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Sums of Balancing and Lucas-Balancing Numbers with Binomial Coefficients

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Abstract

For $a, b \in \mathbb{R}$ we study binomial sums of the form $\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} B_{jk+m}$ and $\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} C_{jk+m}$, where $(B_n)_{n\geq 0}$ and $(C_n)_{n\geq 0}$ are Balancing and Lucas-Balancing numbers, respectively. We provide closed form solutions for many types of these sums. We also express these sums in a different combinatorial way. This enables us to state new combinatorial expressions for B_n and C_n.

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Keywords: Binomial Sum, Balancing number, Lucas-Balancing number

1 Introduction and Preliminaries

In 1999, Behera and Panda [1] introduced the notion of Balancing numbers $(B_n)_{n\geq 0}$ as solutions to a certain Diophantine equation. They have shown that a Balancing number B_n satisfies the recurrence relation $B_{n+1} = 6B_n - B_{n-1}, n \geq 1$, with initial terms $B_0 = 0$ and $B_1 = 1$. Another result about Balancing numbers is, that B_n is a Balancing number, if and only if B_n^2 is a triangular number, i.e., $8B_n^2 + 1$ is a perfect square. The sequence $C_n = \sqrt{8B_n^2 + 1}$ is called a Lucas-Balancing number. It satisfies the same recurrence

[1]Disclaimer: Statements and conclusions made in this article are entirely those of the author. They do not necessarily reflect the views of LBBW.

relation as B_n: $C_{n+1} = 6C_n - C_{n-1}, n \geq 1$, with initial terms $C_0 = 1$ and $C_1 = 3$. The Binet forms for B_n and C_n are given by, respectively,

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \qquad \text{and} \qquad C_n = \frac{1}{2}(\lambda_1^n + \lambda_2^n),$$

with $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$.

Ab initio, both B_n and C_n have gained popularity and are still the subject of research (see for instance [2], [4]-[10] and [12]-[16]). $(B_n)_{n\geq 0}$ is sequence A001109 in the OEIS [17], whereas $(C_n)_{n\geq 0}$ has the id-number A001541 in OEIS.

Expressions for binomial sums involving B_n and C_n may be derived using their Binet forms. The maybe most obvious examples are

$$\sum_{k=0}^{n}\binom{n}{k}B_{2k+m} = 6^n B_{n+m} \qquad \text{and} \qquad \sum_{k=0}^{n}\binom{n}{k}C_{2k+m} = 6^n C_{n+m}. \tag{1}$$

Other examples appear in [8] and [16]. Some hybrid variants are stated in [6]. In this article, we study the four parameter sums

$$S_n(B) = S_n(B; a, b, j, m) = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} B_{jk+m}, \tag{2}$$

and

$$S_n(C) = S_n(C; a, b, j, m) = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} C_{jk+m}, \tag{3}$$

where $a, b \in \mathbb{R}$ and $j, m \in \mathbb{N}$.

2 A First Result

The first theorem may be seen as an analogue of results from [3]. See also [11]. The theorem answers the following question: Take two integers p and q with $p \neq q$. For what values of a and b does the identity

$$\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} B_{qk+m} = B_{pn+m}$$

hold? We also find the answer if B_n is replaced by C_n. We will need the following lemmas:

Lemma 2.1. *The generating functions for the sequences $(B_{jn+m})_{n\geq 0}$ and $(C_{jn+m})_{n\geq 0}$ are given by*

$$f_{B_{jn+m}}(x) = \sum_{n=0}^{\infty} B_{jn+m}x^n = \frac{B_m + B_{j-m}x}{1 - 2C_j x + x^2}, \tag{4}$$

and

$$f_{C_{jn+m}}(x) = \sum_{n=0}^{\infty} C_{jn+m}x^n = \frac{C_m - C_{j-m}x}{1 - 2C_j x + x^2}. \tag{5}$$

PROOF: The proof is obvious using the Binet forms and the geometric series. Another proof can be found in [8]. □

Lemma 2.2. *The generating functions for $S_n(B)$ and $S_n(C)$ are*

$$f_{S_n(B)}(x) = \sum_{n=0}^{\infty} S_n(B)x^n = \frac{B_m + (aB_{j-m} - bB_m)x}{1 - (2b + 2aC_j)x + (a^2 + 2C_j ab + b^2)x^2}, \tag{6}$$

and

$$f_{S_n(C)}(x) = \sum_{n=0}^{\infty} S_n(C)x^n = \frac{C_m - (aC_{j-m} + bC_m)x}{1 - (2b + 2aC_j)x + (a^2 + 2C_j ab + b^2)x^2}. \tag{7}$$

PROOF: Applying Theorem 1 from [11] it follows that

$$f_{S_n(B)}(x) = \frac{1}{1 - bx} f_{B_{jn+m}}\Big(\frac{ax}{1 - bx}\Big),$$

and

$$f_{S_n(C)}(x) = \frac{1}{1 - bx} f_{C_{jn+m}}\Big(\frac{ax}{1 - bx}\Big).$$

Using the equations for $f_{B_{jn+m}}(x)$ and $f_{C_{jn+m}}(x)$ form the first Lemma and simplifying proves the relations. □

Theorem 2.3. *We have*

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} B_{qk+m} = B_{pn+m} \tag{8}$$

if and only if $a = B_p/B_q$ and $b = B_{q-p}/B_q$. Analogously, it holds that

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} C_{qk+m} = C_{pn+m} \tag{9}$$

if and only if $a = B_p/B_q$ and $b = B_{q-p}/B_q$.

PROOF: Comparing (4) with (6) we get the following system of equations:

$$\begin{aligned} C_p &= b + aC_q, \\ B_{p-m} &= aB_{q-m} - bB_m, \\ 1 &= a^2 + 2C_j ab + b^2. \end{aligned}$$

The first two equations produce

$$a = \frac{B_{p-m} + C_p B_m}{B_{q-m} + C_q B_m} = \frac{B_p}{B_q},$$

where we have used $B_{n-m} = B_n C_m - C_n B_m$. This gives

$$b = C_p - C_q \frac{B_p}{B_q} = \frac{B_{q-p}}{B_q}.$$

The verification of the third equation leads to $B_q^2 - B_p^2 = B_{q+p} B_{q-p}$, which is known as the Catalan identity for Balancing numbers. The proof of the second part of the theorem is very similar and omitted. □

Choosing $p = 1$ and $q = 2$ gives (1). For $p = 2$ and $q = 1$ we obtain

$$\sum_{k=0}^{n} \binom{n}{k} (-6)^k B_{k+m} = (-1)^n B_{2n+m} \quad \text{and} \quad \sum_{k=0}^{n} \binom{n}{k} (-6)^k C_{k+m} = (-1)^n C_{2n+m}. \tag{10}$$

The first example in (10) appears in [16]. It is worth to remark that the candidates in (1) and (10) are connected via the binomial transform. As a further example, we choose $q = 3$ and $p = 1$ and the result is

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{6}\right)^k B_{3k+m} = \left(\frac{35}{6}\right)^n B_{n+m} \quad \text{and} \quad \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{6}\right)^k C_{3k+m} = \left(\frac{35}{6}\right)^n C_{n+m}. \tag{11}$$

3 Some Special Sums

From (6) and (7) evaluations of some special binomial sums can be inferred.

Theorem 3.1. *We have*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(\frac{1}{C_j}\right)^k B_{jk+m} = \begin{cases} 8^{\frac{n}{2}} B_m \left(\frac{B_j}{C_j}\right)^n & \text{if } n \text{ is even} \\ -8^{\frac{n-1}{2}} C_m \left(\frac{B_j}{C_j}\right)^n & \text{if } n \text{ is odd.} \end{cases} \tag{12}$$

Especially,

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(\frac{1}{C_j}\right)^k B_{jk} = \begin{cases} 0 & \text{if } n \text{ is even} \\ -8^{\frac{n-1}{2}} \left(\frac{B_j}{C_j}\right)^n & \text{if } n \text{ is odd.} \end{cases} \tag{13}$$

PROOF: If we choose $a = -b/C_j$, then $(2b + 2aC_j)x$ vanishes from (6). Next,

$$a^2 + 2C_jab + b^2 = \frac{b^2(1 - C_j^2)}{C_j^2} = -b^2\frac{8B_j^2}{C_j^2},$$

and

$$aB_{j-m} - bB_m = -b\frac{B_jC_m}{C_j}.$$

Hence,

$$f_{S_n(B)}(x) = \sum_{n=0}^{\infty} B_m\Big(\frac{8B_j^2}{C_j^2}\Big)^n b^{2n}x^{2n} - \sum_{n=0}^{\infty} \frac{B_jC_m}{C_j}\Big(\frac{8B_j^2}{C_j^2}\Big)^n b^{2n+1}x^{2n+1}.$$

Comparing the coefficients of x^n proves the stated identity. □

The companion result for Lucas-Balancing numbers is stated without proof.

Theorem 3.2. *We have*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\Big(\frac{1}{C_j}\Big)^k C_{jk+m} = \begin{cases} 8^{\frac{n}{2}}C_m\Big(\frac{B_j}{C_j}\Big)^n & \text{if } n \text{ is even} \\ -8^{\frac{n+1}{2}}B_m\Big(\frac{B_j}{C_j}\Big)^n & \text{if } n \text{ is odd.} \end{cases} \tag{14}$$

Especially,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\Big(\frac{1}{C_j}\Big)^k C_{jk} = \begin{cases} 8^{\frac{n}{2}}\Big(\frac{B_j}{C_j}\Big)^n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{15}$$

Theorem 3.3. *For $n \geq 1$ the following identities are valid:*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\lambda_1^{jk}B_{jk+m} = (-1)^n(2\sqrt{8})^{n-1}\lambda_1^{jn+m}B_j^n, \tag{16}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\lambda_2^{jk}B_{jk+m} = -(2\sqrt{8})^{n-1}\lambda_2^{jn+m}B_j^n. \tag{17}$$

PROOF: To derive the identities we choose a such that $a^2+2C_jab+b^2 = 0$ in (6). We have

$$a_{1/2}(j, b) = b(-C_j \pm \sqrt{C_j^2 - 1}) = b(-C_j \pm \sqrt{8}B_j).$$

Thus, $a_1(j, b) = -b\lambda_2^j$ and $a_2(j, b) = -b\lambda_1^j$. Also,

$$2b + 2a_1(j, b)C_j = b2\sqrt{8}\lambda_2^jB_j,$$

and

$$2b + 2a_2(j,b)C_j = -b2\sqrt{8}\lambda_1^j B_j.$$

This gives

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(a_2(b,j))^k b^{n-k}B_{jk+m}x^n = B_m x^0$$
$$+\sum_{n=1}^{\infty}\Big((-1)^n(2\sqrt{8}B_m\lambda_1^j B_j + B_m + \lambda_1^j B_{j-m})(2\sqrt{8}\lambda_1^j B_j)^{n-1}\Big)b^n x^n,$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(a_1(b,j))^k b^{n-k}B_{jk+m}x^n = B_m x^0$$
$$+\sum_{n=1}^{\infty}\Big((2\sqrt{8}B_m\lambda_2^j B_j - B_m - \lambda_2^j B_{j-m})(2\sqrt{8}\lambda_2^j B_j)^{n-1}\Big)b^n x^n.$$

Finally, the Binet form for B_n can be used to show that

$$2\sqrt{8}B_m\lambda_1^j B_j + B_m + \lambda_1^j B_{j-m} = \lambda_1^{j+m}B_j,$$

and

$$2\sqrt{8}B_m\lambda_2^j B_j - B_m - \lambda_2^j B_{j-m} = -\lambda_2^{j+m}B_j.$$

□

Corollary 3.4. *For $n \geq 1$ we have*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k B_{jk}B_{jk+m} = \begin{cases} 2(2\sqrt{8})^{n-2}B_j^n C_{jn+m} & \text{if } n \text{ is even} \\ -(2\sqrt{8})^{n-1}B_j^n B_{jn+m} & \text{if } n \text{ is odd,} \end{cases} \tag{18}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k C_{jk}B_{jk+m} = \begin{cases} \frac{1}{2}(2\sqrt{8})^{n}B_j^n B_{jn+m} & \text{if } n \text{ is even} \\ -(2\sqrt{8})^{n-1}B_j^n C_{jn+m} & \text{if } n \text{ is odd.} \end{cases} \tag{19}$$

The analogue results for C_n can be inferred from (7) and are stated without proof.

Theorem 3.5. *For $n \geq 1$ the following identities are valid:*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\lambda_1^{jk}C_{jk+m} = \frac{(-1)^n}{2}(2\sqrt{8})^n\lambda_1^{jn+m}B_j^n, \tag{20}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\lambda_2^{jk}C_{jk+m} = \frac{1}{2}(2\sqrt{8})^n\lambda_2^{jn+m}B_j^n. \tag{21}$$

Corollary 3.6. *For $n \geq 1$ we have*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k B_{jk} C_{jk+m} = \begin{cases} \frac{1}{2}(2\sqrt{8})^n B_j^n B_{jn+m} & \text{if } n \text{ is even} \\ -(2\sqrt{8})^{n-1} B_j^n C_{jn+m} & \text{if } n \text{ is odd,} \end{cases} \tag{22}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k C_{jk} C_{jk+m} = \begin{cases} \frac{1}{2}(2\sqrt{8})^n B_j^n C_{jn+m} & \text{if } n \text{ is even} \\ -\frac{1}{4}(2\sqrt{8})^{n+1} B_j^n B_{jn+m} & \text{if } n \text{ is odd.} \end{cases} \tag{23}$$

Note that

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k C_{jk} B_{jk+m} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k B_{jk} C_{jk+m}. \tag{24}$$

4 Combinatorial Identities for $S_n(B)$ and $S_n(C)$

Theorem 4.1. *The following combinatorial identity is valid*

$$S_n(B) = \delta(n) + \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-l-1}{l} (-1)^l (v+bu)^l u^{n-1-2l} \Big(\frac{n}{n-2l} (b+aC_j) B_m + aC_m B_j \Big), \tag{25}$$

where $u = 2(b + aC_j)$, $v = a^2 - b^2$ and

$$\delta(n) = \begin{cases} B_m (-1)^{\lfloor \frac{n}{2} \rfloor} (v+bu)^{\lfloor \frac{n}{2} \rfloor} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{26}$$

PROOF: For notational brevity we set $w = aB_{j-m} - bB_m$. Then, again from (6) we have

$$\begin{aligned} f_{S_n(B)}(x) &= (B_m + wx) \sum_{n=0}^{\infty} x^n (u - (v+bu)x)^n \\ &= B_m \sum_{n=0}^{\infty} \sum_{s=0}^{n} \binom{n}{s} (-1)^s (v+bu)^s u^{n-s} x^{n+s} \\ &\quad + w \sum_{n=0}^{\infty} \sum_{s=0}^{n} \binom{n}{s} (-1)^s (v+bu)^s u^{n-s} x^{n+s+1} \\ &= B_m \sum_{r=0}^{\infty} \sum_{l=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r-l}{l} (-1)^l (v+bu)^l u^{r-2l} x^r \\ &\quad + w \sum_{r=1}^{\infty} \sum_{l=0}^{\lfloor \frac{r-1}{2} \rfloor} \binom{r-l-1}{l} (-1)^l (v+bu)^l u^{r-2l-1} x^r. \end{aligned}$$

Comparing the coefficients gives the relation

$$S_n(B) \;=\; \delta(n) + \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^l (v+bu)^l u^{n-2l-1} \Big(uB_m \binom{n-l}{l} + w \binom{n-l-1}{l} \Big),$$

where $\delta(n)$ is defined above. We have $w = -\frac{1}{2}uB_m + aC_mB_j$. The statement now follows since

$$\binom{n-l}{l} = \frac{n-l}{n-2l}\binom{n-l-1}{l},$$

and

$$\binom{n-l}{l} - \frac{1}{2}\binom{n-l-1}{l} = \frac{n}{2(n-2l)}\binom{n-l-1}{l}.$$

□

The analogue result for $S_n(C)$ is stated without proof.

Theorem 4.2. *The following combinatorial identity is valid*

$$S_n(C) = \delta^*(n) + \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-l-1}{l} (-1)^l (v+bu)^l u^{n-1-2l} \Big(\frac{n}{n-2l}(b+aC_j)C_m + 8aB_mB_j \Big), \tag{27}$$

where $u = 2(b + aC_j)$, $v = a^2 - b^2$ *and*

$$\delta^*(n) = \begin{cases} C_m(-1)^{\lfloor \frac{n}{2} \rfloor}(v+bu)^{\lfloor \frac{n}{2} \rfloor} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{28}$$

Identities (25) and (27) contain a range of combinatorial formulas for B_n and C_n as special cases. We give three examples of such formulas: From $S_n(B; 1, 1, 2, 0)$ we can easily deduce the known identity

$$B_n = \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-l-1}{l} (-1)^l 6^{n-2l-1}. \tag{29}$$

Similarly, $S_n(C; 1, 1, 2, 0)$ gives

$$C_n = 3 \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-l-1}{l} (-1)^l 6^{n-2l-1} \frac{n}{n-2l} + d(n), \tag{30}$$

where

$$d(n) = \begin{cases} (-1)^{\lfloor \frac{n}{2} \rfloor} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Also, using the relations (see [8])

$$\sum_{k=0}^{2n} \binom{2n}{k} (-1)^k B_k = 4^n B_n \qquad \text{and} \qquad \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k C_k = 4^n C_n, \tag{31}$$

we get from $S_n(B; 1, -1, 1, 0)$ and $S_n(C; 1, -1, 1, 0)$ the combinatorial results $(n \geq 1)$

$$B_n = \sum_{l=0}^{\lfloor \frac{2n-1}{2} \rfloor} \binom{2n-l-1}{l} 4^{n-l-1}, \tag{32}$$

and

$$C_n = 1 + 2 \sum_{l=0}^{\lfloor \frac{2n-1}{2} \rfloor} \binom{2n-l-1}{l} 4^{n-l-1} \frac{n}{n-l}. \tag{33}$$

Finally, $S_n(B; 3, -1, 1, 0)$ and $S_n(C; 3, -1, 1, 0)$ can be combined with two other identities from [8] to get

$$B_{2n} = 3 \sum_{l=0}^{\lfloor \frac{2n-1}{2} \rfloor} \binom{2n-l-1}{l} 2^{5(n-l)-4}, \tag{34}$$

$$B_{2n+1} = \sum_{l=0}^{n} \binom{2n-l}{l} 2^{5(n-l)} \frac{2n+1}{2(n-l)+1}, \tag{35}$$

$$C_{2n} = \sum_{l=0}^{\lfloor \frac{2n-1}{2} \rfloor} \binom{2n-l-1}{l} 2^{5(n-l)-1} \frac{n}{n-l} + 1, \tag{36}$$

and

$$C_{2n+1} = 3 \sum_{l=0}^{n} \binom{2n-l}{l} 2^{5(n-l)}. \tag{37}$$

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