frac{2\pi i}{24}} \). Stacks of zeros of \( T_{n,q,\zeta}(x, \lambda) \) for \( 1 \leq n \leq 30 \) from a 3-D structure are presented(Figure 2). In Figure 2(left), we choose \( 1 \leq n \leq 30 \) and \( \zeta = e^{\frac{2\pi i}{2}}, q = 1/2, \lambda = 1/10. \) In Figure 2(right), we choose \( 1 \leq n \leq 30 \) and \( \zeta = e^{\frac{2\pi i}{3}}, q = 1/2, \lambda = 1/10. \) Our numerical results for approximate solutions of real zeros of \( T_{n,q,\zeta}(x, \lambda) \) are displayed(Tables 1, 2).
\begin{figure}
\centering
\includegraphics[width=\textwidth]{zeros.png}
\caption{Zeros of $T_{n,q,\zeta}(x, \lambda)$}
\end{figure}

Table 1. Numbers of real and complex zeros of $T_{n,q,\zeta}(x, \lambda)$

<table>
<thead>
<tr>
<th>degree $n$</th>
<th>$\zeta = e^{\frac{2\pi i}{n}}, q = \frac{1}{2}, \lambda = \frac{1}{10}$</th>
<th>$\zeta = e^{\frac{2\pi i}{n}}, q = \frac{1}{2}, \lambda = \frac{1}{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>real zeros</td>
<td>complex zeros</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
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<td>3</td>
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<tr>
<td>10</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>
For $q = \frac{1}{2}$ and $\lambda = \frac{1}{10}$, a plot of real zeros of $T_{n,q,\zeta}(x, \lambda)$ for $1 \leq n \leq 30$ structure are presented (Figure 3).

We observe a remarkably regular structure of the complex roots of the Appell-type degenerate twisted $q$-tangent polynomials $T_{n,q,\zeta}(x, \lambda)$ (see Table 1). Next, we calculated an approximate solution satisfying $T_{n,q,\zeta}(x, \lambda) = 0$ for $q = \frac{1}{2}, \lambda = \frac{1}{10}, x \in \mathbb{C}$. The results are given in Table 2.

**Table 2.** Approximate solutions of $T_{n,q,\zeta}(x, \lambda) = 0, w = e^{2\pi i}, x \in \mathbb{C}$

<table>
<thead>
<tr>
<th>degree $n$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$0.59968 + 0.41731i$</td>
</tr>
<tr>
<td>2</td>
<td>$-0.38816 + 0.26932i, 1.5875 + 0.5653i$</td>
</tr>
<tr>
<td>3</td>
<td>$-1.03755 - 0.00382i, 0.42299 + 0.70680i, 2.4136 + 0.5490i$</td>
</tr>
<tr>
<td>4</td>
<td>$-1.4774 - 0.4068i, -0.56189 + 0.72195i$ $1.3325 + 0.9280i, 3.1055 + 0.4261i$</td>
</tr>
</tbody>
</table>
4 Conclusions and future developments

This study introduced the Appell-type degenerate twisted $q$-tangent polynomials. We have derived several formulas for the Carlitz-type $(h, q)$-tangent numbers and polynomials. Some properties for the Appell-type degenerate twisted $q$-tangent polynomials are also obtained. Moreover, the results of [5] can be derived from ours as special cases when $q \to 1$. By numerical experiments, we will make a series of the following conjectures: Many more values of $n$ have been checked. It still remains unknown if the conjecture holds or fails for any value $n$ (see Figure 1, Table 1). Prove or disprove that $T_{n,q,ζ}(x, λ) = 0$ has $n$ distinct solutions. In the notations: $R_{T_{n,q,ζ}(x, λ)}$ denotes the number of real zeros of $T_{n,q,ζ}(x, λ)$ lying on the real plane $Im(x) = 0$ and $C_{T_{n,q,ζ}(x, λ)}$ denotes the number of complex zeros of $T_{n,q,ζ}(x, λ)$. Since $n$ is the degree of the polynomial $T_{n,q,ζ}(x, λ)$, the number of real zeros $R_{T_{n,q,ζ}(x, λ)}$ lying on the real plane $Im(x) = 0$ is then $R_{T_{n,q,ζ}(x, λ)} = n - C_{T_{n,q,ζ}(x, λ)}$. See Table 1 for tabulated values of $R_{T_{n,q,ζ}(x, λ)}$ and $C_{T_{n,q,ζ}(x, λ)}$.

References


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