Existence of Solutions for a Class of $p(x)$-Biharmonic Problems without (A-R) Type Conditions

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Abstract

In this paper, we study the existence and multiplicity of nontrivial solutions for a class of $p(x)$-biharmonic problems. The interesting point lines in the fact that we do not need the usual Ambrosetti-Rabinowitz type condition for the nonlinear term $f$. The proofs are essentially based on the mountain pass theorem and its $\mathbb{Z}_2$ symmetric version.

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1 Introduction and preliminary results

In this paper, we are interested in the existence of nontrivial solutions for a class of $p(x)$-biharmonic problems

$$\begin{cases}
\Delta^2_{p(x)} u = K(x)f(u), & x \in \Omega, \\
u = \Delta u = 0, & x \in \Omega,
\end{cases}$$

(1)

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geq 2$, $p \in C(\bar{\Omega})$ with $\max\{2, \frac{2N}{N+2}\} < p^- := \inf_{x \in \Omega} p(x) \leq p^+ := \sup_{x \in \Omega} p(x)$, and $\Delta^2_{p(x)} u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the operator of fourth order called the $p(x)$-biharmonic operator, $K \in L^\infty(\Omega)$ and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function.

Recently, Navier problems involving the biharmonic operator have been studied by some authors, see [5, 6, 14] and references therein. In [4], A. Ayoujil and A.R. El Amrouss first studied the spectrum of a fourth order elliptic equation with variable exponent. After that, many authors studied the existence of solutions for problems of this type, see for examples [1, 3, 11, 12, 13, 16]. In [3], A. El Amrouss et al. used the mountain pass theorem to study the existence of nontrivial solutions. For this purpose, the authors need the Ambrosetti-Rabinowitz (A-R) type condition (see [2]) to prove the energy functional satisfies the Palais-Smale (PS) condition. In [12, 16], the authors studied the multiplicity of solutions for a class of Navier boundary value problems involving the $p(x)$-biharmonic operator. We also refer the readers to recent papers [1, 11, 13], in which the authors study the existence of eigenvalues of the $p(x)$-biharmonic operator. Motivated by the ideas introduced in [7] and some properties of the $p(x)$-biharmonic operator in [3, 4, 16], we study the existence and multiplicity of nontrivial solutions for a class of $p(x)$-biharmonic problems without (A-R) type conditions. Our result here is different from one introduced in the previous paper [15].

For the reader’s convenience, we recall some necessary background knowledge and propositions concerning the generalized Lebesgue-Sobolev spaces. We refer the reader to the papers [8, 9, 10]. Let $\Omega$ be a bounded domain of $\mathbb{R}^N$, denote $C_+(\bar{\Omega}) := \{p(x); p(x) \in C(\bar{\Omega}), p(x) > 1, \forall x \in \bar{\Omega}\}$, $p^+ := \max_{x \in \Omega} p(x)$, $p^- := \min_{x \in \Omega} p(x)$ and define the space

$L^{p(x)}(\Omega) := \left\{ u : \Omega \to \mathbb{R}; u \text{ is a measurable such that } \int_\Omega |u(x)|^{p(x)} \, dx < +\infty \right\},$

with the norm

$$\|u\|_{L^{p(x)}(\Omega)} = |u|_{p(x)} := \inf \left\{ \mu > 0; \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} \, dx \leq 1 \right\}.$$
Denote

\[ p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N, \end{cases} \]

\[ p_k^*(x) = \begin{cases} \frac{Np(x)}{N-kp(x)} & \text{if } kp(x) < N, \\ +\infty & \text{if } kp(x) \geq N \end{cases} \]

for any \( x \in \overline{\Omega}, k \geq 1 \).

**Proposition 1.1** (see [10]). The space \( (L^{p(x)}(\Omega), |.|_{p(x)}) \) is separable, uniformly convex, reflexive and its conjugate space is \( L^{q(x)}(\Omega) \) where \( q(x) \) is the conjugate function of \( p(x) \), i.e., \( \frac{1}{p(x)} + \frac{1}{q(x)} = 1 \), for all \( x \in \Omega \). For \( u \in L^{p(x)}(\Omega) \) and \( v \in L^{q(x)}(\Omega) \), we have

\[
\left| \int_{\Omega} uv \, dx \right| \leq \left( \frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)} \leq 2|u|_{p(x)} |v|_{q(x)}. \]

The Sobolev space with variable exponent \( W^{k,p(x)}(\Omega) \) is defined as

\[ W^{k,p(x)}(\Omega) = \{ u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq k \}, \]

where \( D^\alpha u = \frac{\partial^{\alpha}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}} u \), with \( \alpha = (\alpha_1, \ldots, \alpha_N) \) is a multi-index and \( |\alpha| = \sum_{i=1}^N \alpha_i \). The space \( W^{k,p(x)}(\Omega) \) equipped with the norm

\[ \| u \|_{k,p(x)} := \sum_{|\alpha| \leq k} |D^\alpha u|_{p(x)}, \]

also becomes a separable and reflexive Banach space. For more details, we refer to [10].

**Proposition 1.2** (see [10]). For \( p, r \in C_+(\overline{\Omega}) \) such that \( r(x) \leq p_k^*(x) \) for all \( x \in \overline{\Omega} \), there is a continuous embedding \( W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega) \). If we replace \leq \) with \(< \), the embedding is compact.

We denote by \( W^{k,p(x)}_0(\Omega) \) the closure of \( C_0^\infty(\Omega) \) in \( W^{k,p(x)}(\Omega) \). Note that the weak solutions of problem (1) are considered in the generalized Sobolev space \( X = W^{2,p(x)}(\Omega) \cap W^{1,p(x)}_0(\Omega) \) equipped with the norm

\[ \| u \| := \inf \left\{ \mu > 0 : \int_{\Omega} \left| \frac{\Delta u(x)}{\mu} \right|^{p(x)} \, dx \leq 1 \right\}. \]

**Proposition 1.3** (see [16]). If \( \Omega \subset \mathbb{R}^N \) is a bounded domain, then the embedding \( X \hookrightarrow C(\overline{\Omega}) \) is compact whenever \( \frac{N}{2} < p^- \).

From Proposition 1.3, there exists a positive constant \( c \) depending on \( p(x), N \) and \( \Omega \) such that

\[ |u|_\infty := \sup_{x \in \overline{\Omega}} |u| \leq c \| u \|, \quad \forall u \in X. \]
Remark 1.4. According to [17], the norm $\| \cdot \|_{2,p(x)}$ is equivalent to the norm $|\Delta|_{p(x)}$ in the space $X$. Consequently, the norms $\| \cdot \|_{2,p(x)}$, $\| \cdot \|$ and $|\Delta|_{p(x)}$ are equivalent.

Proposition 1.5 (see [3]). If we denote $\rho(u) = \int_{\Omega} |\Delta u|^{p(x)} \, dx$, then for $u, u_n \in X$, we have the following assertions:

(i) $\|u\| < 1$ (respectively $= 1$; $> 1$) $\iff$ $\rho(u) < 1$ (respectively $= 1$; $> 1$);
(ii) $\|u\| \leq 1 \Rightarrow \|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$;
(iii) $\|u\| \geq 1 \Rightarrow \|u\|^{p^+} \leq \rho(u) \leq \|u\|^{p^-}$;
(iv) $\|u_n\| \to 0$ (respectively $\to +\infty$) $\iff$ $\rho(u_n) \to 0$ (respectively $\to +\infty$).

Let us define the functional

$$I(u) = \int_{\Omega} \frac{1}{p(x)} |\Delta u|^{p(x)} \, dx.$$ 

It is well known that $J$ is well defined, even and $C^1$ in $X$. Moreover, the operator $L = I' : X \to X^*$ defined as

$$\langle L(u), v \rangle = \int_{\Omega} |\Delta u|^{p(x)-2} \Delta u \Delta v \, dx$$

for all $u, v \in X$ satisfies the following assertions.

Proposition 1.6 (see [3]). (i) $L$ is continuous, bounded and strictly monotone.
(ii) $L$ is a mapping of $(S_+)$ type, namely, $u_n \to u$ and $\limsup_{n \to +\infty} L(u_n)(u_n - u) \leq 0$, implies $u_n \to u$.
(iii) $L$ is a homeomorphism.

In order to establish the existence of infinitely many solutions for the problem (1), we will use the following $\mathbb{Z}_2$ version of the mountain pass theorem in [2].

Proposition 1.7. Let $X$ be an infinite-dimensional Banach space, and let $J \in C^1(X, \mathbb{R})$ be even, satisfy the $(PS)$ condition, and have $J(0)$. Assume that $X = V \oplus Y$, where $V$ is finite dimensional and

(i) There are constants $\rho, \alpha > 0$ such that $\inf_{\partial B_{\rho} \cap Y} J \geq \alpha$;
(ii) For each finite-dimensional subspace $\tilde{X} \subset X$, there is $R = R(\tilde{X})$ such that $J(u) \leq 0$ on $\tilde{X} \setminus B_{R(\tilde{X})}$.

Then the functional $J$ possesses an unbounded sequence of critical values.

2 Main result

In this section, we state and prove the main result of the paper. We shall use $c_i$ to denote general positive constants whose values may be changed from line to line. We first make the definition of weak solutions for (1).
Definition 2.1. We say that $u \in X$ is a weak solution of problem (1) if for all $v \in X$, it holds that

$$
\int_{\Omega} |\Delta u|^{p(x) - 2} \Delta u \Delta v \, dx - \int_{\Omega} K(x) f(u) v \, dx = 0
$$

The main result of the paper can be formulated as follows.

Theorem 2.2. Assume that $K, f$ satisfy the following conditions:

(H1) $K \in L^\infty(\Omega)$ and there exists $k_0 > 0$ such that $K(x) \geq k_0$ for all $x \in \Omega$;

(H2) $f \in C(\mathbb{R}, \mathbb{R})$ and there exist a constant $s_0 \geq 0$ and a decreasing function $\theta(s) \in C(\mathbb{R}\setminus(-s_0, s_0), \mathbb{R})$ such that

$$
0 < (p^+ + \inf_{|s| \geq s_0} \theta(s)) F(s) \leq f(s) s, \quad \forall |s| \geq s_0,
$$

where $\theta(s) > 0$ and $\lim_{|s| \to +\infty} \theta(s) |s| = +\infty$, $\lim_{|s| \to +\infty} \int_{s_0}^{s} \frac{\theta(t)}{t} \, dt = +\infty$,

$$
F(s) = \int_{0}^{s} f(t) \, dt;
$$

(H3) $\lim_{s \to +\infty} \frac{f(s)}{|s|^{p^+-1}} = 0$.

Then problem (1) has a nontrivial weak solution. If further, $f$ is odd, then (1) has infinitely many pairs of weak solutions.

If $\inf_{|s| \geq s_0} \theta(s) > 0$, then it follows from condition (H2) that

$$
0 < (p^+ + \inf_{|s| \geq s_0} \theta(s)) F(s) \leq f(s) s, \quad \forall |s| \geq s_0,
$$

and thus we have the well-known Ambrosetti-Rabinowitz type condition as in [3]. In this paper, we are interested in the case $\inf_{|s| \geq s_0} \theta(s) = 0$. For this reason, we may assume throughout this work that $s_0 \geq 1$ and there is a constant $N_0 > 0$ such that $|\theta(s)| \leq N_0$ for all $s \in \mathbb{R}\setminus(-s_0, s_0)$. Note that the result here is different from one introduced in [15].

Our idea is to prove Theorem 2.2 by using the mountain pass theorem and its $\mathbb{Z}_2$ symmetric version stated in the celebrated paper [2]. For this purpose, we introduce the following energy functional $J : X \to \mathbb{R}$ defined by

$$
J(u) = \int_{\Omega} \frac{1}{p(x)} |\Delta u|^{p(x)} \, dx - \int_{\Omega} K(x) F(u) \, dx,
$$

where $F(s) = \int_{0}^{s} f(t) \, dt$. Then $J$ is of $C^1(X, \mathbb{R})$ and its derivative is given by

$$
J'(u)(v) = \int_{\Omega} |\Delta u|^{p(x) - 2} \Delta u \Delta v \, dx - \int_{\Omega} K(x) f(u) v \, dx
$$

for all $u, v \in X$. Hence, weak solutions of problem (1) are exactly the critical points of the functional $J$. 
Lemma 2.3. There exist positive constants $\rho$ and $\alpha$ such that $J(u) \geq \alpha$ for all $u \in X$ with $\|u\| = \rho$.

Proof. From (2) we have that $|u|_\infty \to 0$ if $\|u\| \to 0$. By the hypothesis (H3), for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(s)| \leq \epsilon |s|^{p-1}, \quad \forall |s| < \delta.$$ 

Hence,

$$|F(s)| \leq \frac{\epsilon}{p^+} |s|^{p^+}, \quad \forall |s| < \delta. \quad (3)$$

Combining (H1) with Proposition 1.5, we deduce for $u \in X$ with $\|u\| < \min\{1, \frac{\delta}{c}\}$ ($c$ is given by (2) that

$$J(u) = \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} \, dx - \int_\Omega K(x) F(u) \, dx$$

$$\geq \frac{1}{p^+} \|u\|^{p^+} - \int_\Omega K(\infty) F(u) \, dx$$

$$\geq \frac{1}{p^+} \|u\|^{p^+} - \frac{\epsilon \|K\|_\infty}{p^+} \int_\Omega |u|^{p^+} \, dx$$

$$\geq \frac{1}{p^+} \|u\|^{p^+} - \frac{\epsilon \|K\|_\infty \mu(\Omega)}{p^+} \|u\|^{p^+}$$

$$\geq \frac{1}{p^+} \|u\|^{p^+} - \frac{\epsilon \|K\|_\infty \mu(\Omega) c^{p^+}}{p^+} \|u\|^{p^+}$$

$$= \left( \frac{1}{p^+} - \frac{\epsilon \|K\|_\infty \mu(\Omega) c^{p^+}}{p^+} \right) \|u\|^{p^+}. \quad (4)$$

From (4), there exist positive constants $\rho$ and $\alpha$ such that $J(u) \geq \alpha$ for all $u \in X$ with $\|u\| = \rho$. \hfill \Box

Let $S = \{w \in X : \|w\| = 1\}$. We note that, for all $w \in S$ and a.e. $x \in \Omega$ we have $|w(x)| \leq L$ for some $L > 0$. There is a number $s_\lambda \in \{s \in \mathbb{R} : |s| \leq |\lambda L|\}$ such that $\theta(s_\lambda) = \min_{s_0 \leq |s| \leq |\lambda L|} \theta(s)$. Then $|\lambda| \geq \frac{|s_\lambda|}{L}$ and $|s_\lambda| \to +\infty$ when $|\lambda| \to +\infty$. When $|s| \geq s_0$ we have

$$0 < (p^+ + \theta(s)) F(s) \leq f(s).$$

Hence,

$$F(s) \geq C_1 |s|^{p^+} \exp \left( \int_{s_0}^{|s|} \frac{\theta(t)}{t} \, dt \right) = C_1 |s|^{p^+} G(|s|), \quad (5)$$

where $G(|s|) = \exp \left( \int_{s_0}^{|s|} \frac{\theta(t)}{t} \, dt \right)$. Then by (H2), it follows that $G(|s|)$ increases when $|s|$ increases and $\lim_{|s| \to +\infty} G(|s|) = +\infty$. 
Lemma 2.4. For any $w \in S$ there exist $\delta_w > 0$ and $\lambda_w > 0$ such that, for all $v \in S \cap B(w, \delta_w)$ and for all $|\lambda| \geq \lambda_w$, we have $J(\lambda v) < 0$, where $B(w, \delta_w) = \{v \in X : \|v - w\| < \delta_w\}$.

Proof. Fix $w \in S$. By $\|w\| = 1$, we know that $\mu(\{x \in \Omega : w(x) \neq 0\}) > 0$ and that there exists a $\lambda_w > s_0$ such that $\mu(\{x \in \Omega : \lambda_w w(x) \geq s_0\}) > 0$, where $\mu$ is the Lebesgue measure. Let $\Omega_w^1 := \{x \in \Omega : \lambda_w w(x) < s_0\}$, $\Omega_w^2 := \{x \in \Omega : \lambda_w w(x) \geq s_0\}$.

Then $\mu(\Omega_w^1) > 0$. When $x \in \Omega_w^1$ we have $|w(x)| \geq \frac{s_0}{\lambda_w}$. Let $\delta_w = \frac{s_0}{2\lambda_w}$. Then, for any $v \in S \cap B(w, \delta_w)$,

$$\|v - w\|_{\infty} \leq L\|v - w\| < \frac{s_0}{2\lambda_w}. $$

Hence, when $x \in \Omega_w^1$, we observe that $v(x) \geq \frac{s_0}{2\lambda_w}$ and

$$|v(x)|^p^+ \geq \left(\frac{s_0}{2\lambda_w}\right)^{p^+} = C_2. \quad (6)$$

When $|\lambda| \geq 2\lambda_w$, one has $|\lambda v| \geq s_0$ in $\Omega_w^2$. By the condition $(H1)$ and (5), (6) we know that

$$|\lambda|^{-p^+} \int_{\Omega_w^2} K(x)F(\lambda v) \, dx \geq C_1 \int_{\Omega_w^2} K(x)|v|^p^+ G(|\lambda v|) \, dx$$

$$\geq C_1 C_2 \int_{\Omega_w^2} K(x)G(|\lambda v|) \, dx$$

$$\geq C_1 C_2 \mu(\Omega_w^2) k_0 G \left(\frac{s_0}{2\lambda_w} |\lambda|\right), \quad (7)$$

since $G(|s|)$ increases when $|s|$ increases and $|\lambda v(x)| \geq \frac{s_0}{2\lambda_w} |\lambda|$. There exists a $C_3 > 0$ such that $F(s) \geq -C_3$ when $|s| \leq s_0$. However, $F(s) > 0$ if $|s| \geq s_0$, so

$$\int_{\Omega_w^1} K(x)F(\lambda v) \, dx \geq \int_{\Omega_w^1 \cap \{x \in \Omega : |\lambda v(x)| \leq s_0\}} K(x)F(\lambda v) \, dx \geq -C_3 \|K\|_{\infty}. $$

Hence, by Proposition 1.5, for any $v \in S \cap B(w, \delta_w)$ and $|\lambda| > 1$, we have

$$J(\lambda v) = \int_{\Omega} \frac{1}{p(x)} |\Delta(\lambda v)|^{p(x)} \, dx - \int_{\Omega} K(x)F(\lambda v) \, dx$$

$$\leq \frac{1}{p} |\lambda|^{p^+} \|v\|^{p^+} - \int_{\Omega_w^1} K(x)F(\lambda v) \, dx - \int_{\Omega_w^2} K(x)F(\lambda v) \, dx$$

$$= |\lambda|^{p^+} \left(\frac{1}{p} - |\lambda|^{-p^+} \int_{\Omega_w^1} K(x)F(\lambda v) \, dx \right) - \int_{\Omega_w^1} K(x)F(\lambda v) \, dx$$
\[ |\lambda|^p \left( \frac{1}{p} - C_1 C_2 \mu(\Omega^2_{w}) k_0 G \left( \frac{s_0}{2\lambda_w} |\lambda| \right) \right) + C_3 \|K\|_\infty. \]  \tag{8}

From (8), \(J(\lambda v) \to -\infty\) uniformly for \(v \in S \cap B(w, \delta_w)\) as \(|\lambda| \to +\infty\). Therefore, there exists a \(\lambda_w > 2\lambda_w\) such that \(J(\lambda v) < 0\) for any \(v \in S \cap B(w, \delta_w)\) and \(|\lambda| \geq \lambda_w\).

\textbf{Lemma 2.5.} The functional \(J\) satisfies the (PS) condition.

\textit{Proof.} Let \(\{u_m\}\) be a (PS) sequence of the functional \(J\), that is,
\[ |J(u_m)| \leq c \quad \text{and} \quad |J'(u_m)(v)| \leq \epsilon_m \|v\| \tag{9} \]
for all \(v \in X\) with \(\epsilon_m \to 0\) as \(m \to \infty\). We shall prove that \(\{u_m\}\) is bounded in \(X\). Indeed, if \(\{u_m\}\) is not bounded, we may assume that \(\|u_m\| \to +\infty\) as \(m \to \infty\). Let \(\{\lambda_m\} \subset \mathbb{R}\) such that \(u_m = \lambda_m w_m, \ w_m \in S\). Then \(|\lambda_m| \to +\infty\) as \(m \to \infty\).

Let us define the sets
\[ \Omega^1_m = \{x \in \Omega: |\lambda_m w_m(x)| \geq L\} \quad \text{and} \quad \Omega^2_m = \{x \in \Omega: |\lambda_m w_m(x)| < L\}. \]
Then we have
\[ -\epsilon_m |\lambda_m| = -\epsilon_m \|u_m\| \leq J'(u_m)(u_m) \]
\[ = \int_{\Omega} |\Delta u_m|^{p(x)} \, dx - \int_{\Omega} K(x) f(u_m) u_m \, dx \]
\[ = \int_{\Omega} |\Delta (\lambda_m w_m)|^{p(x)} \, dx - \int_{\Omega^1_m} K(x) f(\lambda_m w_m) \lambda_m w_m \, dx \]
\[ - \int_{\Omega^2_m} K(x) f(\lambda_m w_m) \lambda_m w_m \, dx, \tag{10} \]
which implies that
\[ \int_{\Omega^1_m} K(x) f(\lambda w_m) \lambda_m w_m \, dx \leq \int_{\Omega} |\Delta (\lambda_m w_m)|^{p(x)} \, dx + \epsilon_m |\lambda_m| \]
\[ - \int_{\Omega^2_m} K(x) f(\lambda_m w_m) \lambda_m w_m \, dx \]
\[ \leq p^+ \int_{\Omega} \frac{1}{p(x)} |\Delta (\lambda_m w_m)|^{p(x)} \, dx + \epsilon_m |\lambda_m| \]
\[ - \int_{\Omega^2_m} K(x) f(\lambda_m w_m) \lambda_m w_m \, dx. \tag{11} \]
We know that
\[ 0 < (p^+ + \theta(s \lambda_m)) F(\lambda_m w_m) \leq f(\lambda_m w_m) \lambda_m w_m \text{ in } \Omega^1_m. \]
Combining this with (11) we then have
\[ J(u_m) = J(\lambda_m w_m) \]
\[ = \int_{\Omega} \frac{1}{p(x)} |\Delta(\lambda_m w_m)|^{p(x)} \, dx - \int_{\Omega_m^2} K(x) F(\lambda_m w_m) \, dx - \int_{\Omega_m^2} K(x) F(\lambda_m w_m) \, dx \]
\[ \geq \int_{\Omega} \frac{1}{p(x)} |\Delta(\lambda_m w_m)|^{p(x)} \, dx - \frac{p^+}{p^+ + \theta(s_{\lambda_m})} \int_{\Omega} \frac{1}{p(x)} |\Delta(\lambda_m w_m)|^{p(x)} \, dx \]
\[ - \int_{\Omega_m^2} K(x) F(\lambda_m w_m) \, dx \]
\[ \geq \int_{\Omega} \frac{1}{p(x)} |\Delta(\lambda_m w_m)|^{p(x)} \, dx - \frac{p^+}{p^+ + \theta(s_{\lambda_m})} \int_{\Omega} \frac{1}{p(x)} |\Delta(\lambda_m w_m)|^{p(x)} \, dx \]
\[ - \int_{\Omega_m^2} K(x) F(\lambda_m w_m) \, dx \]
\[ = \frac{\theta(s_{\lambda_m})}{p^+ + \theta(s_{\lambda_m})} \int_{\Omega} \frac{1}{p(x)} |\Delta(\lambda_m w_m)|^{p(x)} \, dx - \frac{\epsilon_m |\lambda_m|}{p^+ + \theta(s_{\lambda_m})} + \psi(\lambda_m w_m) \]
\[ \geq \frac{\theta(s_{\lambda_m})}{(p^+ + N_0)p^+} |\lambda_m|^{p^+ - 1} - \frac{\epsilon_m |\lambda_m|}{p^+} + \psi(\lambda_m w_m) \]
\[ \geq |\lambda_m| \left( \frac{\theta(s_{\lambda_m})}{(p^+ + N_0)p^+} |\lambda_m|^{p^+ - 1} - \frac{\epsilon_m}{p^+} \right) + \psi(\lambda_m w_m), \quad (12) \]

where
\[ \psi(\lambda_m w_m) = \int_{\Omega_m^2} \left( \frac{1}{p^+ + \theta(s_{\lambda_m})} K(x) f(\lambda_m w_m) \lambda_m w_m - K(x) F(\lambda_m w_m) \right) \, dx. \]

By the condition \((H2)\), the sequence \(\{\psi(\lambda_m w_m)\}\) is bounded from below. On the other hand, we know that \(|\lambda_m| \to +\infty\), and so \(|s_{\lambda_m}| \to +\infty\) as \(m \to +\infty\). By \((H2)\),
\[ \lim_{m \to +\infty} |\lambda_m|^{p^+ - 1} \theta(s_{\lambda_m}) \geq \lim_{m \to +\infty} |s_{\lambda_m}| \theta(s_{\lambda_m}) = +\infty. \]
Hence, \(J(u_m) \to +\infty\), and we obtain the contradiction. Now, \(\{u_m\}\) is bounded in \(X\). Since \(X\) is compactly embedded into \(C(\Omega)\) there exist a function \(u \in X\) and a subsequence still denoted by \(\{u_m\}\) of \(\{u_m\}\) such that it converges strongly towards \(u\) in \(C(\Omega)\). From this and the continuity of \(f\), we then have
\[ \left| \int_{\Omega} K(x) f(u_m) (u_m - u) \, dx \right| \leq |K|_{\infty} \max_{|s| \leq \|u\|_{\infty} + 1} |f(s)| \|u_m - u\|_{\infty} \to 0 \quad (13) \]
when \(m \to +\infty\). Combining (9) and (13) imply that
\[ \int_{\Omega} |\Delta u_m|^{p(x)-2} \Delta u_m (\Delta u_m - \Delta u) \, dx \to 0 \text{ as } m \to +\infty. \quad (14) \]
By Proposition 1.6, the sequence \( \{u_m\} \) converges strongly to \( u \) in \( X \) and the functional \( J \) satisfies the \((PS)\) condition.

\[ \]

\textbf{Proof of Theorem 2.2.} By Lemmas 2.3, 2.4 and 2.5, the functional \( J \) satisfies the conditions of the classical mountain pass theorem due to Ambrosetti and Rabinowitz [2]. Thus, we obtain a nontrivial weak solution of problem (1).

If, further, \( f \) is odd, then \( J \) is even. By Lemma 2.3, the functional \( J \) satisfies Proposition 1.7(i) and the \((PS)\) condition. For any finite-dimensional subspace \( \hat{X} \subset X \), \( S \cap \hat{X} = \{ w \in \hat{X} : \|w\| = 1 \} \) is compact. From Lemma 2.5 and the finite covering theorem, it is easy to verify that \( J \) satisfies condition (ii) of Proposition 1.7. Therefore, \( J \) has a sequence of critical points \( \{u_m\} \). That is, problem (1) has infinitely many pairs of solutions.

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\textbf{References}


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