Fixed Point Theorem in Locally K-Convex Space

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Abstract

In the present paper, we obtain some new fixed point theorems for set-valued $p$–contractive mappings in the setting of locally $K$–convex spaces. Our theorems complement, generalize and extend some well known results of Petalas and Vidalis [A fixed point theorem in non-Archimedean vector spaces, Proc. Amer. Math. Soc 118(1993), 819–821.]

1. Introduction

In 1922, Banach established the most famous fundamental fixed point theorem (the so-called the Banach contraction principle ) which has played an important role in various fields of applied mathematical analysis. Due to its importance and simplicity, several authors have obtained many interesting extensions and generalizations of the Banach contraction principle. Ciric [2], introduced quasi contraction, which is a generalization of Banach contraction principle.

In this paper, motivated by the work of Kirk WA, Shahzad N [14], we prove that every $p$–contractive mapping on a spherically complete locally $K$–convex space has a fixed point.

A. F. Monna in [5] introduced the concept of locally convex space. C.Petals et al [3]. (1993) proved a fixed point theorem on non-Archimedean normed space using a contractive condition. We first recall some basic notions in locally


2. Preliminaries

A non-Archimedean valued field is a field \( \mathbb{K} \) equipped with a function (valuation) \( |\cdot| \) from \( \mathbb{K} \) into \([0,\infty)\) such that \( a = 0 \) if and only if \( |a| = 0 \), \( |a + b| \leq \max\{|a|, |b|\} \) and \( |ab| = |a||b|, a, b \in \mathbb{K} \). Clearly, \( |−1| = 1 \) and \( |n.1_{\mathbb{K}}| \leq 1 \) for all \( n \in \mathbb{N} \).

**Definition 1.** Let \( X \) be a topological vector space over \( \mathbb{K} \). A subset \( A \subset X \) is called absolutely \( \mathbb{K} \)-convex (in the sense of Monna [5]) if \( ax + by \in A \) for all \( x, y \in X \) and \( a, b \in B_{\mathbb{K}} \) where \( B_{\mathbb{K}} = \{a \in \mathbb{K} / |a| \leq 1\} \).

A locally convex space is defined as a topological vector space whose topology has a base consisting of convex sets. These base sets are automatically clopen, so that a locally convex space is zerodimensional.

One can also start from the notion of a seminorm.

**Definition 2.** A seminorm on a \( \mathbb{K} \)-vector space is a function a map \( p : X \rightarrow [0,\infty) \) satisfying the requirements

1. \( p(\alpha x) = |\alpha| p(x) \) for all \( x \in X \) and \( \alpha \in \mathbb{K} \)
2. \( p(x + y) \leq \max\{p(x), p(y)\} \) for all \( x, y \in X \)

the \( p \)-balls \( \{x \in X | p(x - x_0) \leq r\} \) are convex sets. the center of such a ball is not solely determined: any point of the ball can serve as the center. the balls have curious properties: if two balls have a non-empty intersection, one of these balls is contained in the other.

If \( \Gamma \) is a collection of seminorms on a vector space \( X \), then the weakest topology that makes all elements of \( \Gamma \)-continuous renders \( X \) a locally convex space. Conversely, every locally convex space can be obtained in this way from a set of seminorms. (The connection between convex sets and seminorms, however, is just a little more complicated than in the Archimedean).

Let \( X \) be a locally convex space whose topology is determined by a \( \Gamma \) family of semi-norms n.a.

**Proposition 3.** if \( x_1 \in B_p(x_0, r) \) we have \( B_p(x_0, r) = B_p(x_1, r) \)

**Proof.** See [5]

**Proposition 4.** Let \( A \) be a family of \( p \)-balls such that any two of these balls have a non-empty intersection; then \( A \) is a totally ordered set by the relation of inclusion.

**Proof.** See [5]

**Definition 5.** Space \( X \) is called \( p \)-spherically complete when any family of \( p \)-balls that is totally ordered by inclusion has a non-empty intersection.

3. Main Results

Lemma 6. (Zorn’s lemma). Let $S$ be a partially ordered set. If every totally ordered subset of $S$ has an upper bound, then $S$ contains a maximal element.

Theorem 7. Let $(X, \Gamma)$ be a spherically complete Hausdorff locally $K$-convex space if $T : X \to X$ is a $p-$contractive mapping then $T$ has a unique fixed point in $\bar{x}$ in $X$

Proof. Let $B_a = \{a, p(a - Ta)\}$ denote the closed $p-$spheres centered at $a$ with radius $p(a - Ta)$, and let $\mathcal{A}$ be the collection of these $p-$spheres for all $a \in X$.

The relation

$$B_a \leq B_b \iff B_b \subseteq B_a$$

is a partial order. Consider a totally ordered subfamily $\mathcal{A}_1$ of $\mathcal{A}$. From the spherical completeness of $X$, we have

$$\bigcap_{B_a \in \mathcal{A}} B_a = B \neq \emptyset$$

Let $b \in B$ and $B_a \subset \mathcal{A}_1$. Let $x \in B_b$

Then

$$p(x - a) \leq \max \{p(b - a), p(b - x)\} \leq p(a - Ta)$$

since $p(b - a) \leq p(a - Ta)$ and

$$p(x - b) \leq p(b - Tb) \leq \max \{p(b - a), p(a - Ta), p(Ta - Tb)\} = p(a - Ta) \quad (1)$$

In opposite case, $p(Ta - Tb) > p(a - Ta)$ and from (1) follows that

$$p(x - b) \leq p(b - Tb) \leq p(Ta - Tb) \leq \max \{p(a - b), p(a - Ta), p(b - Tb)\} = \max \{p(a - Ta), p(b - Tb)\}$$

Now for $p(b - Tb) \leq p(a - Ta)$ we have

$$p(x - b) \leq p(a - Ta)$$

The inequality $p(b - Tb) > p(a - Ta)$ implies that $p(b - Tb) < p(b - Tb)$ which is a contradiction.

So we have proved that for $x \in B_a$

This proves that $B_b \subseteq B_a$ for every $B_a \in \mathcal{A}_1$. Thus $B_b$ is an upper bound in $\mathcal{A}$ for the family $\mathcal{A}_1$. by Zorn’s lemma $\mathcal{A}$ has a maximal element, say $B_z$, for some $z \in Z$, we claim $z = Tz$

Suppose $z \neq Tz$. Since

$$p(Tz - T^2z) < p(z - Tz) \quad (T^2 = T_0T) \quad (\ast)$$

and

$$Tz \in B(Tz, p(Tz - T^2z)) \cap B(z, p(z - Tz))$$

we have $B_{Tz} \subseteq B_z$. But $z \notin B_{Tz}$ by $(\ast)$, so $B_{Tz} \not\subseteq B_z$ and this contradicts the maximality of $B_z$. Therefore $T$ has a fixed point, obviously unique. □
Proposition 8. Let \((X, \Gamma)\) be a spherically complete Hausdorff locally \(\mathbb{K}\) convex space and
\[
B(x_0, r) = \{x \in X : p(x - x_0) \leq r\} \quad \text{where} \quad x_0 \text{and} \quad r > 0
\]
Suppose \(T : B(x_0, r) \to X\) is a \(p\)-contraction and for all \(x, y \in B(x_0, r)\) with
\[
p(Tx_0 - x_0) \leq (1 - \gamma_p) r
\]
Then \(T\) has a unique fixed point in \(B(x_0, r)\)

Proof: Let \(x \in B(x_0, r)\)
Then
\[
p(Tx - x_0) \leq \max \{p(Tx - Tx_0), p(Tx_0 - x_0)\}
\]
\[
\leq \max \{\gamma_p p(x - x_0), p(Tx_0 - x_0)\}
\]
\[
\leq \max \{\gamma_p r, (1 - \gamma_p) r\} \leq r
\]
and consequently \(Tx \in B(x_0, r)\)
so we have \(T : B(x_0, r) \to B(x_0, r)\) is a \(p\)-contraction mapping
We can now apply Theorem 7 to deduce that \(T\) has a unique fixed

Proposition 9. Suppose \((X, \Gamma)\) is a spherically complete Hausdorff locally \(\mathbb{K}\) convex space and suppose \(T : M \to M\) is a mapping for which \(T^n\) is a \(p\)-contraction mapping for some positive integer \(n\). Then \(T\) has a unique fixed point.

Proof: By Theorem 7 \(T^n\) has a unique fixed point \(x\).
However
\[
T^{n+1}(x) = T(T^n(x)) = T(x)
\]
So \(T(x)\) is also a fixed point of \(T^n\). Since the fixed point of \(T^n\) is unique, it must be the case that \(T(x) = x\). Also, if \(T(z) = z\) then \(T^n(z) = z\) proving (again by uniqueness) that \(x = z\)

Theorem 10. Let \((X, \Gamma)\) be a spherically complete Hausdorff locally \(\mathbb{K}\) convex space and \(T : X \to X\) is a mapping such that
\[
p(Tx - Ty) < \max \{p(x - y), p(x - Tx), p(y - Ty)\}
\]
for all \(x, y \in X, x \neq y\) and \(p \in \Gamma\)
Then \(T\) has a unique fixed point in \(X\).

Proof: Let \(B_a = \{a, p(a - Ta)\}\) denote the closed \(p\)-spheres centered at \(a\) with radius \(p(a - Ta)\), and let \(\mathcal{A}\) be the collection of these \(p\)-spheres for all \(a \in X\).
The relation
\[
B_a \leq B_b \iff B_b \subseteq B_a
\]
is a partial order. Consider a totally ordered subfamily $A_1$ of $A$. From the spherical completeness of $(X, \Gamma)$, we have

$$\bigcap_{B_a \in A} B_a = B \neq \emptyset$$

Let $b \in B$ and $B_a \subset A_1$. Let $x \in B_b$

Then

$$p(x - b) \leq p(b - T b) \leq \max \{p(b - a), p(a - Ta), p(Ta - T b)\}$$

$$= \max \{p(a - Ta), p(Ta - T b)\} \ (2)$$

Then

$$\max \{p(a - Ta), p(b - T b)\} = p(a - Ta)$$

In opposite case, $p(Ta - T b) > p(a - Ta)$ and from (2) follows that

$$p(x - b) \leq p(b - T b) \leq p(Ta - T b) < \max \{p(a - b), p(a - Ta), p(b - T b)\}$$

Now for $p(b - T b) \leq p(a - Ta)$ we have

$$p(x - b) \leq p(a - Ta)$$

The inequality $p(b - T b) > p(a - Ta)$ implies that $p(b - T b) < p(b - T b)$ which is a contradiction.

So we have proved that for $x \in B_a$

This proves that $B_b \subseteq B_a$ for every $B_a \in A_1$. Thus $B_b$ is an upper bound in $A$ for the family $A$. By Zorn’s lemma $A$ has a maximal element, say $B_z$, for some $z \in Z$, we claim $z = T z$

Let $w$ be different fixed point. For $w \neq z$ we have that

$$p(z - w) = p(T z - T w) < \max \{p(z - w), p(z - T z), p(w - T w)\} = p(z - w)$$

which is a contradiction.

The proof is completed.

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**References**


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