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Bounded Subsets of the Zygmund F-Algebra

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Abstract

We will consider some characterizations of boundedness in the Zygmund F-algebra $N\log^\alpha N(X)$ $(\alpha > 0)$ of holomorphic functions f on the unit polydisk or the unit ball that satisfy

$$\sup_{0\le r<1}\int_{\partial X}\varphi_\alpha(\log^+|f(r\zeta)|)\,d\sigma(\zeta)<\infty,$$

where $\varphi_\alpha(t) = t\{\log(c_\alpha + t)\}^\alpha$ for $t \ge 0$ and $c_\alpha = \max\{e\,,\,e^\alpha\}$.

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1 Introduction

Let $n \in \mathbb{N}$ and $\mathbb{C}^n = \{z = (z_1, \ldots, z_n) \,|\, z_j \in \mathbb{C}, 1 \le j \le n\}$ be the space of n-complex variables. The unit polydisk $\{z \in \mathbb{C}^n \,|\, |z_j| < 1, 1 \le j \le n\}$ is denoted by U^n and the distinguished boundary $\mathbb{T}^n$ is $\{\zeta \in \mathbb{C}^n \,|\, |\zeta_j| = 1, 1 \le j \le n\}$. The unit ball $\{z \in \mathbb{C}^n \,|\, \sum_{j=1}^n |z_j|^2 < 1\}$ is denoted by B_n and $S_n = \{\zeta \in \mathbb{C}^n \,|\, \sum_{j=1}^n |\zeta_j|^2 = 1\}$ is its boundary. In this paper, X denotes the unit polydisk or the unit ball and ∂X denotes $\mathbb{T}^n$ for $X = U^n$ or S_n for $X = B_n$. The normalized Lebesgue measure on ∂X is denoted by $d\sigma$.

$H^q(X)\,(0 < q \le \infty)$ denotes the Hardy space on X. We denote the Nevanlinna class on X by $N(X)$, which consists of all holomorphic functions f on X such that

$$\sup_{0\le r<1} \int_{\partial X} \log^+ |f(r\zeta)|\, d\sigma(\zeta) < \infty$$

holds. It is well-known that each function $f \in N(X)$ has the nontangential limit $f^*(\zeta) = \lim\limits_{r\to 1^-} f(r\zeta)$ (a.e. $\zeta \in \partial X$).

The Smirnov class $N_*(X)$ is defined as the set of all $f \in N(X)$ which satisfy

$$\sup_{0\le r<1} \int_{\partial X} \log^+ |f(r\zeta)|\, d\sigma(\zeta) = \int_{\partial X} \log^+ |f^*(\zeta)|\, d\sigma(\zeta).$$

The metric in the class $N_*(X)$ is introduced by

$$d_{N_*(X)}(f,g) = \int_{\partial X} \log(1 + |f^*(\zeta) - g^*(\zeta)|)\, d\sigma(\zeta) \quad (f,g \in N_*(X)).$$

With the metric $d_{N_*(X)}(\cdot,\cdot)$ $N_*(X)$ is an F-algebra, that is, a complete linear metric space with multiplication continuous.

The Privalov class $N^p(X)$, $1 < p < \infty$, is the set of all holomorphic functions f on X satisfying

$$\sup_{0\le r<1} \int_{\partial X} \left(\log^+ |f(r\zeta)|\right)^p d\sigma(\zeta) < \infty.$$

It is known that $N^p(X)$ is a subalgebra of $N_*(X)$, hence each function $f \in N^p(X)$ has the nontangential limit almost everywhere on ∂X. Under the metric defined by

$$d_{N^p(X)}(f,g) = \left(\int_{\partial X} (\log(1 + |f^*(\zeta) - g^*(\zeta)|))^p\, d\sigma(\zeta)\right)^{\frac{1}{p}} \quad (f,g \in N^p(X)),$$

$N^p(X)$ is also an F-algebra (cf. [7]).

For $0 < p < \infty$, the class $M^p(X)$ consists of holomorphic functions f on X for which

$$\int_{\partial X} \left(\log^+ Mf(\zeta)\right)^p d\sigma(\zeta) < \infty,$$

where $Mf(\zeta) := \sup\limits_{0\le r<1} |f(r\zeta)|$. Define a metric

$$d_{M^p(X)}(f,g) = \left\{\int_{\partial X} (\log(1 + M(f-g)(\zeta)))^p\, d\sigma(\zeta)\right\}^{\frac{\alpha_p}{p}} \quad (f,g \in M^p(X)),$$

where $\alpha_p = min(1,p)$. With this metric $M^p(X)$ is also an F-algebra (see [2]).

It is well-known that it holds the following inclusion relations:

$$H^q(X) \subsetneq N^p(X) \subsetneq M^1(X) \subsetneq N_*(X) \subsetneq N(X) \quad (0 < q \leq \infty\,,\, p > 1).$$

Moreover, it is known that $N^p(X) = M^p(X)$ $(p > 1)$ and $N(X) \subsetneq M^p(X)$ $(0 < p < 1)$.

We shall define the Zygmund F-algebra $N\log^\alpha N(X)$ $(\alpha > 0)$. The class $N\log^\alpha N(X)$ is the set of all holomorphic functions f on X such that

$$\sup_{0\leq r<1} \int_{\partial X} \varphi_\alpha(\log^+ |f(r\zeta)|)\, d\sigma(\zeta) < \infty, \tag{1}$$

where $\varphi_\alpha(t) = t\{\log(c_\alpha + t)\}^\alpha$ for $t \geq 0$ and $c_\alpha = \max\{e\,,\, e^\alpha\}$. It is verified that (1) is equivalent to the condition

$$|f|_\alpha := \sup_{0\leq r<1} \int_{\partial X} \varphi_\alpha(\log(1 + |f(r\zeta)|))\, d\sigma(\zeta) < \infty. \tag{2}$$

This class was considered by Zygmund first [10]. Further, the topological properties of this class were studied in [1, 3, 8]. It is known that the following relations hold:

$$\bigcup_{q>0} H^q(X) \subset N\log^\alpha N(X) \subset N_*(X) \quad (\alpha > 0).$$

This implies that every $f \in N\log^\alpha N(X)$ has a finite nontangential limit almost everywhere on ∂X. Therefore the characteristic defined by (2) satisfies the relation

$$|f|_\alpha = \int_{\partial X} \varphi_\alpha(\log(1 + |f^*(\zeta)|))\, d\sigma(\zeta).$$

For $f, g \in N\log^\alpha N(X)$, we can define a metric

$$d_{N\log^\alpha N(X)}(f, g) := |f - g|_\alpha = \int_{\partial X} \varphi_\alpha(\log(1 + |f^*(\zeta) - g^*(\zeta)|))\, d\sigma(\zeta).$$

With this metric $N\log^\alpha N(X)$ becomes an F-algebra (see [1, 3]).

A subset L of a linear topological space A is said to be *bounded* if for any neighborhood U of zero in A there exists a real number λ, $0 < \lambda < 1$, such that $\lambda L = \{\lambda f\,;\, f \in L\} \subset U$. Yanagihara investigated the properties of boundedness in $N_*(X)$ in the case $n = 1$ [9]. As for $N^p(X)$ with $p > 1$ in the case $n \geq 1$, Subbotin described some characterizations of boundedness [7]. As for $M^p(X)$ with $p = 1$ in the case $n = 1$, Kim characterized bounded subsets of the class (see [5]). For $p > 1$ and $n = 1$, these characterizations were described by Meštrović [6]. In recent paper [4], the author considered bounded subsets of $M^p(X)$ with $0 < p < \infty$ in the case $n \geq 1$.

In this paper, some characterizations of boundedness in the Zygmund F-algebra $N\log^\alpha N(X)$ with $\alpha > 0$ in the case $n \geq 1$ will be described.

2 The results

Theorem 2.1. *Let $\alpha > 0$. $L \subset N\log^{\alpha} N(X)$ is bounded if and only if*
(i) there exists a $K < \infty$ such that

$$\int_{\partial X} \varphi_{\alpha}(\log^{+}|f^{*}(\zeta)|)\, d\sigma(\zeta) < K$$

for any $f \in L$;
(ii) for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_{E} \varphi_{\alpha}(\log^{+}|f^{*}(\zeta)|)\, d\sigma(\zeta) < \varepsilon,$$

for any $f \in L$ and for any measurable set $E \subset \partial X$ with the Lebesgue measure $|E| < \delta$.

Proof. We follow [4].
Necessity. Let L be a bounded subset of $N\log^{\alpha} N(X)$.
(i) For any $\eta > 0$, we can find a number $\lambda_0 = \lambda_0(\eta)$ $(0 < \lambda_0 < 1)$ such that

$$d_{N\log^{\alpha} N(X)}(\lambda f, 0) = \int_{\partial X} \varphi_{\alpha}(\log(1 + |\lambda f^{*}(\zeta)|))\, d\sigma(\zeta) < \eta$$

for all $f \in L$ and $|\lambda| \le \lambda_0$. Since

$$|fg|_{\alpha} \le 2^{\alpha+2}(|f|_{\alpha} + |g|_{\alpha}), \tag{3}$$

which is derived from the definition of φ_{α} and the elementary inequality

$$(a+b)^{\alpha} \le 2^{\alpha}(a^{\alpha} + b^{\alpha}) \quad (a \ge 0\,,\, b \ge 0\,,\, \alpha > 0),$$

we obtain

$$\int_{\partial X} \varphi_{\alpha}(\log^{+}|f^{*}(\zeta)|)\, d\sigma(\zeta)$$

$$\le \int_{\partial X} \varphi_{\alpha}\left(\log\left(1 + \left|\lambda_0^{-1}\right| |\lambda_0 f^{*}(\zeta)|\right)\right) d\sigma(\zeta) \le 2^{\alpha+2}\left(|\lambda_0^{-1}|_{\alpha} + |\lambda_0 f|_{\alpha}\right)$$

$$= 2^{\alpha+2}\left(\varphi_{\alpha}\left(\log(1 + |\lambda_0^{-1}|)\right) + \eta\right) = K = \mathit{constant}.$$

Thus the condition (i) is satisfied.
(ii) Given $\varepsilon > 0$, take η as $\eta < \varepsilon/2^{\alpha+3}$ and $\lambda_0 = \lambda_0(\eta)$ as above. Next we take $\delta > 0$ such that

$$\delta \cdot \varphi_{\alpha}\left(\log(1 + |\lambda_0^{-1}|)\right) < \frac{\varepsilon}{2^{\alpha+3}}.$$

If $|E| < \delta$ for each set $E \subset \partial X$ and for any $f \in L$, we have

$$\int_E \varphi_\alpha \left(\log^+ |f^*(\zeta)|\right) \, d\sigma(\zeta)$$

$$\leq \int_E \varphi_\alpha \left(\log(1 + \left|\lambda_0^{-1}\right| |\lambda_0 f^*(\zeta)|)\right) \, d\sigma(\zeta)$$

$$\leq 2^{\alpha+2} \left(\varphi_\alpha \left(\log(1 + |\lambda_0^{-1}|)\right) |E| + \eta\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon.$$

Therefore the condition (ii) holds.

Sufficiency. Let

$$V = \{g \in N \log^\alpha N(X)\,;\, d_{N \log^\alpha N(X)}(g\,,\,0) < \eta\}$$

be a neighborhood of zero in $N \log^\alpha N(X)$. We take $\varepsilon > 0$ such that

$$\varphi_\alpha \left(\log(1+\varepsilon)\right) + 2^{\alpha+2}\varphi_\alpha(\log 2)\varepsilon + 2^{\alpha+2}\varepsilon < \eta.$$

Then we can find a δ $(0 < \delta < \varepsilon)$ so that (ii) is satisfied. For $f \in L$, there is a measurable set $E_f \subset \partial X$ such that

$$|\partial X \backslash E_f| < \delta, \quad \varphi_\alpha \left(\log^+ |f^*(\zeta)|\right) \leq \frac{K}{\delta} \quad \text{on } E_f$$

by Chebyshev's inequality. Thus we obtain

$$|f^*(\zeta)| \leq \exp\left(\varphi_\alpha^{-1}\left(\frac{K}{\delta}\right)\right) = A(\delta) = A \quad \text{on } E_f.$$

Take λ such that $0 < \lambda < \varepsilon/A$. Then, using inequalities

$$\log(1+x) \leq \log 2 + \log^+ x \quad (x > 0)$$

and

$$\varphi_\alpha(x+y) \leq 2^{\alpha+2}\left(\varphi_\alpha(x) + \varphi_\alpha(y)\right) \quad (x\,,\,y \geq 0),$$

which is derived in the same way of (3), we have, for any $f \in L$,

$$d_{N \log^\alpha N(X)}(\lambda f, 0)$$

$$= \int_{\partial X} \varphi_\alpha \left(\log(1 + |\lambda f^*(\zeta)|)\right) \, d\sigma(\zeta) = \int_{E_f} + \int_{\partial X \backslash E_f}$$

$$\leq \int_{E_f} \varphi_\alpha \left(\log(1+\varepsilon)\right) \, d\sigma(\zeta) + \int_{\partial X \backslash E_f} \varphi_\alpha \left(\log(1 + |f^*(\zeta)|)\right) \, d\sigma(\zeta)$$

$$\leq \varphi_\alpha(\log(1+\varepsilon)) + 2^{\alpha+2}\left(\int_{\partial X\setminus E_f}\varphi_\alpha(\log 2)\,d\sigma(\zeta) + \int_{\partial X\setminus E_f}\varphi_\alpha\left(\log^+|f^*(\zeta)|\right)\,d\sigma(\zeta)\right)$$

$$\leq \varphi_\alpha(\log(1+\varepsilon)) + 2^{\alpha+2}\varphi_\alpha(\log 2)\delta + 2^{\alpha+2}\varepsilon < \eta.$$

Hence we get $d_{N\log^\alpha N(X)}(\lambda f, 0) < \eta$. Therefore L is a bounded subset of $N\log^\alpha N(X)$.

This completes the proof.

□

Remark. Note that the characterization of boundedness in $N\log^\alpha N(X)$ $(\alpha > 0)$ has the same conditions as the characterization of boundedness in the Smirnov class $N_*(X)$ in the case $n = 1$ [9, Theorem 1], the Privalov class $N^p(X)\,(1 < p < \infty)$ [7, Theorem 5], and the class $M^p(X)\,(0 < p < \infty)$ [4, Theorem 1] [5, Theorem 4.1].

Next we will show a standard example of a bounded set of $N\log^\alpha N(X)$.

Theorem 2.2. *Let $\alpha > 0$. If $f \in N\log^\alpha N(X)$, then $f_r(z) = f(rz)$ $(z \in X,\ 0 \leq r < 1)$ form a bounded set in $N\log^\alpha N(X)$.*

To prove Theorem 2.2, we utilize the following result by Eminyan.

Theorem 2.3. (see [1, Theorem 3]) *Let $\alpha > 0$. If $f \in N\log^\alpha N(X)$ and $f_r(z) = f(rz)$ for $z \in X$ and $0 \leq r < 1$, then $f_r \to f$ as $r \to 1^-$ in the metric $d_{N\log^\alpha N(X)}$.*

Proof of Theorem 2.2. Let

$$V = \{g \in N\log^\alpha N(X)\,;\, d_{N\log^\alpha N(X)}(g\,,\,0) < \eta\}$$

be a neighborhood of zero in $N\log^\alpha N(X)$. Choose λ', $0 < \lambda' < 1$, so that $d_{N\log^\alpha N(X)}(\lambda' f\,,\,0) < \eta/2$.

Let r_0 be sufficiently near to 1 such that $d_{N\log^\alpha N(X)}(f\,,\,f_r) < \eta/2$ for $r_0 \leq r < 1$ by Theorem 2.3. Then we have $d_{N\log^\alpha N(X)}(\lambda' f_r\,,\,\lambda' f) < \eta/2$. For $r \geq r_0$ we obtain

$$d_{N\log^\alpha N(X)}(\lambda' f_r\,,\,0) \leq d_{N\log^\alpha N(X)}(\lambda' f_r\,,\,\lambda' f) + d_{N\log^\alpha N(X)}(\lambda' f\,,\,0) < \eta.$$

For $0 \leq r \leq r_0$ we can find λ'' so small that $d_{N\log^\alpha N(X)}(\lambda'' f_r\,,\,0) < \eta$. Therefore, if $\lambda = \min(\lambda'\,,\,\lambda'')$, we have $\{\lambda f_r\}_{0\leq r<1} \subset V$.

□

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