KKM-Type Theorems for Best Proximal Points in Normed Linear Space

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Abstract

We generalize the $R$-KKM theorem in [4] to that for the intersectionally closed-valued maps and using this, we prove best proximity pair theorems for multimaps with (unionly) open fibers in normed linear spaces which generalize the results of [4].

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1 Introduction and Preliminaries

A multimap (or map) $F : X \to Y$ is a function from a set $X$ into the power set $2^Y$ of $Y$; that is, a function with the values $F(x) \subset Y$ for $x \in X$. For $A \subset X$, let $F(A) = \bigcup \{F(x) : x \in A\}$. Let $\bar{F}$ denote the closure of $F$.

Let $(M, d)$ be a metric space and $A$ and $B$ be nonempty subsets of $M$.

Consider a multimap $F : A \to M$. A point $x_0 \in A$ is called a best proximal point of $F$ if $d(x_0, F(x_0)) = d(A, B)$. In this case, $(x_0, F(x_0))$ is called a best proximity pair for $F$. Note that if $d(A, B) = 0$ and $F$ is a single valued map, then the best proximal point is a fixed point of $F$. 
The following notations are used in the sequel.

\[ A_0 = \{ x \in A : d(x, y) = d(A, B) \text{ for some } y \in B \}, \]

\[ B_0 = \{ y \in B : d(x, y) = d(A, B) \text{ for some } x \in A \}. \]

The pair \((A, B)\) is said to be a proximal pair if, for each \((x, y) \in A \times B\), there exists \((\tilde{x}, \tilde{y}) \in A \times B\) such that \(d(x, \tilde{y}) = d(\tilde{x}, y) = d(A, B)\). A pair \((A, B)\) is a proximal pair if and only if \(A = A_0\) and \(B = B_0\).

Sankar Raj and Somasundaram [4] introduced \(R\)-KKM maps as follows:

Let \((A, B)\) be a proximal pair of a normed linear space \(M\) and \(A\) be convex. The map \(F : B \rightrightarrows A\) is said to be an \(R\)-KKM map if for any \(\{y_1, \ldots, y_m\} \subset B\), there exists \(\{x_1, \ldots, x_m\} \subset A\) with \(\|x_j - y_j\| = d(A, B)\) for each \(j = 1, \ldots, m\) such that \(\text{co}\{x_1, \ldots, x_m\} \subset \bigcup_{j=1,\ldots,m} F(y_j)\).

They proved an extended version of the Fan-Browder multivalued fixed point theorem for the best proximal point setting;

**Theorem 1.1.** Let \((A, B)\) be a nonempty compact convex proximal pair in a normed linear space \(M\). Let \(F : A \rightrightarrows B\) be a multimap such that

1. \(F(x)\) is convex for each \(x \in A\); and
2. \(F^{-1}(y)\) is open for each \(y \in B\).

Then there is a \(w \in A\) such that \(d(w, F(w)) = d(A, B)\).

Recently the author [1] showed that \(R\)-KKM maps are generalized KKM maps, applied \(R\)-KKM property to the map defined on product spaces, and proved new best proximal point theorems in normed linear spaces using the generalized KKM theorem.

In this paper, we just generalize the \(R\)-KKM theorem of [4] to that for the intersectionally closed-valued maps and prove fixed points of best proximity pair theorem for maps with unionly open fibers in normed linear spaces in [1] with different method. We also prove fixed points of best proximity pair theorems for a family of maps with open fibers in normed linear spaces. These theorems generalize the results of [4].

## 2 Main Results

When \(A\) is a nonempty subset of a normed linear space \(M\), let \(P_A(y) = \{ x \in A : \|x - y\| = d(y, A) \}\).

The author [1] generalized \(R\)-KKM maps in [4] as follows:

For \(I = \{1, \ldots, n\}\), let \((A, B_i)\) be a proximal pair of a normed linear space \(M\), \(A\) be convex, and \(B := \prod_i B_i\). The map \(F : B \rightrightarrows A\) is said to be an \(R\)-KKM map if for any \(\{y^1, \ldots, y^m\} \subset B\),
there exists \( \{x_1, \ldots, x_m\} \subset A \) with \( x_j \in \bigcap_{i \in I} P_A(y_i^j) \) for each \( j = 1, \ldots, m \) such that \( \text{co}\{x_1, \ldots, x_m\} \subset \bigcup_{j=1,\ldots,m} F(y^j) \).

When \( Z \) is a set and \( X \) is a topological space, consider the following related three conditions for \( F : Z \twoheadrightarrow X \):

(a) \( \bigcap_{z \in Z} F(z) = \bigcap_{z \in Z} F(z) \) (\( F \) is intersectionally closed-valued).

(b) \( \bigcap_{z \in Z} F(z) = \bigcap_{z \in Z} F(z) \) (\( F \) is transfer closed-valued).

(c) \( F \) is closed-valued.

Luc et al. [3] noted that (c) \( \implies \) (b) \( \implies \) (a).

A multimap \( F : X \twoheadrightarrow Y \) is said to be unionly open-valued (resp., transfer open-valued) on \( X \) if and only if the multimap \( G : X \twoheadrightarrow Y \), defined by \( G(x) = Y \setminus F(x) \) for every \( x \in X \), is intersectionally closed-valued (resp., transfer closed-valued) on \( X \). See [3] and Tian [5].

**Lemma 2.1.** Let \( Z \) be a set, \( X \) be a topological space, \( Y \) be a closed subset of \( X \) and \( F : Z \twoheadrightarrow X \) be a transfer closed-valued map. And let \( S : Z \twoheadrightarrow Y \) be a map defined by \( S(z) = F(z) \cap Y \) for \( z \in Z \). Then \( S \) is transfer closed-valued.

**Proof.** Since \( \bigcap_{z \in Z} F(z) = \bigcap_{z \in Z} F(z), \bigcap_{z \in Z} F(z) = \bigcap_{z \in Z} F(z) \bigcap Y = \bigcap_{z \in Z} F(z) \bigcap Y \neq \emptyset \bigcap S(z). \)

The following KKM type theorem is Theorem 3.2 in [4];

**Theorem 2.2.** Let \((A, B)\) be a nonempty proximal pair in a normed linear space \( M \) and \( F : B \twoheadrightarrow A \) be an \( R \)-KKM map. If, for each \( x \in B, F(z) \) is closed in \( M \) and there exists at least one \( z_0 \in B \) such that \( F(z_0) \) is compact in \( M \), then \( \bigcap_{z \in Z} F(z) \neq \emptyset \).

**Theorem 2.3.** Let \((A, B)\) be a nonempty proximal pair of a normed linear space \( M \), and \( F : B \twoheadrightarrow A \) be a multimap satisfying

1. \( \overline{F} : B \twoheadrightarrow M \) is an \( R \)-KKM map;
2. \( F \) is intersectionally closed-valued; and
3. there exists at least one \( z_0 \in B \) such that \( \overline{F(z_0)} \) is compact in \( M \).

Then \( \bigcap_{z \in B} F(z) \neq \emptyset \).

**Proof.** Since \( \overline{F} \) is an \( R \)-KKM map with closed values, by Theorem 2.2, we have \( \bigcap_{z \in B} \overline{F(z)} \neq \emptyset \). Since \( F \) is intersectionally closed-valued, we have

\[ \bigcap_{z \in B} F(z) = \bigcap_{z \in B} \overline{F(z)} \neq \emptyset . \]

The following is an existence theorem for the best proximity pairs:
Theorem 2.4. Let $I = \{1, \ldots, n\}$ and for each $i \in I$, $(A, B_i)$ be a nonempty convex proximal pair in a normed linear space $M$. Let $A$ be compact. For each $i \in I$, let $F_i : A \rightharpoonup B_i$ be a multimap such that

1. $F_i(x)$ is convex for each $x \in A$;
2. $F_i^{-1}(y_i)$ is open for each $y_i \in B_i$; and
3. $\bigcap_{i \in I} P_A(y_i) \neq \emptyset$ for each $(y_1, \ldots, y_n) \in B = \prod B_i$.

Then there is a $w \in A$ such that $d(w, F_i(w)) = d(A, B_i)$ for each $i \in I$.

Proof. Define $F : A \rightharpoonup B$ by $F(x) = \prod F_i(x)$ and $G : B \rightharpoonup A$ by $G(y_1, \ldots, y_n) = A \setminus F^{-1}(y_1, \ldots, y_n)$. Then $G(y_1, \ldots, y_n) = \{x \in A : y_i \notin F_i(x)$ for some $i \in I\} = A \setminus \bigcap_{i \in I} F_i^{-1}(y_i)$.

(a) Suppose that for some $(y_1, \ldots, y_n) \in B$, $G(y_1, \ldots, y_n) = \emptyset$. Then $F^{-1}(y_1, \ldots, y_n) = A$, that is, $(y_1, \ldots, y_n) \in F(x)$ for all $x \in A$. Since $\bigcap_{i \in I} P_A(y_i) \neq \emptyset$ and $y_i \in B_i = B_0$, there exists a $w \in A$ such that $||y_i - w|| = d(y_i, A) = d(A, B_i)$ for all $i \in I$. Note that $(y_1, \ldots, y_n) \in F(w)$ and for each $i \in I$,

$$d(A, B_i) \leq d(w, F_i(w)) \leq ||w - y_i|| = d(A, B_i).$$

Therefore $d(w, F_i(w)) = d(A, B_i)$ for each $i \in I$.

(b) Assume that $G(y_1, \ldots, y_n)$ is nonempty for each $(y_1, \ldots, y_n) \in B$. By the hypothesis, $G$ is nonempty closed-valued and

$$\bigcap_{(y_1, \ldots, y_n) \in B} G(y_1, \ldots, y_n) = \bigcap_{(y_1, \ldots, y_n) \in B} \bigcap_{i \in I} (A \setminus F_i^{-1}(y_i))$$

$$= \bigcap_{i \in I} \bigcap_{(y_1, \ldots, y_n) \in B} (A \setminus F_i^{-1}(y_i))$$

$$= \bigcap_{i \in I} (A \setminus \bigcup F_i^{-1}(y_i)) = \bigcap_{i \in I} \emptyset = \emptyset.$$

Therefore $G$ is not an $R$-KKM map.

Hence there exist $\{y_j = (y^1_j, \ldots, y^m_j), \ldots, y^m_j = (y^1_j, \ldots, y^m_j)\}$ and $x_j \in \bigcap_{i \in I} P_A(y^i_j)$ for $j = 1, \ldots, m$ such that $co\{x_1, \ldots, x_m\} \not\subset \bigcup_{j=1,\ldots,m} G(y^j)$. Choose $w = \sum_j \lambda_j x_j \in co\{x_1, \ldots, x_m\} \setminus \bigcup_{j=1,\ldots,m} G(y^j)$, i.e., $w \in \bigcap_{j=1}^m F^{-1}(y^j)$. Therefore $y^j_i \in F_i(w)$ for all $j = 1, \ldots, m$ and $i \in I$. Put $z_i := \sum_j \lambda_j y^j_i$. Since $F_i(w)$ is convex and $y^j_i \in F_i(w)$, $z_i \in F_i(w)$ for each $i \in I$. For each $i \in I$,

$$d(A, B_i) \leq d(w, F_i(w)) \leq ||w - z_i|| = ||\sum_j \lambda_j x_j - \sum_j \lambda_j y^j_i|| \leq \sum_j \lambda_j ||x_j - y^j_i|| = \sum_j \lambda_j d(A, y^j_i) = d(A, B_i).$$

Therefore $d(w, F_i(w)) = d(A, B_i)$ for all $i \in I$.

\[ \square \]

If $I$ is a singleton, then $P_A(y) \neq \emptyset$ for each $y \in B$, since $B = B_0$. Therefore the condition (3) in Theorem 2.4 is automatically satisfied. Using Theorem 2.3 and the argument of the proof in Theorem 2.4, we obtain the following theorem which extends Theorem 1.1;
Theorem 2.5. Let \((A, B)\) be a nonempty convex proximal pair in a normed linear space \(M\). Let \(A\) be compact and \(F : A \rightarrow B\) be a multimap such that
(1) \(F(x)\) is convex for each \(x \in A\); and
(2) \(F^{-1}(y)\) is unionly open for each \(y \in B\).
Then there is a \(w \in A\) such that \(d(w, F(w)) = d(A, B)\).

Theorem 2.5 is the case that \((A, B)\) is a proximal pair. The following existence theorem is for \(A \neq A_0\) or \(B \neq B_0\):

Theorem 2.6. Let \(A\) and \(B\) be nonempty compact convex subsets of a normed linear space \(M\), \(A_0 \neq \emptyset\) and \(F : A \rightarrow B\) be a multimap such that
(1) \(F(x) \cap B_0 \neq \emptyset\) for each \(x \in A_0\);
(2) \(F(x)\) is convex for each \(x \in A_0\); and
(3) \(F^{-1}(y)\) is transfer open for each \(y \in B_0\).
Then there is a \(w \in A\) such that \(d(w, F(w)) = d(A, B)\).

Proof. The first part of this proof is proved by [2].

Let \(x_1, x_2 \in A_0\). Then there exist \(y_1, y_2 \in B_0\) such that \(||x_1 - y_1|| = ||x_2 - y_2|| = d(A, B)\). For \(\lambda \in (0, 1)\), let \(w = \lambda x_1 + (1 - \lambda)x_2\) and \(z = \lambda y_1 + (1 - \lambda)y_2\). Then \(w \in A\), \(z \in B\) and \(||z - w|| \leq \lambda||x_1 - y_1|| + (1 - \lambda)||x_2 - y_2|| = d(A, B)\).

So \(w \in A_0\) and hence \(A_0\) is convex.

In order to prove \(A_0\) is compact, it is enough to show that \(A_0\) is closed. Let \(\{x_n\}_{n \in \mathbb{N}}\) be a sequence in \(A_0\) which converges to \(x \in A\). Then there exists \(\{y_n\} \subset B_0 \subset B\) such that \(||x_n - y_n|| = d(A, B)\). Since \(B\) is compact, there exists a convergent subsequence \(\{y_{n_k}\}\) of \(\{y_n\}\) which converges to \(y \in B\). Then \(d(A, B) \leq ||x - y|| \leq ||x - x_{n_k}|| + ||x_{n_k} - y_{n_k}|| + ||y_{n_k} - y||\). Since \(||x - x_{n_k}||\) converges to 0, \(d(A, B) = ||x - y||\). Therefore \(x \in A_0\) and so \(A_0\) is compact.

Similarly the convexity and compactness of \(B_0\) can be proved.

For each \(x \in A_0\), there exists a \(y \in B_0\) such that \(||x - y|| = d(A, B)\). Since \(d(A, B) = d(A_0, B_0) \leq ||x - y||, ||x - y|| = d(A, B) = d(A_0, B_0)\). Put \(C = A_0\) and \(D = B_0\). The above argument shows that \(C = C_0\). By the same method we can show that \(D = D_0\). Therefore \((A_0, B_0)\) is a proximal pair of \(M\).

Define \(S : A_0 \rightarrow B_0\) by \(S(x) = F(x) \cap B_0\) for each \(x \in A_0\). By Lemma 2.1, \(S^{-1}(y)\) is transfer open for each \(y \in B_0\) and \(S\) has convex values. By Theorem 2.5, there is a \(w \in A_0\) such that \(d(w, S(w)) = d(A_0, B_0)\). Since \(d(w, S(w)) \geq d(w, F(w)) \geq d(A, B) = d(A_0, B_0)\), \(d(w, F(w)) = d(A, B)\).

Theorem 2.7. For each \(i \in I = \{1, \cdots, n\}\), let \(A\) and \(B_i\) be nonempty compact convex subsets of a normed linear space \(M\), and \(F_i : A \rightarrow B_i\) be a multimap such that
(1) \(C := \bigcap_{i \in I} A_{0i} \neq \emptyset\) where \(A_{0i} := \{x \in A : ||x - y|| = d(A, B_i)\} for some \(y \in B_i\}\);
(2) \( F_i(x) \cap B_{\partial} \neq \emptyset \) for each \( x \in C \);
(3) \( F_i(x) \) is convex for each \( x \in C \);
(4) \( F_i^{-1}(y_i) \) is open for each \( y_i \in B_{\partial} \); and
(5) \( \bigcap_{i \in I} P_{\lambda}(y_i) \neq \emptyset \) for each \( (y_1, \cdots, y_n) \in \prod B_{\partial} \).
Then there is a \( w \in A \) such that \( d(w, F_i(w)) = d(A, B_i) \) for each \( i \in I \).

**Proof.** Let \( i \in I \). According to the proof of Theorem 2.6, \( A_{0i} \) and \( B_{\partial} \) are compact and convex. Therefore, \( C \) is compact and convex. For each \( x \in C \), there exists a \( y_i \in B_{\partial} \) such that \( ||x - y_i|| = d(A, B_i) \) by (1). And \( d(A, B_i) \leq d(C, B_{\partial}) \leq ||x - y_i|| \), so \( ||x - y_i|| = d(C, B_{\partial}) \), that is, \( C = \{ x \in C : ||x - y_i|| = d(C, B_{\partial}) \} \) for some \( y_i \in B_{\partial} \). Note that \( d(C, B_{\partial}) = d(A, B_i) \).

For \( (y_1, \cdots, y_n) \in \prod B_{\partial} \), there exists an \( x \in A \) such that \( ||x - y_i|| = d(A, y_i) \) for each \( i \in I \), by condition (5). Since \( y_i \in B_{\partial} \), \( d(A, y_i) = d(A, B_i) \). And \( ||x - y_i|| = d(C, B_{\partial}) \) as \( d(C, B_{\partial}) = d(A, B_i) \). That is, \( x \in \bigcap_{i \in I} P_C(y_i) \) for each \( (y_1, \cdots, y_n) \in \prod B_{\partial} \). And \( B_i = \{ y_i \in B_i : ||x - y_i|| = d(C, B_{\partial}) \} \) for some \( x \in C \) for each \( i \in I \). Hence \( (C, B_{\partial}) \) is a proximity pair of \( M \) for each \( i \in I \).

Define \( S_i : C \rightarrow B_{\partial} \) by \( S_i(x) = F_i(x) \cap B_{\partial} \) for each \( x \in C \). Since \( S_i \) satisfies all the conditions of Theorem 2.4, there is a \( w \in C \) such that \( d(w, S_i(w)) = d(C, B_{\partial}) \). So \( d(A, B_i) \leq d(w, F_i(w)) \leq d(w, S_i(w)) = d(C, B_{\partial}) = d(A, B_i) \) for each \( i \in I \). Therefore the conclusion holds. \( \square \)

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