An Application of Fractional RPS Approach to Class of Linear IVPs of Order $2\beta$

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Abstract

In this paper, an efficient analytical technique, called residual power series (RPS) method, is provided for finding the approximate solution of a class of fractional IVPs of order $2\beta$, $0 < \beta \leq 1$, subject to suitable initial conditions. This technique is presented based upon the use of fractional Taylor and residual functions under the Caputo differentiability. The RPS methodology provides the solution in the form of a convergent sequence of approximate functions with easily computable and easily verifiable components. For potentiality, superiority, applicability, and validity of the RPS method, an illustrative example is given. It is observed that suggested technique is highly reliable and may be extended to other highly nonlinear differential models of fractional order.

Keywords: Fractional differential equations, Generalized Taylor formula; Initial value problems; Analytical technique

1. Introduction

Many phenomena in engineering and applied sciences such as image processing, fluid flow, viscoelastic, control theory, and so forth, can be modeled very accurately by using the operations of differentiation and integration of fractional
order [1–6]. Generally, in many numeric simulations and mathematical modeling of various dynamical systems and processes, including fluid dynamics, biological models, chemical kinetics, and community medicine models, the fractional differential equations appear that is mostly nonlinear and complex issues. Anyhow, most of these models cannot be solved analytically, so effective and reliable numerical or approximate methods are required [7-12] and [22-26].

Unlike the ordinary derivatives and integrals, which have unique definitions and clear geometrical and physical interpretations, there are many concepts for the derivatives and integrals of fractional order such that Riemann-Liouville, Caputo, Riesz, Grünwald-Letnikov, and Riesz-Caputo. In this paper, the application of fractional RPS method has been extended in finding the approximate series solution for both linear and nonlinear differential equations of fractional order. To reach our goal, we will construct symbolic approximate PS solutions for the following fractional IVP:

\[
D^{2\beta} \varphi(x) = f \left( x, \varphi(x), D^{\beta} \varphi(x) \right), a \leq x \leq b, \tag{1.1}
\]

subject to the initial conditions

\[
\varphi(a) = \mu_0, D^{\beta} \varphi(a) = \mu_1, \tag{1.2}
\]

where \(0 < \beta \leq 1, \mu_0, \mu_1 \in \mathbb{R}, D^{\beta}\) denotes the Caputo fractional derivative of order \(\beta, f\) is linear or nonlinear real valued function, and \(\varphi(x)\) is an unknown analytical function of independent variable \(x\) to be determined. Throughout this paper, we assume that \(\varphi\) has a FPS expansion about the initial point \(a\), and \(f\) satisfies all the necessary requirements for the existence of a unique solution.

The RPS technique is an effective approximate method for handling different kinds of ODEs, PDEs, fuzzy DEs, fractional DEs, integral and integrodifferential equations [13-17]. This method has a lot of advantages as follows: First, its nature is global according to the obtainable solutions along with being able to solve numerous problems such as mathematical, physical and engineering ones. Second, it is easily noted that it is precise, needs few efforts to have the required results achieved, alongside being developed for nonlinear problems and cases. As for the third advantage, it can be said that any point in the interval of interest will be possibly picked, in addition, to have the approximate solutions applied. Fourth, the method does not need the variables’ discretization, also it is not implemented by computational round off errors.

The rest of this paper is organized as follows. In the next section, the required definitions and properties related to Caputo concept and the fractional PS representations are given. In section 3, the analysis of the FRPS method is presented as well as the representation of RPS solution for a class of FIVP is constructed. In section 4, two numerical examples are given to demonstrate the efficiency and capability of the FRPS method. Finally, the conclusion is discussed in the last section.
2. Caputo fractional derivative

In this section, some essential definitions and properties of popular fractional operators, Riemann-Liouville fractional integral and Caputo fractional derivative, are revisited. Meanwhile, the FPS representation is given.

**Definition 2.1**: The Riemann-Liouville fractional integral operator of order \( \beta \) for a function \( \varphi(x) \) is defined by

\[
I^\beta \varphi(x) = \frac{1}{\Gamma(\beta)} \int_0^x \frac{\varphi(t)}{(x-t)^{1-\beta}} \, dt, \quad 0 < t < x, \beta > 0.
\]

**Definition 2.2**: The Caputo fractional derivative of order \( \beta \) is defined by

\[
D^\beta \varphi(x) = \frac{1}{\Gamma(n-\beta)} \int_0^x \frac{\varphi^{(n)}(\eta)}{(x-t)^{\beta-n+1}} \, dt, \quad 0 < t < x, \quad n-1 < \beta \leq n.
\]

For the operator \( D^\beta \) there are the following properties:

1. \( D^\beta c = 0 \), for any constant.
2. \( D^\beta x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\beta)} x^{\gamma-\beta}, \quad \gamma > -1. \)
3. \( D^\beta I^\beta \varphi(x) = \varphi(x). \)

**Definition 2.3**: [18] A power series representation of the form

\[
\sum_{n=0}^{\infty} c_n(x-a)^n \beta = c_0 + c_1(x-a)^\beta + c_2(x-a)^{2\beta} + \cdots,
\]

is called a fractional power series (FPS) expansion about \( a \), where \( m-1 < \beta \leq m \), and \( c_n \)'s are constants called the coefficients of the series.

**Corollary 2.1**: [18] Suppose that \( \varphi \) has a FPS representation at \( x = a \) in the form

\[
\varphi(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \beta, \quad m-1 < \beta \leq m, \quad a \leq x < b.
\]

If \( \varphi(x) \) is continuous on \([a,b]\) and \( D^\beta \varphi(x) \) exists over \((a,b)\) for \( n = 0,1,2, \ldots, \)

then the coefficients \( c_n \) can be written as

\[
c_n = D^{n\beta} \varphi(a), \quad n = 0,1,2, \ldots, \text{where } D^{n\beta} = D^\beta \cdot D^\beta \cdots D^\beta \quad (n\text{-times}).
\]

3. Fractional power series method

This section aims to adapt the generalized Taylorbasis with the RPS method to
obtain the analytical solutions for FIVP (1.1) and (1.2). So, we suppose that the solution for the FVIP (1.1) and (1.2) has the following FPS expansion about the initial point

\[ \varphi(x) = \sum_{n=0}^{\infty} c_n(x - a)^n \beta, \]  

(3.1)

and the \( k \)-th truncated series

\[ \varphi_k(x) = \sum_{n=0}^{k} c_n(x - a)^n \beta. \]  

(3.2)

Since \( \varphi(x) \) satisfy the initial conditions \( \varphi(a) = \mu_0 \), and \( D^\beta \varphi(a) = \mu_1 \), then \( c_1 = \mu_0 \) and \( \alpha_1 = \frac{\mu_1}{\Gamma(\beta + 1)} \). Thus, Eq. (3.2) can be rewritten by

\[ \varphi_k(x) = \mu_0 + \frac{\mu_1}{\Gamma(\beta + 1)}(x - a)\beta + \sum_{n=2}^{k} c_n(x - a)^n \beta, \quad n = 2,3,\ldots,k. \]  

(3.3)

Now, we define the \( k \)-th residual function \( Res^k_\varphi(x) \) about \( x = a \) as follows

\[ Res^k_\varphi(x) = D^{2\beta} \varphi_k(x) - f(x, \varphi_k(x), D^\beta \varphi_k(x)). \]  

(3.4)

Note here \( \lim_{k \to \infty} Res^k_\varphi(x) = Res(x) \) and \( Res(x) = 0 \) for each \( x \in (a,b) \). This show that \( D^\beta Res(x) = 0 \) for each \( x \in (a,b) \). Also, \( D^{n\beta} Res(a) = D^{n\beta} Res^k_\varphi(a) = 0 \) for each \( n = 0,1,2,\ldots,k \). However, to obtain the value of the coefficient \( a_n, n = 2,3,\ldots,k \), we need to solve the following algebraic relation

\[ D^{(k-2)\beta} Res^k_\varphi(a) = 0, \quad k = 2,3,\ldots \]  

(3.5)

For more details about the Generalized Taylors series and its application, we refer to [19-21].

4. Numerical Results

In this section, numerical example for fractional differential equations (FDEs) is given to illustrate the efficiency and applicability of the FPS algorithm. This algorithm provides accurate numerical solutions without discretization or perturbation. The results obtained indicate that the FRPS method is powerful and separable to handle such problem. All computations are calculated by using Mathematica 10.

Consider the following linear FDE:

\[ D^{2\beta} \varphi(x) - \varphi(x) = 0, x \geq 0, \beta \in (0,1], \]  

(4.1)
subject to the initial conditions
$$\varphi(0) = 1, D^\beta \varphi(0) = 1.$$ \hfill (4.2)

To apply the FPS approach, use the fractional generalized Taylor series about the initial point $a = 0$. According to Eq. (4.2), the FPS solution for the FVIP (4.1) and (4.2) takes the following form
$$\varphi(x) = 1 + \frac{1}{\Gamma(\beta + 1)} x^\beta + \sum_{n=2}^{\infty} c_n x^{n\beta}.$$ 

Thus, the $k$-th-truncated series of Eq. (4.1) is
$$\varphi_k(x) = \left(1 + \frac{1}{\Gamma(\beta + 1)} x^\beta + c_2 x^{2\beta} + \cdots + c_k x^{k\beta}\right),$$
and the $k$-th residual function $Res_\varphi^k(x)$ about $x = 0$ is
$$Res_\varphi^k(x) = D^{2\beta} \varphi_k(x) - \varphi_k(x)
= D^{2\beta} \left(1 + \frac{1}{\Gamma(\beta + 1)} x^\beta + c_2 x^{2\beta} + \cdots + c_k x^{k\beta}\right)$$ \hfill (4.3)
$$- \left(1 + \frac{1}{\Gamma(\beta + 1)} x^\beta + c_2 x^{2\beta} + \cdots + c_k x^{k\beta}\right).$$

Consequently, the $\beta(k - 2)$-th derivative of Eq. (4.3) is given by
$$D^{(k-2)\beta} Res_\varphi^k(x) = D^{(k-2)\beta} \left(D^{2\beta} \varphi_k(x) - \varphi_k(x)\right), \ k = 2, 3, ... \hfill (4.4)$$

Thus, the 2nd-truncated series is
$$\varphi_2(x) = 1 + \frac{1}{\Gamma(\beta + 1)} x^\beta + c_2 x^{2\beta}$$
and the 2nd residual function is
$$Res_\varphi^2(x) = D^{2\beta} \varphi_2(x) - \varphi_2(x)
= D^{2\beta} \left(1 + \frac{1}{\Gamma(\beta + 1)} x^\beta + c_2 x^{2\beta}\right)
- \left(1 + \frac{1}{\Gamma(\beta + 1)} x^\beta + c_2 x^{2\beta}\right)
= c_2 \Gamma(2\beta + 1) - \left(1 + \frac{1}{\Gamma(\beta + 1)} x^\beta + c_2 x^{2\beta}\right).$$

To find the coefficient $c_2$, use the fact of Eq. (4.4) that $Res_\varphi^2(0) = 0$. Hence,
$c_2 = \frac{1}{\Gamma(2\beta + 1)}$ and then
$$\varphi_2(x) = 1 + \frac{1}{\Gamma(\beta + 1)} x^\beta + \frac{1}{\Gamma(2\beta + 1)} x^{2\beta}. $$

Now, to find the coefficient $c_3$ of $\varphi_3(x) = 1 + \frac{1}{\Gamma(\beta + 1)} x^\beta + \frac{1}{\Gamma(2\beta + 1)} x^{2\beta} + c_3 x^{3\beta}$, apply $D^\beta$ on both sides of Eq. (4.3) such that
\[ D^\beta \text{Res}_\varphi^3(x) = D^\beta \left( D^{2\beta} \varphi_3(x) - \varphi_3(x) \right) \]
\[ = D^\beta \left[ D^{2\beta} \left( 1 + \frac{1}{\Gamma(\beta + 1)} x^\beta + \frac{1}{\Gamma(2\beta + 1)} x^{2\beta} + c_3 x^{3\beta} \right) \right. \]
\[ - \left( 1 + \frac{1}{\Gamma(\beta + 1)} x^\beta + \frac{1}{\Gamma(2\beta + 1)} x^{2\beta} + c_3 x^{3\beta} \right) \]
\[ = D^\beta \left( 1 + c_3 \frac{\Gamma(3\beta + 1)}{\Gamma(\beta + 1)} x^\beta \right) \]
\[ - \left( 1 + \frac{1}{\Gamma(\beta + 1)} x^\beta + c_3 \frac{\Gamma(3\beta + 1)}{\Gamma(2\beta + 1)} x^{2\beta} \right) \]
\[ = c_3 \Gamma(3\beta + 1) - \left( 1 + \frac{1}{\Gamma(\beta + 1)} x^\beta + c_3 \frac{\Gamma(3\beta + 1)}{\Gamma(2\beta + 1)} x^{2\beta} \right). \]

and then using the fact that \( D^\beta \text{Res}_\varphi^3(0) = 0 \), which leads to \( c_3 = \frac{1}{\Gamma(3\beta + 1)}. \)

Therefore, \( \varphi_3(x) = 1 + \frac{1}{\Gamma(\beta + 1)} x^\beta + \frac{1}{\Gamma(2\beta + 1)} x^{2\beta} + \frac{1}{\Gamma(3\beta + 1)} x^{3\beta}. \) By continuing with this technique and using Eq. (4.4), one can get the following recurrence relation
\[ c_k = \frac{\Gamma((k-1)\beta + 1)}{\Gamma(k\beta + 1)} c_{k-1}, k = 2, 3, \ldots . \]

Hence, the fractional series solution for the FIVP (4.1) and (4.2) has the following form \( \varphi(x) = \lim_{k \to \infty} \varphi_k(x) = \lim_{k \to \infty} \sum_{n=0}^{k} \frac{x^{n\beta}}{\Gamma(1+k\beta)} \)
which coincides with the exact solution \( E_\beta(x) \) “the Mittag-Leffler function”.

For numerical simulation, Table 1 shows the representations of the 6th FPS series solutions for \( \beta = \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \) and \( \beta = 1 \). Table 2 shows the absolute and relative errors of the FPS approximate solutions at \( \beta = 1 \) with step size 0.2. While Table 2 shows the approximation of the FPS approximate solutions at different values of \( \beta \) with step size 0.2. In Figure 1, the solution behavior of the exact solution, \( \varphi(x) \), and the 6th FPS solutions, \( \varphi_6(x) \), are plotted for different values of \( \beta, \beta \in \{0.6, 0.7, 0.8, 0.9, 1.0\} \), over the interval \([0, 3]\) with step size 0.01. From these results, it found that the numerical approximations are in good agreement in each other.

<table>
<thead>
<tr>
<th>Table 1: Representation of the 6th FPS approximation.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta )</td>
</tr>
<tr>
<td>1/4</td>
</tr>
<tr>
<td>1/2</td>
</tr>
</tbody>
</table>
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Table 1: Representation of the 6th FPS approximation.

\[
\begin{align*}
\frac{3}{4} \quad \varphi_6(x) &= 1 + \frac{4x^{3/2}}{3\sqrt{\pi}} + \frac{x^3}{6} + \frac{32x^{9/2}}{945\sqrt{\pi}} + \frac{x^{3/4}}{\Gamma\left(\frac{3}{4}\right)} + \frac{x^{9/4}}{\Gamma\left(\frac{15}{4}\right)} \\
1 \quad \varphi_6(x) &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720}
\end{align*}
\]

Table 2: Numerical errors of the FPS approximate solutions at \(\beta = 1\).

<table>
<thead>
<tr>
<th>(x)</th>
<th>Exact</th>
<th>(\varphi_6(x))</th>
<th>Absolute error</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>1.221402758160169</td>
<td>1.2214027555555556</td>
<td>2.604614 \times 10^{-9}</td>
<td>2.132477 \times 10^{-9}</td>
</tr>
<tr>
<td>0.4</td>
<td>1.4918246976412703</td>
<td>1.4918243555555555</td>
<td>3.420857 \times 10^{-7}</td>
<td>2.293069 \times 10^{-7}</td>
</tr>
<tr>
<td>0.6</td>
<td>1.8221188003905090</td>
<td>1.8221128000000000</td>
<td>6.000391 \times 10^{-6}</td>
<td>3.293084 \times 10^{-6}</td>
</tr>
<tr>
<td>0.8</td>
<td>2.2255409284924680</td>
<td>2.2254947555555558</td>
<td>4.617293 \times 10^{-5}</td>
<td>2.074683 \times 10^{-5}</td>
</tr>
<tr>
<td>1.0</td>
<td>2.7182818284590450</td>
<td>2.7180555555555554</td>
<td>2.262729 \times 10^{-4}</td>
<td>8.321411 \times 10^{-5}</td>
</tr>
</tbody>
</table>

Table 3: Numerical values of the FPS approximate solutions at different values of \(\beta\)’s.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(\beta = 1)</th>
<th>(\beta = 0.9)</th>
<th>(\beta = 0.75)</th>
<th>(\beta = 0.5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>1.2214027555555556</td>
<td>1.2805331250059868</td>
<td>1.4046752986949018</td>
<td>1.7986260543715458</td>
</tr>
<tr>
<td>0.4</td>
<td>1.4918243555555555</td>
<td>1.5937422512295514</td>
<td>1.804050825752347</td>
<td>2.4250719370997450</td>
</tr>
<tr>
<td>0.6</td>
<td>1.8221128000000000</td>
<td>1.9688104843731777</td>
<td>2.2624672614256145</td>
<td>3.1235619617326007</td>
</tr>
<tr>
<td>0.8</td>
<td>2.2254947555555558</td>
<td>2.4227191207191807</td>
<td>2.8145357441367620</td>
<td>3.9251004603422746</td>
</tr>
<tr>
<td>1.0</td>
<td>2.7180555555555554</td>
<td>2.97399715935319052</td>
<td>3.47865175558241380</td>
<td>4.8481997230513240</td>
</tr>
</tbody>
</table>

Fig. 1: The FPS behavior of Example 4.1 at different values of \(\beta\) and \(x \in [0,3]\).
5. Conclusion

In this work, we have analyzed a class of fractional IVPs. Using the FRPS method, we have shown that we can acquire an investigative answer for linear fractional IVPs. The applicability and accuracy of the proposed method have been shown for obtaining approximate solutions to such problems. In the numerical simulation, the level of calculation difficulties can be reduced by utilizing the FRPS method. The results obtained show that the approximate solution converged with rapidly sequence function to the exact solution in few steps.

References


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