A Note on Hölder Estimates of Solutions to a Degenerate Diffusion Equation

Ricardo Cano

Universidad de La Sabana, Facultad de Ingeniería, Chía, Colombia

Leonardo Rendón Arbeláez

Universidad Nacional de Colombia, Facultad de Ciencias
Departamento de Matemáticas, Bogotá, Colombia

Juan C. Juajibioy

Universidad Pedagógica y Tecnológica de Colombia, Tunja, Colombia

Copyright © 2018 Ricardo Cano, Leonardo Rendón Arbeláez and Juan C. Juajibioy.
This article is distributed under the Creative Commons Attribution License, which permits
unrestricted use, distribution, and reproduction in any medium, provided the original work
is properly cited.

Abstract

In this paper we are concerned with the Hölder regularity of weak solutions of the Cauchy problem for the general degenerate parabolic equations in (1). By using the vanishing viscosity method and the maximum principle some explicit Hölder exponent are obtained.

In this paper, we are interested in the Hölder estimates for weak solutions of the Cauchy problem for the general degenerate parabolic equations in (1)

$$\begin{cases} u_t = \Delta G(u) + \sum_{j=1}^N f_j(u)_{x_j} + g(x, u, t) & \text{in } \mathbb{R}^N \times [0, T] \\ u(x, 0) = u_0(x) & \text{on } \mathbb{R}^N, \end{cases} \quad (1)$$

where $t \in [0, T]$, for a fixed time given T , $x := (x_1, \dots, x_N) \in \mathbb{R}^N$ and $G : \mathbb{R} \rightarrow \mathbb{R}$ is a nondecreasing smooth function.

The kind of equations in (1), arise in several applications such as flows on porous medium and biological models. Notice that, when G is a constant function, equation in (1), reduces to the multidimensional scalar conservation laws with a source term. In particular, when the functions $f_j(u)$ are nonlinear, the solutions blowup in finite time. This paper use principally the technique developed in [12], where the author consider the Hölder regularity for the source term $h(u)$ that satisfies $|h'(u)| \leq c G'(u)$, where c is a positive constant. We consider only the condition $g(x, u, t) \in C_0^2(\mathbb{R}^N \times [0, T])$.

The most important result of this paper is the following theorem.

Theorem 1. *Suppose that $G'(u) \geq 0$ and the solutions u is bounded: $|u| \leq \bar{u}$. If*

$$\left| \frac{GG''}{G'^2} \right| \leq \beta, \quad \beta^2 \leq \frac{1}{2N}, \quad (2)$$

and

$$|f_j''(u)| \leq c |G''(u)|, \quad g(x, u, t) \in C_0^2(\mathbb{R}^N \times [0, T]) \quad (3)$$

for a suitable positive constant c , then the solutions u of the Cauchy problem (1), satisfy the global estimates

$$|\nabla G^\alpha(u)| \leq M \quad \text{on } \mathbb{R}^N \times \mathbb{R}^+, \quad (4)$$

if the initial condition in (1), satisfies the same regular estimates (4), where M is a positive constant depending only on the initial condition, and

$$\alpha > 1 - \frac{1 + \sqrt{1 - 2N\beta^2}}{4}. \quad (5)$$

If $f_j(u) \equiv 0$ and $g(x, u, t) \equiv 0$, the equality in (5) is true. For details see [10].

Proof. Without loss of generality, let $G(u) \geq 0$. Consider the alternative Cauchy problem related to (1),

$$\begin{cases} u_t = \Delta G^\varepsilon(u) + \sum_{j=1}^N f_j(u)_{x_j} + g(x, u, t) \\ u^\varepsilon(x, 0) = u_0(x) + \varepsilon_1, \end{cases} \quad (6)$$

where $G^\varepsilon(u) = G(u) + \varepsilon(u + \bar{u})$, with ε and ε_1 small nonnegative constants. For the porous medium equation, we choose $\varepsilon = 0, \varepsilon_1 > 0$; for other cases, we choose $\varepsilon_1, \varepsilon > 0$. Thus for fixed ε and ε_1 , the equation in (6) is strictly parabolic. We can resolve the Cauchy problem (6) and obtain the approximate solutions $u^{\varepsilon, \varepsilon_1}$. Since we assume $u^{\varepsilon, \varepsilon_1}$ to be bounded, there exists a subsequence $u^{\varepsilon^l, \varepsilon_1^l}$ converging weakly to a bounded function u as $\varepsilon^l, \varepsilon_1^l \rightarrow 0^+$, on any bounded region. If we can prove the Hölder continuity of $G(u^{\varepsilon, \varepsilon_1})$, then $G(u^{\varepsilon^l, \varepsilon_1^l})$ converges

weakly to $G(u)$ by applying the Minty device with the condition $G'(u) \geq 0$. From the conditions in (3), we can prove that $f_j(u^{\varepsilon^l, \varepsilon_1^l})$ and $g(x, u^{\varepsilon^l, \varepsilon_1^l}, t)$ converge weakly to $f_j(u)$ and $g(x, u, t)$ respectively. Therefore the limit u is a weak solution of the Cauchy problem (1) and $G(u)$ satisfies the same Hölder estimates. This technique is standard, and the details can be found in [11]. Here we only give the proof of the uniform estimate (4).

Make the transformation

$$w = \frac{1}{2} \sum_{i=1}^N (G^\varepsilon(u)_{x_i})^2, \quad (7)$$

and set $P(u) = G^\varepsilon(u)$. So $P_{x_i} = P'(u)u_{x_i}$ and $u_{x_i} = \frac{P_{x_i}}{P'(u)}$. From (6), we obtain

$$\begin{aligned} P_t &= P'(u)u_t = P'(u) \left(\Delta G^\varepsilon(u) + \sum_{j=1}^N f_j(u)_{x_j} + g(x, u, t) \right) \\ &= P'(u)\Delta G^\varepsilon(u) + \sum_{j=1}^N f'_j(u)P'(u)u_{x_j} + P'(u)g(x, u, t) \\ &= P'(u)\Delta P + \sum_{j=1}^N f'_j(u)P_{x_j} + P'(u)g(x, u, t), \quad (8) \\ (P_{x_i})_t &= \left(P'(u)u_{x_i} \right)_t = P''(u)u_{x_i}u_t + P'(u)(u_{x_i})_t \\ &= \frac{P''(u)}{P'(u)}P_{x_i} \left(\Delta P + \sum_{j=1}^N f_j(u)_{x_j} + g(x, u, t) \right) \\ &\quad + P'(u) \left(\Delta P + \sum_{j=1}^N f_j(u)_{x_j} + g(x, u, t) \right)_{x_i} \\ &= \frac{P''(u)}{P'(u)}P_{x_i}\Delta P + \frac{P''(u)}{P'(u)}P_{x_i} \sum_{j=1}^N f_j(u)_{x_j} + \frac{P''(u)}{P'(u)}P_{x_i}g(x, u, t) \\ &\quad + P'(u)\Delta(P_{x_i}) + P'(u) \left(\sum_{j=1}^N f_j(u)_{x_j} \right)_{x_i} + P'(u)g(x, u, t)_{x_i} \\ &= P'(u)\Delta(P_{x_i}) + \frac{P''(u)}{P'(u)}P_{x_i}\Delta P + \left(P'(u)g(x, u, t) \right)_{x_i} \\ &\quad + P'(u)_{x_i} \sum_{j=1}^N f_j(u)_{x_j} + P'(u) \left(\sum_{j=1}^N f_j(u)_{x_j} \right)_{x_i} \\ &= P'(u)\Delta(P_{x_i}) + \frac{P''(u)}{P'(u)}P_{x_i}\Delta P + \left(P'(u)g(x, u, t) \right)_{x_i} \\ &\quad + \left(P'(u) \sum_{j=1}^N f_j(u)_{x_j} \right)_{x_i} \end{aligned}$$

$$\begin{aligned}
&= P'(u)\Delta(P_{x_i}) + \frac{P''(u)}{P'(u)}P_{x_i}\Delta P + \left(\sum_{j=1}^N f'_j(u)P_{x_j} \right)_{x_i} \\
&\quad + \left(P'(u)g(x, u, t) \right)_{x_i}
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
w_t &= \left(\frac{1}{2} \sum_{i=1}^N (P_{x_i})^2 \right)_t = \sum_{i=1}^N P_{x_i} (P_{x_i})_t \\
&= \sum_{i=1}^N P_{x_i} \left(P'(u)\Delta(P_{x_i}) + \frac{P''(u)}{P'(u)}P_{x_i}\Delta P + \left(P'(u)g(x, u, t) \right)_{x_i} \right) \\
&\quad + \sum_{i=1}^N P_{x_i} \left(\sum_{j=1}^N f'_j(u)P_{x_j} \right)_{x_i} \\
&= P'(u) \sum_{i=1}^N P_{x_i}\Delta(P_{x_i}) + \frac{P''(u)}{P'(u)}\Delta P \sum_{i=1}^N (P_{x_i})^2 \\
&\quad + \sum_{i=1}^N P_{x_i} \left(P'(u)g(x, u, t) \right)_{x_i} + \sum_{i=1}^N P_{x_i} \left(\sum_{j=1}^N f'_j(u)P_{x_j} \right)_{x_i} \\
&= P'(u) \left(\sum_{i,j=1}^N P_{x_i}P_{x_ix_jx_j} + \sum_{i,j=1}^N (P_{x_ix_j})^2 - \sum_{i,j=1}^N (P_{x_ix_j})^2 \right) \\
&\quad + \sum_{i=1}^N P_{x_i} \left(P''(u)u_{x_i}g(x, u, t) + P'(u)(g_{x_i}(x, u, t) + g_u(x, u, t)u_{x_i}) \right) \\
&\quad + \frac{2P''(u)}{P'(u)}w\Delta P + \sum_{i=1}^N P_{x_i} \sum_{j=1}^N (f'_j(u)P_{x_j})_{x_i} \\
&= P'(u)\Delta w - P'(u) \sum_{i,j=1}^N (P_{x_ix_j})^2 + \frac{2P''(u)}{P'(u)}w\Delta P \\
&\quad + \frac{2P''(u)}{P'(u)}wg(x, u, t) + 2wg_u(x, u, t) \\
&\quad + P'(u) \sum_{i=1}^N P_{x_i}g_{x_i}(x, u, t) + \sum_{i=1}^N P_{x_i} \sum_{j=1}^N (f''_j(u)u_{x_i}P_{x_j} + f'_j(u)P_{x_jx_i}) \\
&= P'(u)\Delta w - P'(u) \sum_{i,j=1}^N (P_{x_ix_j})^2 + \frac{2P''(u)}{P'(u)}w\Delta P \\
&\quad + 2 \left(\frac{P''(u)g(x, u, t)}{P'(u)} + g_u(x, u, t) \right) w + P'(u) \sum_{i=1}^N P_{x_i}g_{x_i}(x, u, t)
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^N P_{x_i} \left(\sum_{j=1}^N \frac{f_j''(u)}{P'(u)} P_{x_j} P_{x_i} + \sum_{j=1}^N f_j'(u) P_{x_j x_i} \right) \\
& = P'(u) \Delta w - P'(u) \sum_{i,j=1}^N (P_{x_i x_j})^2 + \frac{2P''(u)}{P'(u)} w \Delta P \\
& + 2 \left(\frac{P''(u)g(x, u, t)}{P'(u)} + g_u(x, u, t) \right) w + P'(u) \sum_{i=1}^N P_{x_i} g_{x_i}(x, u, t) \\
& + \sum_{j=1}^N \frac{2f_j''(u)}{P'(u)} P_{x_j} w + \sum_{j=1}^N f_j'(u) w_{x_j}. \tag{10}
\end{aligned}$$

Let $z = P^s w$, s be a constant. Then $w = P^{-s} z$,

$$\begin{aligned}
w_{x_j} &= z_{x_j} P^{-s} - s P^{-s-1} P_{x_j} z, \\
w_{x_j x_j} &= z_{x_j x_j} P^{-s} - s P^{-s-1} P_{x_j} z_{x_j} - s P^{-s-1} P_{x_j} z_{x_j} \\
&\quad - s \left((-s-1) P^{-s-2} (P_{x_j})^2 + P^{-s-1} P_{x_j x_j} \right) z \\
&= z_{x_j x_j} P^{-s} - 2s P^{-s-1} P_{x_j} z_{x_j} + s(s+1) P^{-s-2} (P_{x_j})^2 z \\
&\quad - s P^{-s-1} P_{x_j x_j} z. \\
\sum_{j=1}^N w_{x_j x_j} &= P^{-s} \sum_{j=1}^N z_{x_j x_j} - 2s P^{-s-1} \sum_{j=1}^N P_{x_j} z_{x_j} \\
&\quad + 2s(s+1) P^{-s-2} z \left(\frac{1}{2} \sum_{j=1}^N (P'(u) u_{x_j})^2 \right) - s P^{-s-1} z \sum_{j=1}^N P_{x_j x_j},
\end{aligned}$$

and

$$\begin{aligned}
\Delta w &= P^{-s} \Delta z - 2s P^{-s-1} \sum_{j=1}^N P_{x_j} z_{x_j} + 2s(s+1) P^{-s-2} z^2 \\
&\quad - s P^{-s-1} z \Delta P. \tag{11}
\end{aligned}$$

Then we have by (8)-(11) that

$$\begin{aligned}
z_t &= P^s w_t + s P^{s-1} P_t w \\
&= P^s \left(P'(u) \Delta w - P'(u) \sum_{i,j=1}^N (P_{x_i x_j})^2 + \frac{2P''(u)}{P'(u)} w \Delta P \right) \\
&\quad + P^s \left(\left(\frac{2P''(u)g(x, u, t)}{P'(u)} + g_u(x, u, t) \right) w + P'(u) \sum_{i=1}^N P_{x_i} g_{x_i}(x, u, t) \right)
\end{aligned}$$

$$\begin{aligned}
& + P^s \left(\sum_{j=1}^N \frac{2f_j''(u)}{P'(u)} P_{x_j} w + \sum_{j=1}^N f_j'(u) w_{x_j} \right) + sP^{s-1} P_t(P^{-s} z) \\
& = P^s P'(u) \Delta w - P^s P'(u) \sum_{i,j=1}^N (P_{x_i x_j})^2 + \frac{2P''(u)}{P'(u)} z \Delta P \\
& + 2 \left(\frac{P''(u)g(x, u, t)}{P'(u)} + g_u(x, u, t) \right) z + 2z \sum_{j=1}^N \frac{f_j''(u)}{P'(u)} P_{x_j} \\
& + P^s \sum_{j=1}^N f_j'(u) w_{x_j} + P^s P'(u) \sum_{i=1}^N P_{x_i} g_{x_i}(x, u, t) + sP^{-1} P_t z \\
& = P^s P'(u) \left(P^{-s} \Delta z - 2sP^{-s-1} \sum_{i=1}^N P_{x_i} z_{x_i} + 2s(s+1)P^{-2s-2} z^2 \right) \\
& - P^s P'(u) \left(sP^{-s-1} z \Delta P \right) - P^s P'(u) \sum_{i,j=1}^N (P_{x_i x_j})^2 \\
& + \frac{2P''(u)}{P'(u)} z \Delta P + 2 \left(\frac{P''(u)g(x, u, t)}{P'(u)} + g_u(x, u, t) \right) z + 2z \sum_{j=1}^N \frac{f_j''(u)}{P'(u)} P_{x_j} \\
& + P^s \sum_{j=1}^N f_j'(u) w_{x_j} + P^s P'(u) \sum_{i=1}^N P_{x_i} g_{x_i}(x, u, t) + sP^{-1} z P'(u) u_t \\
& = P'(u) \Delta z - 2sP^{-1} P'(u) \sum_{i=1}^N P_{x_i} z_{x_i} + 2s(s+1)P^{-s-2} P'(u) z^2 \\
& - sP^{-1} P'(u) z \Delta P - P^s P'(u) \sum_{i,j=1}^N (P_{x_i x_j})^2 + \frac{2P''(u)}{P'(u)} z \Delta P \\
& + 2 \left(\frac{P''(u)g(x, u, t)}{P'(u)} + g_u(x, u, t) \right) z + 2z \sum_{j=1}^N \frac{f_j''(u)}{P'(u)} P_{x_j} + P^s \sum_{j=1}^N f_j'(u) w_{x_j} \\
& + P^s P'(u) \sum_{i=1}^N P_{x_i} g_{x_i}(x, u, t) \\
& + sP^{-1} P'(u) z \left(\Delta P + \sum_{j=1}^N f_j(u)_{x_j} + g(x, u, t) \right) \\
& = P'(u) \Delta z - 2sP^{-1} P'(u) \sum_{i=1}^N P_{x_i} z_{x_i} + 2s(s+1)P^{-s-2} P'(u) z^2 \\
& - sP^{-1} P'(u) z \Delta P - P^s P'(u) \sum_{i,j=1}^N (P_{x_i x_j})^2 + \frac{2P''(u)}{P'(u)} z \Delta P
\end{aligned}$$

$$\begin{aligned}
& + 2 \left(\frac{P''(u)g(x, u, t)}{P'(u)} + g_u(x, u, t) \right) z + 2z \sum_{j=1}^N \frac{f_j''(u)}{P'(u)} P_{x_j} + P^s \sum_{j=1}^N f_j'(u) w_{x_j} \\
& + P^s P'(u) \sum_{i=1}^N P_{x_i} g_{x_i}(x, u, t) \\
& + s P^{-1} P'(u) z \Delta P + s P^{-1} P'(u) z \sum_{j=1}^N f_j(u)_{x_j} + s P^{-1} P'(u) g(x, u, t) z \\
& = P'(u) \Delta z - 2s P^{-1} P'(u) \sum_{i=1}^N P_{x_i} z_{x_i} + 2s(s+1) P^{-s-2} P'(u) z^2 \\
& - \sum_{i,j=1}^N P^s P'(u) (P_{x_i x_j})^2 + \frac{2P''(u)}{P'(u)} z \Delta P \\
& + \left(s P^{-1} P'(u) g(x, u, t) + 2g_u(x, u, t) + \frac{2P''(u)g(x, u, t)}{P'(u)} \right) z \\
& + \sum_{j=1}^N \frac{2f_j''(u)}{P'(u)} P_{x_j} z + \sum_{j=1}^N f_j'(u) z_{x_j} + P^s P'(u) \sum_{i=1}^N P_{x_i} g_{x_i}(x, u, t). \tag{12}
\end{aligned}$$

Since

$$\sum_{i,j=1}^N (P_{x_i x_j})^2 \geq \sum_{i=1}^N (P_{x_i x_i})^2 \geq \frac{1}{N} (\Delta P)^2, \tag{13}$$

we choose s such that

$$s(s+1) + \frac{N}{2} \left(\frac{P''(u)P(u)}{(P'(u))^2} \right)^2 < 0; \tag{14}$$

then $-1 < s < 0$ and there exists a suitable constant C such that

$$\begin{aligned}
& 2s(s+1) P^{-s-2} P'(u) z^2 - \sum_{i,j=1}^N P^s P'(u) (P_{x_i x_j})^2 + \frac{2P''(u)}{P'(u)} z \Delta P \\
& \leq 2s(s+1) P^{-s-2} P'(u) z^2 - \frac{1}{N} P^s P'(u) (\Delta P)^2 + \frac{2P''(u)}{P'(u)} z \Delta P \\
& \leq -C P^{-s-2} P'(u) z^2. \tag{15}
\end{aligned}$$

From (12) and (15), we have

$$\begin{aligned}
z_t &\leq P'(u)\Delta z - 2sP^{-1}P'(u) \sum_{i=1}^N P_{x_i} z_{x_i} - CP^{-s-2}P'(u)z^2 + \sum_{j=1}^N f'_j(u)z_{x_j} \\
&\quad + \left(sP^{-1}P'(u)g(x, u, t) + 2g_u(x, u, t) + \frac{2P''(u)g(x, u, t)}{P'(u)} \right) z \\
&\quad + \sum_{j=1}^N \frac{2f''_j(u)}{P'(u)} P_{x_j} z + P^s P'(u) \sum_{i=1}^N P_{x_i} g_{x_i}(x, u, t). \tag{16}
\end{aligned}$$

If the conditions in Theorem 1 are satisfies, then there exist constant $c_i > 0$, $i = 1, 2, 3, 4, 5, 6, 7$, such that

$$\begin{aligned}
\left| \sum_{j=1}^N \frac{2f''_j(u)}{P'(u)} P_{x_j} z \right| &\leq 2c \frac{|G''(u)|}{P'(u)} \sum_{j=1}^N |P_{x_j}| z \\
&\leq c_1 \frac{|G''(u)|}{P'(u)} w^{\frac{1}{2}} z \\
&= c_1 \frac{|G''(u)|}{(P'(u))^2} w^{\frac{1}{2}} z P^{\frac{s}{2}} P^{-\frac{s}{2}} P^{-s-2} P^{s+2} P'(u) \\
&= c_1 \frac{P^{\frac{s}{2}+2} |G''(u)|}{(P'(u))^2} P^{-s-2} P'(u) z^{\frac{3}{2}} \\
&\leq c_2 \frac{P(u) |G''(u)|}{(P'(u))^2} P^{-s-2} P'(u) z^{\frac{3}{2}} \\
&\leq c_2 M(\varepsilon) P^{-s-2} P'(u) z^{\frac{3}{2}}, \tag{17}
\end{aligned}$$

and $M(\varepsilon) \rightarrow \beta$ as $\varepsilon \rightarrow 0$. So $M(\varepsilon)$ is uniformly bounded. Moreover

$$\begin{aligned}
|sP^{-1}P'(u)g(x, u, t)z| &= |sP^{-1}P'(u)P^{s+2}P^{-s-2}g(x, u, t)z| \\
&= |sP^{s+1}g(x, u, t)(P^{-s-2}P'(u)z)| \\
&\leq c_3 P^{-s-2}P'(u)z, \tag{18}
\end{aligned}$$

$$\begin{aligned}
|2g_u(x, u, t)z| &= \left| \frac{2g_u(x, u, t)P^{s+2}}{P'(u)} (P^{-s-2}P'(u)) \right| \\
&\leq c_4 P^{-s-2}P'(u), \tag{19}
\end{aligned}$$

$$\begin{aligned}
\left| \frac{2P''(u)g(x, u, t)}{P'(u)z} \right| &= \left| \frac{2P''(u)}{(P'(u))^2} g(x, u, t) P^{s+2} P^{-s-2} P'(u)z \right| \\
&= \left| 2P^{s+1} g(x, u, t) \left(\frac{P(u)P''(u)}{(P'(u))^2} \right) P^{-s-2} P'(u)z \right| \\
&\leq c_5 \left(\frac{P(u)|G''(u)|}{(P'(u))^2} \right) P^{-s-2} P'(u)z \\
&\leq c_5 M(\varepsilon) P^{-s-2} P'(u)z
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
\left| P^s P'(u) \sum_{i=1}^N P_{x_i} g_{x_i}(x, u, t) \right| &\leq c_6 \sum_{i=1}^N |P_{x_i}| P'(u) \\
&= c_6 w^{\frac{1}{2}} P^{\frac{s}{2}} P^{-\frac{s}{2}} P^{s+2} (P^{-s-2} P'(u)) \\
&\leq c_7 P^{-s-2} P'(u) z^{\frac{1}{2}}.
\end{aligned} \tag{21}$$

We have from (16)-(21) that

$$\begin{aligned}
z_t &\leq P'(u) \Delta z - 2s P^{-1} P'(u) \sum_{i=1}^N P_{x_i} z_{x_i} + \sum_{j=1}^N f'_j(u) z_{x_j} \\
&\quad + \left(-Cz^2 + c_2 M(\varepsilon) z^{\frac{3}{2}} + (c_3 + c_4 + c_5 M(\varepsilon))z + c_7 z^{\frac{1}{2}} \right) P^{-s-2} P'(u).
\end{aligned} \tag{22}$$

Where $M(\varepsilon)$ is uniformly bounded. Thus we have $z \leq M$ for a suitable large positive constant M by using the maximum principle in (22).

From the condition (2),

$$\left(\frac{P''P}{P'^2} \right)^2 = \left(\frac{G''(u)(G(u) + \varepsilon(u + \bar{u}))}{(G'(u) + \varepsilon)^2} \right)^2 \leq \beta^2 + \beta(\varepsilon), \tag{23}$$

where $\beta(\varepsilon) > 0$ is a function of ε and $\beta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. If we choose s^* to be the smaller root of the equation

$$s(s+1) + \frac{N}{2}(\beta^2 + \beta(\varepsilon)) = 0, \tag{24}$$

then (14) is satisfied. Thus we have

$$P^{s^*} w \leq M, \tag{25}$$

where

$$s^* = \frac{-1 - \sqrt{1 - 2N(\beta^2 + \beta(\varepsilon))}}{2}. \tag{26}$$

Letting $\varepsilon \rightarrow 0$ in (26), we obtain the estimate (4), and with it the proof of Theorem 1. $\square$

References

- [1] S.B. Angenent and D.G. Aronson, The focusing problem for the radially symmetric porous medium equation, *Communications in Partial Differential Equations*, **20** (1995), 1217-1240.
<https://doi.org/10.1080/03605309508821130>
- [2] H. Brezis and M.G. Crandall, Uniqueness of solution of the initial value problem for $u_t - \Delta G(u) = 0$, *J. Pure Appl. Math.*, **58** (1979), 153-163.
- [3] L.A. Caffarelli, J.L. Vazques and N.I. Wolanski, Lipschitz continuity of solutions and interfaces of the N-dimensional porous medium equation, *Indiana Univ. Math. Journal*, **36** (1987), 373-401.
<https://doi.org/10.1512/iumj.1987.36.36022>
- [4] E. DiBenedetto, *Degenerate Parabolic Equations*, Springer-Verlag, 1993.
<https://doi.org/10.1007/978-1-4612-0895-2>
- [5] E. DiBenedetto and V. Vespri, On the singular Equation $\beta(u)_t = \Delta u$, *Arch. Rat. Mech. Anal.*, **132** (1995), 247-309.
<https://doi.org/10.1007/bf00382749>
- [6] A. Friedman, *Partial Differential Equations of Parabolic Type*, Englewood Cliffs, N.J.: Prentice-Hall Inc. 1964.
- [7] W. Jäger and Y.G. Lu, Global regularity of solutions for general degenerate parabolic equations in 1-D, *J. Diff. Eqs.*, **140** (1997), 365-377.
<https://doi.org/10.1006/jdeq.1997.3313>
- [8] A.S. Kalashnikov, Some problems of the qualitative theory of non-linear degenerate second order parabolic equations, *Russian Math. Surveys*, **42** (1987), 169-222. <https://doi.org/10.1070/rm1987v042n02abeh001309>
- [9] Y.G. Lu, Hölder Estimates of Solutions of Some Doubly Nonlinear Degenerate Parabolic Equations, *Communications in Partial Differential Equations*, **24** (1999), 895-913.
<https://doi.org/10.1080/03605309908821452>
- [10] Y.G. Lu, I. Mantilla and L. Rendón, Hölder Estimates of Solutions on the Equation $u_t = \Delta G(u)$, *Applicable Analysis*, **81** (2002), no. 2, 1-7.
- [11] Y.G. Lu and W. Jäger, On Solutions to Nonlinear Reaction-Diffusion-Convection Equations with Degenerate Diffusion, *J. Diff. Equations*, **170** (2001), no. 1, 1-21. <https://doi.org/10.1006/jdeq.2000.3800>

- [12] Y.G. Lu, Hölder Estimates of Solutions to a Degenerate Diffusion Equation, *American Mathematical Society*, **130** (2002), no. 5, 1339-1343.

Received: August 1, 2018; Published: August 30, 2018