A Fixed Point Theorem for the Nonexpansive Mappings in Smooth Reflexive Banach Spaces whose Dual Having the Kadec-Klee Property

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Abstract

We propose a proof of the following result: Let $X$ be a smooth reflexive banach space whose dual having the Kadec-Klee property. Let $K$ be a nonempty convex bounded closed subset of a $X$; then every nonexpansive mapping $T : K \rightarrow K$ has a fixed point. This result generalizes the theorem of Baillon (1978-1979) where $X$ is uniformly smooth Banach space.

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I. Introduction

We consider only Banach spaces over the reals. $X = (X, | . |)$ and $X' = (X', || . ||)$ respectively denote a Banach space and its topological dual. For $C$ a nonempty subset of a Banach space $X$, a mapping $T : C \rightarrow C$ is
nonexpansive if $|Tx - Ty| \leq |x - y|$ for every $x, y \in C$. A nonempty subset $C$ of a Banach space $X$ has the fixed point property FPP if every nonexpansive self mapping $T$ of $C$ has a fixed point, that is, an $x \in C$ such that $T(x) = x$. A Banach space $X$ has the fixed point property FPP if every closed bounded convex nonempty subset has the FPP.

The purpose of this paper is to demonstrate that all smooth reflexive Banach space, whose dual having the Kadec-Klee property, has the FPP. Recall that the definition of the following notions:

A Banach space $X$ is said strictly convex if for all $x, y \in X$ such that $|x| = |y| = 1$ and $x \neq y$, then $|\lambda x + (1 - \lambda)y| < 1$ for all $\lambda \in (0, 1)$.

Also, a Banach space $X$ is said to have the Kadec-Klee property (KKP) if every weakly convergent sequence on the unit sphere is convergent in norm; which is equivalent to:

if every sequence $\{x_n\}_n$ in $X$ converges weakly to some $x \in X$ and $\lim_{n\to\infty} |x_n| = |x|$ then $\lim_{n\to\infty} |x_n - x| = 0$.

Many Banach spaces have this property: the locally uniformly convex (introduced by Lovaglia (1955)), the space $l^1$ which satisfies the Shur property. On the other hand, $L^1[0, 1], c_0$ and $l^\infty$ fail the KKP.

For any reflexive Banach space $X$, it is even possible to find an equivalent norm such that $X$ and $X'$ with the corresponding dual norm are locally uniformly convex (Trojanski (1972)). Also, let us recall that a locally uniformly Banach space satisfies two properties, i.e., the strict convexity and Kadec-Klee property.

Also, recall the notion of Duality map which will be used in the proof below. Let $X$ be a Banach space and $\mathcal{P}(X')$ is the set of all subset of $X'$. $J : X \to \mathcal{P}(X')$, defined by:

$$Jx = \left\{ x' \in X' : x'(x) = |x|^2 = \|x'\|^2 \right\},$$

is called the duality map of $X$. Recall that If $X'$ is strictly convex, then the duality map $J : X \to X'$ is single-valued. (for the remarkable properties of duality map see, for example, Deimling (1984)).

The definition of Smooth Banach spaces is related to the duality map. A Banach space $X$ is said smooth if, and only if, its norm is $G-$ differentiable on $X \setminus \{0\}$, which means that the duality map $J : X \to X'$ is single-valued, and for all $x, y \in X$, with $y \neq 0$,

$$\lim_{t \to 0} t^{-1} \left[ |y + tx| - |y| \right] = \langle \frac{y}{|y|}, x \rangle \quad (0.1)$$

i.e., the $G$- differential, noted $D$, of the norm $| \cdot |$ at $y \neq 0$ is a $D(|y|) = \frac{Jy}{|y|} \in X'$. Evidently, if $X'$ is strictly convex then $X$ is a smooth. If the Banach $X$ space is reflexive, we verify that $X$ is smooth (resp. strictly convex), if, and only if, the dual space $X'$ is strictly convex (resp. smooth).

An Banach space $X$ is said **uniformly Gâteaux smooth** if, and only if, the limit (0.1) is uniform with respect to the set $\{y : |y| = 1\}$ for each $x \in S_X$.

Also, an Banach space $X$ is said **uniformly smooth** if, and only if, the limit (0.1) is uniform with respect to the set $\{(x, y) : |x| = |y| = 1\}$. (For the informations on smooth Banach spaces, see for example Köthe (1969) pp 345-366). We remark that uniformly smooth $\Rightarrow$ uniformly Gâteaux smooth $\Rightarrow$ smooth.

The following result of this paper generalizes the theorem of Baillon (1978–1979); first of all, the space considered, in this theorem, is uniformly smooth Banach space who is reflexive; recall that its dual is a uniformly convex Banach space which strictly convex and having the Kadec–Klee property. Note also that a uniformly smooth space has the normal structure (see Goebel and Kirk (1990) p.70-71).

On the other hand, recall the theorem 1 of Johanis and Rychtar (2007): *Let $X$ be a separable Banach space. Then there exists an equivalent uniformly Gâteaux smooth norm lacking asymptotic normal structure*. As an obvious consequence of the above result, a separable reflexive Banach space, provided with the norm mentioned above, is smooth but lacking the normal structure. What shows the result proposed in this paper is different from the important result in the fixed point theory due to Kirk (1965), where a reflexive Banach space, having the normal structure, possesses FPP (For the information on the notion of normal structure, see Brodski and Milman (1948), Baillon, J.B.-Schonerberg, R. (1981) or Goebel and Kirk (1990)).

**II. Theorem 1**

All smooth reflexive Banach space whose dual having the Kadec-Klee property has the FPP.

For the demonstration below, we recall the following results. The first result is a consequence of Hahn-Banach theorem:
Lemma 1
Let $X$ be a Banach space, $M$ is a closed subspace of $X$ and $x_0 \in X$ such that $x_0 \notin M$. Then there exist $f \in X'$ satisfying

$(\alpha)$

$(a) \| f \| = 1$, $(b) < f, x_0 > = \text{dist}(x_0, M)$ and $(c) M \subset \text{Ker} f$;

where $\text{Ker} f = \{ x \in X : f(x) = 0 \}$.

$(\beta)$ If the dual space $X'$ is strictly convex, then $f$ is a unique.

Recall some steps of the proof of lemma 1:

$(\alpha)$ Let $E = \{ \alpha x_0 : \alpha \in \mathbb{R} \}$ and $L = M + E = \{ y + \alpha x_0 : y \in M \text{ et } \alpha \in \mathbb{R} \}$. Since $x_0 \notin M$, for each $z \in L$ it exists $\alpha \in \mathbb{R}$ unique and $y \in M$ such that $z = y + \alpha x_0$.

Let an application, denoted $f_0'$, of $L$ in $\mathbb{R}$ defined by $f_0'(y + \alpha x_0) = \alpha$ for all $y \in M$ and $\alpha \in \mathbb{R}$. Then $f_0'(x_0) = 1$ and $f_0'(y) = 0$ on $M$. We verify that $f_0'$ is a continuous linear form on $L$, and

$$\| f_0' \| = d^{-1}, \quad d = \text{dist}(x_0, M)$$

By Hahn-Banach theorem’s, there exist a continuous linear form on $X$, denoted $f_0$, prolonging $f_0'$ and such that

$(a_1) \| f_0 \| = d^{-1}, \quad (b_1) < f_0, x_0 > = 1, \quad (c_1) M \subset \text{Ker} f_0$ ;

The function $f$, defined by $f = df_0$, satisfied the properties of lemma 1 $(\alpha)$.

$(\beta)$ Suppose there is another function, denoted $g$, which satisfies the properties of lemma 1; thus,

$$1 = \| f \| = \| f_{|L} \| = \| g \| = \| g_{|L} \| \quad \text{and} \quad f_{|L} = g_{|L} = h \text{ where } L = \mathbb{R} x_0 + M;$$

$f_{|L}$ (resp.$g_{|L}$) denote the restriction of $f$ (resp.$g$) on the closed subspace $L$. On the one hand, $\| f \| + \| g \| = 2$, on the other hand,

$$2 \geq \| f + g \| \geq \| f_{|L} + g_{|L} \| = 2 \| h \| = 2;$$

therefore,

$$2 = \| f + g \| = \| f \| + \| g \| ;$$
$X'$ being strictly convex, then $f = g.$ □

For the proof of lemma 1 (α) see, for example, Martin (1976) P. 7.

Recall that in any Banach space, all subspace of finite dimension has a supplementary topological. We will use the following simple decomposition:

**Lemma 2**

Let $X$ be any Banach space, $x_0 \neq 0$ any element of $X$ and $f \in X'$ such that $f(x_0) \neq 0$. Then, for each $y \in X$, there exist a unique $x \in X$, with $x \in \text{Ker } f$ and

$$y = x + \frac{f(y)}{f(x_0)} x_0$$

Now, we present the **general frame** of the following proof of theorem 1. Let us begin with the reminder of the following definitions and result:

Thereafter, $X = (X, |.|)$ denotes any smooth reflexive Banach provided with a norm denoted $| . |$. The dual space having the **KKP** is noted by $X' = (X', \| . \|)$ provided with the habitual norm denoted $\| . \|. S_X$ and $S_{X'}$ respectively denote the sphere of $X$ and $X'$. $\|\|_\triangleright$ denote the **strong convergence** on $X'$. $\sigma(X, X')$ denote the **weak topology** on $X$; $\rightarrow$ denote the weakly convergence for the weak topology $\sigma(X, X')$ on $X$; similarly, $\sigma(X', X)$ denote the **weak topology** on $X'$; $\rightarrow^*$ denote the weakly convergence for the weak topology $\sigma(X', X)$ on $X'$.

Since the space $X$ is smooth reflexive, then the dual space $X'$ is strictly convex, and from this the duality map of the space $X$ is univocal and noted $J$.

$K$ denote a nonempty bounded closed and convex subset of $X$. $T : K \rightarrow K$ is a nonexpansive mapping.

**III. Proof of theorem 1**

(A) By the standard arguments, one can assume that $K$ is minimal invariant under $T$. By dividing $K$ by its diameter, we can assume that $\text{diam}(K) = 1$. Let $\{u_n\}_{n \geq 1}$ a approximate fixed point sequence of $T$, i.e, $\lim_{n \rightarrow \infty} |u_n - Tu_n| = 0$; $K$ being weakly compact, we can extract a subsequence, again denoted $\{u_n\}_{n \geq 1}$, weakly convergent to $u$. We set

$$v_n = u_n - u$$

Note that for all $\lambda \in [0, 1]$, $v_n + \lambda \omega = u_n - u + \lambda(u - Tu) = u_n - [(1 - \lambda)u + \lambda Tu]$. The convexity of $K$ implies $(1 - \lambda)u + \lambda Tu \in K$. By the property of
Goebel(1975)-Karlovitz(1976):

\[
\lim_{n \to \infty} |v_n + \lambda \omega| = 1 \quad (1)
\]

Thereafter, it is assumed that \(0 < \lambda < 1\) and \(\omega \neq 0\). As \(|\omega| \leq 1\), we have

\[
0 < \lambda|\omega| < 1 \quad (2).
\]

To simplify the writing, we set

\[
\varphi_n = \frac{v_n + \lambda \omega}{|v_n + \lambda \omega|} \in S_X.
\]

The sequence \(\{v_n\}_{n \geq 1}\) is weakly convergent for the topology \(\sigma(X, X')\) to zero and (1) imply

\[
\varphi_n \overset{w}{\longrightarrow} \lambda \omega \quad (3),
\]

The purpose of the following steps is proof that \(\omega = 0\), i.e., \(u\) is a fixed point of \(T\).

In the next step we will define closed vector subspaces, denoted \(C_n\) and \(D_n\), using the vectors \(\frac{\omega}{|\omega|}, \varphi_n, \frac{J\omega}{|\omega|}\) and \(J\varphi_n\).

(B) (a) Thereafter we will use several times the following notations:

\[
A_n = \langle J\varphi_n, \varphi_n \rangle \quad \text{and} \quad B_n = \langle J\varphi_n, \frac{\omega}{|\omega|} \rangle
\]

Thus \(|B_n| \leq 1\) and, from (2) and (3), \(A_n \to \lambda|\omega| < 1\) as \(n \to \infty\). Therefore, by deleting a finite number of terms, we can assume that

\[
0 < A_n < 1 \quad \text{and} \quad 0 \leq A_n|B_n| < 1 \quad \text{for all} \quad n \geq 1 \quad (4)
\]

(b) Show that, for each \(n \geq 1\), \(\varphi_n\) and \(\frac{\omega}{|\omega|}\) are linearly independent; otherwise, there exist a \(\lambda_q \in \mathbb{R} \setminus \{0\}\) such that

\[
\varphi_q = \lambda_q \frac{\omega}{|\omega|},
\]

i.e., \(\varphi_q\) and \(\frac{\omega}{|\omega|}\) are linearly dependent; thus

\[
|\varphi_q| = |\frac{\omega}{|\omega|}| = |\lambda_q| = 1.
\]
The value of both equality members above \( (\varphi_q = \lambda_q \frac{\omega}{|\omega|}) \) at point \( J\omega \) give \( A_q = \lambda_q \).

From (4) \( 0 < A_q < 1 \), for all \( q \); contradiction with the above relation \( A_q = \lambda_q = \pm 1 \). Therefore, \( \varphi_n \) and \( \omega \frac{\omega}{|\omega|} \) are linearly independent for all \( n \geq 1 \).

\( \text{(c)} \) Consider the following closed hyperplans:

\[
\begin{align*}
\text{Ker} J\varphi_n &= \left\{ x_n \in X : < J\varphi_n, x_n >= 0 \right\} \quad \text{and} \\
\text{Ker} J\frac{\omega}{|\omega|} &= \left\{ y \in X : < J\frac{\omega}{|\omega|}, y >= 0 \right\}
\end{align*}
\]

Also, consider the unidimensional subspace of \( X \) noted \( \mathbb{R}(\varphi_n - \frac{\omega}{|\omega|}) \) generated by the vector \( \varphi_n - \frac{\omega}{|\omega|} \). Then, for each \( n \geq 1 \), the sum of the subspace \( \mathbb{R}\left(\varphi_n - \frac{\omega}{|\omega|}\right) + \left(\text{Ker} J\varphi_n \cap \text{Ker} J\frac{\omega}{|\omega|}\right) \) is direct; indeed, let

\[
\alpha_n \in \mathbb{R}(\varphi_n - \frac{\omega}{|\omega|}) \cap \left(\text{Ker} J\varphi_n \cap \text{Ker} J\frac{\omega}{|\omega|}\right)
\]

hence, there exist \( \alpha_n \in \mathbb{R} \) such that

\[
\alpha_n(\varphi_n - \frac{\omega}{|\omega|}) = a_n
\]

thus,

\[
< J\varphi_n, a_n >= 0 = \alpha_n(1 - < J\frac{\omega}{|\omega|}, \varphi_n >) = \alpha_n(1 - A_n)
\]

From (4) \( A_n - 1 \neq 0 \); which implies \( \alpha_n = 0 \Rightarrow a_n = 0 \). So the sum is direct and noted

\[
C_n = \mathbb{R}(\varphi_n - \frac{\omega}{|\omega|}) \oplus \left(\text{Ker} J\varphi_n \cap \text{Ker} J\frac{\omega}{|\omega|}\right)
\]

Since the sum is direct, we have

\[
\mathbb{R}(\varphi_n - \frac{\omega}{|\omega|}) \subset C_n \quad \text{and} \quad \left(\text{Ker} J\varphi_n \cap \text{Ker} J\frac{\omega}{|\omega|}\right) \subset C_n \quad \text{(5)}
\]

\( \text{(d)} \) Let us show that for each \( n \geq 1 \) fixed, the vector subspace \( C_n \) is closed. Indeed, let a sequence \( \{\beta_p\}_{p \geq 1} \subset C_n \) strongly convergent to \( \beta_0 \); obviously, the integers \( p \) depend on the fixed integer \( n \). Hence there exist \( \alpha_p \in \mathbb{R} \) and \( a_p \in \text{Ker} J\varphi_n \cap \text{Ker} J\frac{\omega}{|\omega|} \) such that
\[ \beta_p = \alpha_p(\varphi_n - \frac{\omega}{|\omega|}) + a_p \]

The value of both equality members above at point \( \frac{J\omega}{|\omega|} \) gives

\[ <\beta_p, \frac{J\omega}{|\omega|}> = \alpha_p(<\frac{J\omega}{|\omega|}, \varphi_n > -1) = \alpha_p(A_n - 1) \]

which implies \( \alpha_p = <\beta_p, \frac{J\omega}{|\omega|}> (A_n - 1)^{-1} \). As \( p \to \infty \),

\[ \alpha_p \to \alpha_0 = <\beta_0, \frac{J\omega}{|\omega|}> (A_n - 1)^{-1} \]

According to the above expression of \( \beta_p \), we have that \( a_p = \beta_p - \alpha_p(\varphi_n - \frac{\omega}{|\omega|}) + a_p \);

as \( p \to \infty \) we have \( a_p \to a_0 = \beta_0 - \alpha_0(\varphi_n - \frac{\omega}{|\omega|}) \); \( \text{Ker} J\varphi_n \cap \text{Ker} \frac{J\omega}{|\omega|} \) being closed, hence \( a_0 \in \text{Ker} J\varphi_n \cap \text{Ker} \frac{J\omega}{|\omega|} \); consequently

\[ \beta_0 = \alpha_0(\varphi_n - \frac{\omega}{|\omega|}) + a_0 \in C_n \]

which proves that \( C_n \) is closed and this is valid for all \( n \geq 1 \).

(e) Let us show that \( \frac{\omega}{|\omega|} \notin C_n \) for each \( n \geq 1 \); otherwise, there is a \( \gamma_n \in \mathbb{R} \) and \( b_n \in \text{Ker} J\varphi_n \cap \text{Ker} \frac{J\omega}{|\omega|} \) such that

\[ \frac{\omega}{|\omega|} = \gamma_n(\varphi_n - \frac{\omega}{|\omega|}) + b_n \]

\( \gamma_n = 0 \Rightarrow \frac{\omega}{|\omega|} \in \text{Ker} J\varphi_n \cap \text{Ker} \frac{J\omega}{|\omega|} \), what is absurd \( (<\frac{J\omega}{|\omega|}, \frac{\omega}{|\omega|}> = 1) \); hence

\( \gamma_n \neq 0 \). The value of both equality members above at points \( \frac{J\omega}{|\omega|} \) and \( J\varphi_n \) gives

\[ 1 = \gamma_n(A_n - 1) \text{ and } B_n = \gamma_n(1 - B_n) \].

Thus \( A_n B_n = 1 \); which contradicts (4).

(f) Let

\[ D_n = \bigcap_{1 \leq k \leq n} C_k \; , \; D = \bigcap_{n \geq 1} C_n, \]
The properties of $\{C_n\}_{n \geq 1}$ sequence give: for each $n \geq 1$, $D_n$ is not empty ($0 \in D_n$) and $D_n$ is closed subspace in $X$. We remark that

$$
\text{Ker} J \varphi_n \cap \text{Ker} J_\omega \subset C_n \Rightarrow \bigcap_{1 \leq k \leq n} (\text{Ker} J \varphi_k \cap \text{Ker} J_\omega) \subset \bigcap_{1 \leq k \leq n} C_k = D_n
$$

$X$ being of infinite dimension, it is recalled that the family $\{J \varphi_k\}_{1 \leq k \leq n} \cup \{J_\omega\}$ contains an element $y_0 \in X \setminus \{0\}$ such that $< J_\omega, y_0 > = < J \varphi_k, y_0 > = 0$ for all $k \in \{1, \ldots, n\}$, i.e. $y_0 \in \bigcap_{1 \leq k \leq n} (\text{Ker} J \varphi_k \cap \text{Ker} J_\omega) \subset D_n$. Therefore, for each integer $n \geq 1, D_n$ is not reduced to zero.

The sequence $\{D_n\}_{n \geq 1}$ is a decreasing, thus $D = \bigcap_{n \geq 1} D_n$ is a non-empty, closed subspace and its eventually reduced to zero.

Other hand, we note that $D \subset D_n \subset C_n$; as $\omega \notin C_n$ (see (B)-(d) and (B)-(e)), by the lemma 1 there exist a continuous linear form $Y_n \in X'$ such that

(i) $\| Y_n \| = 1$, (ii) $< Y_n, \omega \rangle = \text{dist}(\omega, C_n)$, (iii) $C_n \subset \text{Ker} Y_n$ ($n \geq 1$).

Thus $\text{R}(\varphi_n - \omega \| \omega \|) \subset C_n$ (see (5)) and the above relation involve that

$$
< Y_n, \omega \rangle = < Y_n, \varphi_n > = \text{dist}(\omega \| \omega \|, C_n) \quad (6)
$$

Similarly, $D_n$ being closed subspace and $\omega \notin D_n$, by the lemma 1, there exist a continuous linear form $Z_n \in X'$ such that

(i) $\| Z_n \| = 1$, (ii) $< Z_n, \omega \rangle = \text{dist}(\omega \| \omega \|, D_n)$, (iii) $D_n \subset \text{Ker} Z_n$ ($n \geq 1$) \quad (7).
$D_n$ being vector subspace of $C_n$, then $\mathbb{R} \frac{\omega}{|\omega|} \oplus D_n \subset \mathbb{R} \frac{\omega}{|\omega|} \oplus C_n$. By taking into account the constructive method of the proof of lemma 1, we have $W_n = Y_n = Z_n$ on $\mathbb{R} \frac{\omega}{|\omega|} \oplus D_n$. $Z_n$ and $Y_n$ are two extensions of $W_n$ on $X$; as the dual space $X'$ is strictly convex, we have $Z_n = Y_n$ on $X$ for all $n \geq 1$; therefore, from (6) and (7) we have,

$$<Z_n, \frac{\omega}{|\omega|}> = <Z_n, \varphi_n> = dist\left(\frac{\omega}{|\omega|}, D_n\right) = dist\left(\frac{\omega}{|\omega|}, C_n\right)$$

$D_n \subset C_n \subset \text{Ker} Z_n$  

(8)

As $X$ is a reflexive and $D_n$ is a nonempty closed subspace, there exist a $x_n \in D_n \subset \text{Ker} Z_n$ (see, for example, Brezis (1983)p.46) such that

$$<Z_n, \frac{\omega}{|\omega|}> = dist\left(\frac{\omega}{|\omega|}, D_n\right) = |\frac{\omega}{|\omega|} - x_n|$$

Hence,

$$<Z_n, \frac{\omega}{|\omega|}> = <Z_n, \frac{\omega}{|\omega|} - x_n> = dist\left(\frac{\omega}{|\omega|}, D_n\right) = |\frac{\omega}{|\omega|} - x_n|$$

Since the space $X'$ is strictly convex, we have

$$Z_n = \frac{J(\frac{\omega}{|\omega|} - x_n)}{|\frac{\omega}{|\omega|} - x_n|} = \frac{J(\frac{\omega}{|\omega|} - x_n)}{dist\left(\frac{\omega}{|\omega|}, D_n\right)}$$

(9)

In the next step, we prove that the sequence $\{Z_n\}_{n \geq 1}$ contains a strongly convergent subsequence, using the Kadec-klee property.

(D) The sequence $\{D_n\}_{n \geq 1}$ is decreasing implies that the sequence $\{dist\left(\frac{\omega}{|\omega|}, D_n\right) = |\frac{\omega}{|\omega|} - x_n|\}_{n \geq 1}$ is increasing. Other hand, $dist\left(\frac{\omega}{|\omega|}, D_n\right) \leq dist\left(\frac{\omega}{|\omega|}, D\right)$ for all $n \geq 1$. $0 \in D$ implies $dist\left(\frac{\omega}{|\omega|}, D\right) \leq |\frac{\omega}{|\omega|}| = 1$. Consequently, the sequence $\{dist\left(\frac{\omega}{|\omega|}, D_n\right)\}_{n \geq 1}$ is convergent to a limit $L \leq dist\left(\frac{\omega}{|\omega|}, D\right) \leq 1$. Other hand the sequence $\{x_n\}_{n \geq 1}$ is bounded, because

$$|x_n| \leq |\frac{\omega}{|\omega|} - x_n| + |\frac{\omega}{|\omega|}| \leq 2$$
X being reflexive, we can extract a subsequence, noted \(\{x_m\}\), weakly convergent to a limit \(x_0 \in X\).

Recall that for each \(m\), \(D_m\) is closed so its weakly closed; the sequence \(\{D_m\}_m\) is decreasing, thus \(x_p \in D_m\) for all \(p \geq m\), therefore \(x_0 \in D_m\) for all \(m\). Consequently, \(x_0 \in \bigcap D_m = \bigcap_{n \geq 1} D_n = D\). Thus

\[
\liminf_m \frac{\omega}{|\omega|} - x_m = \lim_m \frac{\omega}{|\omega|} - x_m \geq \frac{\omega}{|\omega|} - x_0.
\]

Other hand,

\[
d(\frac{\omega}{|\omega|}, D) \geq \lim_m \text{dist}(\frac{\omega}{|\omega|}, D_m) = \lim_m \frac{\omega}{|\omega|} - x_m
\]

\[
\geq \frac{\omega}{|\omega|} - x_0 \geq \text{dist}(\frac{\omega}{|\omega|}, D),
\]

hence,

\[
\lim_m \text{dist}(\frac{\omega}{|\omega|}, D_m) = \lim_m \frac{\omega}{|\omega|} - x_m = \text{dist}(\frac{\omega}{|\omega|}, D) = \frac{\omega}{|\omega|} - x_0 \quad (10)
\]

Consequently,

\[
\frac{\omega}{|\omega|} - x_m \xrightarrow{w^*} \frac{\omega}{|\omega|} - x_0 \quad \text{and} \quad \lim_m \frac{\omega}{|\omega|} - x_m = \frac{\omega}{|\omega|} - x_0.
\]

\(X'\) being reflexive and \(|Z_m| = 1\), then there exist a subsequence, noted again \(\{Z_m\}_m\), weakly convergent to \(Z_0 \in X',\) i.e.,

\[
Z_m \xrightarrow{w^*} Z_0, \text{as } m \to \infty \quad (11)
\]

From (7), (10) and (11),

\[
< Z_0, \frac{\omega}{|\omega|} > = d = \text{dist}(\frac{\omega}{|\omega|}, D) = \frac{\omega}{|\omega|} - x_0.
\]

We show that \(< Z_0, x_0 > = 0\); indeed, the sequence \(\{D_m\}_m\) is decreasing, we note that

\[
D_m \subset \text{Ker} Z_m \Rightarrow \bigcap_{k \leq m} D_k = D_m \subset \bigcap_{k \leq m} \text{Ker} Z_k \Rightarrow D \subset \bigcap_m \text{Ker} Z_m
\]

\(x_0 \in D\) implies that \(< Z_m, x_0 > = 0\) for all \(m\), which implies \(< Z_0, x_0 > = 0\) as \(m \to \infty\). This result implies that

\[
< Z_0, \frac{\omega}{|\omega|} > = < Z_0, \frac{\omega}{|\omega|} - x_0 > = d.
\]
Since the space $X'$ is strictly convex, we have $Z_0 = \frac{J(\frac{\omega}{|\omega|} - x_0)}{d}$; from this equality, (9) and (10), we have

$$Z_m = \frac{J(\frac{\omega}{|\omega|} - x_m)}{d_m} \xrightarrow{w^*} Z_0 = \frac{J(\frac{\omega}{|\omega|} - x_0)}{d}.$$  

Since the sequence $\{Z_m\}_m$ and $Z_0$ are elements of the $S_{X'}$, the Kadec-Klee property gives

$$Z_m \xrightarrow{\|\|} Z_0 \text{ as } m \to \infty \quad (E)$$

Now we finish this proof. From (8),

$$< Z_m, \frac{\omega}{|\omega|} - \varphi_m > = 0 \text{ for all } m.$$  

Because of the strong convergence of the subsequence $\{Z_m\}_m$ to $Z_0$ and the weak convergence of the sequence $\{\frac{\omega}{|\omega|} - \varphi_m\}_m$ to $\frac{\omega}{|\omega|} - \lambda \omega$, we have

$$(1 - \lambda |\omega|) < Z_0, \frac{\omega}{|\omega|} >= 0 \Rightarrow < Z_0, \frac{\omega}{|\omega|} >= \text{dist}(\frac{\omega}{|\omega|}, D) = 0,$$

i.e., $\frac{\omega}{|\omega|} \in D$; what is absurd (see (C)). Consequently $\omega = 0$.□

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**References**


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