On Intuitionistic Fuzzy $\beta$ and $\beta^*$-Normal Spaces

AbdulGawad A. Q. Al-Qubati

Department of Mathematics, Faculty of Sciences and Arts
Najran University, Saudi Arabia

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Abstract

The aim of this paper is to study the class of intuitionistic fuzzy $\beta$-normal spaces (IF$\pi\beta$) with studying the forms of intuitionistic fuzzy continuous functions, IF$\beta^*$-normal spaces with studying the forms of intuitionistic fuzzy generalized $\beta^*$-normal spaces. Also we study the class of intuitionistic fuzzy $\pi g\beta^*$-normal, and $\pi g\beta^*$-normality in subspaces. Moreover, We investigate some of their properties.

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Keywords: intuitionistic fuzzy $\beta$ open set, intuitionistic fuzzy $\beta$ normal space, intuitionistic fuzzy $\beta^*$-normal spaces

1 Introduction

Fuzzy set as proposed by Zadeh [23] in 1965 is a framework to encounter uncertainty, vagueness and partial truth and it represents a degree of membership for each member of the universe of discourse to a subset of it. By adding the degree of non-membership to Fuzzy set, Atanassov [5] proposed intuitionistic fuzzy set in 1986 which looks more accurate to uncertainty quantification and provides the opportunity to precisely model the problem based on the existing knowledge and observations. In 1997, Coker [11] defined the intuitionistic fuzzy topological space using intuitionistic fuzzy sets. Al-Qubati and Al-Qahtani [4]
introduced and studied new types of b-separation axioms (b $T_i$ - space, for $i = 0, 1, 2$ ) in intuitionistic fuzzy topological spaces, many authors investigated the separation axioms in fuzzy topological spaces and intuitionistic fuzzy topological spaces for more details see [3,6,7,8,9,15,16,19]. In a topological space, several new classes of subsets can be obtained by repeatedly using interior and closure operators. In 1983 Abd El-Monsef and et.al. [1,2] introduced $\beta$-open sets and $\beta$-continuous mappings. In 2014, Jayanthi [13] defined generalized $\beta$-closed sets in intuitionistic fuzzy topological spaces. In 2010, Tahiliani [22] introduced the notion of $\pi g\beta$-closed sets and Jenitha and at.al.[14] extended this notion to intuitionistic fuzzy $\pi g\beta$-closed sets. In 2013, palanimani and Parimalazhagan [18] introduced the notion of $\beta^*$-closed sets in a topological spaces.

In this paper, we study the classes of normal spaces, namely $\beta$-normal spaces, $\beta^*$-normal spaces, $\beta^*$ generalized normal spaces and $\pi$ generalized $\beta^*$-normal spaces in intuitionistic fuzzy topological spaces, we obtain some properties of these form in intuitionistic fuzzy topological spaces. Moreover, we study the forms of intuitionistic fuzzy $\pi$ generalized $\beta^*$-normality in subspaces, and investigate some of their properties and characterizations.

2 Preliminaries

Throughout this paper by $(X, \tau)$ or simply by $X$ we mean an intuitionistic fuzzy topological space (Ifts, Shorty).

**Definition 2.1**[5] Let $X$ be a non empty fixed set. An intuitionistic fuzzy set $A$ ($IFS$ for short) in $X$ is an object having the form $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$ where the function $\mu_A : X \to I$ and $\gamma_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\gamma_A(x)$) of each element $x \in X$ to the set $A$, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$, for each $x \in X$.

**Definition 2.2**[5] Let $X$ be anon empty set and the form $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\} \ B = \{(x, \mu_B(x), \gamma_B(x)) : x \in X\}$.

Then

(a) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$.
(b) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$
(c) $A = \{(x, \gamma_A(x), \mu_A(x)) : x \in X\}$.
(d) $A \cap B = \{(x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x)) : x \in X\}$
(e) $A \cup B = \{(x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x)) : x \in X\}$
(f) $0_X = \{(x, , 0, 1) : x \in X\}$ and $1_X = \{(x, 1, 0) : x \in X\}$.
Definition 2.3 [5] Let \( \{A_i : i \in I\} \) be an arbitrary family of IFS in \( X \). Then

(a) \( \bigcap A_i = \{\langle x, \land \mu_{A_i}(x), \lor \gamma_{A_i}(x) : x \in X\} \}

(b) \( \bigcup A_i = \{x, \lor \mu_{A_i}(x), \land \gamma_{A_i}(x) : x \in X\} \}

Definition 2.4[11]. An intuitionistic fuzzy topology (IFT, in short) on a nonempty set \( X \) is a family \( \tau \) of intuitionistic fuzzy sets in \( X \) satisfy the following axioms:

(T1) \( 0_X, 1_X \in \tau \).

(T2) If \( A_1, A_2 \in \tau \), then \( A_1 \cap A_2 \in \tau \).

(T3) If \( A_\lambda \in \tau \) for each \( \lambda \in \Lambda \), then \( \bigcup_{\lambda \in \Lambda} A_\lambda \in \tau \).

In this case the pair \((X, \tau)\) is called an intuitionistic fuzzy topological space (IFTS for short) and each intuitionistic fuzzy set in \( \tau \) is known as an intuitionistic fuzzy open set (IFOS for short) of \( X \). The complement \( A^c \) of an IFOS \( A \) in IFTS \((X, \tau)\) is an intuitionistic fuzzy closed set (IFCS in short) in \( X \).

Definition 2.5. An IFS \( A \) of an IFTS \((X, \tau)\) is an

(i) intuitionistic fuzzy regular-open set (IFros in short) if \( A = int(cl(A)) \) [12]

(ii) intuitionistic fuzzy \( \beta \)-open (resp. \( \beta \)-closed) if \( A \subseteq (cl(int(cl(A)))) \) (resp. \( int(cl(int(A)))) \subseteq A \) [12]

(iii) intuitionistic fuzzy semi-open if \( A \subseteq cl(int(A)) \) [12]

(iv) intuitionistic fuzzy pre-open if \( A \subseteq (int(cl(A))) \) [12]

(v) intuitionistic fuzzy generalized closed (briefly IFgc) if \( cl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is an intuitionistic fuzzy open. [20]

(vi) intuitionistic fuzzy \( \pi g \beta \)-closed set (IF\( \pi g \beta cs \) for short) if \( \beta cl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is an IFS in \((X, \tau)\).

(vii) intuitionistic fuzzy \( \beta^* \)-closed set if \( cl(int(A)) \subseteq U \), whenever \( A \subseteq U \) and \( U \) is an intuitionistic fuzzy g- open.
(viii) intuitionistic fuzzy $\beta^*$ - open set if $F \subseteq int(cl(A))$, whenever $F \subseteq A$ and F is an intuitionistic fuzzy g- closed.

(ix) intuitionistic fuzzy $\beta^*$ generalized - closed set if $cl(int(cl((A))) \subseteq U$ whenever $A \subseteq U$ and U is an intuitionistic fuzzy - open in $(X, \tau)$.

(x) intuitionistic fuzzy $\pi$ generalized $\beta^*$ -closed set (IF$\pi$g$\beta^*$cs for short) if $\beta^* cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an IF$\pi$OS in $(X, \tau)$.

Definition 2.6[13] Let A any intuitionistic fuzzy set. Then,

$$IF\beta cl(A) = \bigcap \{F : A \subseteq F, F is IF\beta CS in X\}$$ is called intuitionistic fuzzy $\beta$-closure.

$$IF\beta int(A) = \bigcup \{U : U \subseteq A, U is IF\beta OS in X\}$$ is called intuitionistic fuzzy $\beta$- interior.

3 Main Results

These are the main results of the paper.

3. Intuitionistic Fuzzy $\beta$-Normal

Definition 3.1 An intuitionistic fuzzy topological space $(X, \tau)$ is said to be

-intuitionistic fuzzy $\beta$ -normal space (or in short IF$\beta$ - N) if for every pair of disjoint intuitionistic fuzzy closed sets $A$ and $B$, there exist two disjoint intuitionistic fuzzy $\beta$ open sets (IF$\beta$OSs)$U$ and $V$ such that $A \subseteq U, B \subseteq V$.

Theorem 3.2 Let $(X, \tau)$ be an intuitionistic fuzzy topological space the following are equivalent :

(1) $X$ is an intuitionistic fuzzy $\beta$ normal space.

(2) For every pair of intuitionistic fuzzy open sets $U$ and $V$ whose union is $1_X$, there exist intuitionistic fuzzy $\beta$-closed sets $A$ and $B$ such that $A \subseteq U$, $B \subseteq V$ and $A \cup B = 1_X$.

(3) For every intuitionistic fuzzy closed set $H$ and every intuitionistic fuzzy open set $K$ containing $H$, there exists an intuitionistic fuzzy $\beta$- open set $U$ such that $H \subseteq U \subseteq \beta - cl(U) \subseteq K$. 
(4) For every pair of intuitionistic fuzzy disjoint $\beta$ closed sets $H$ and $K$ of $X$ there exists an intuitionistic fuzzy $\beta$-open set $U$ of $X$ such that $H \subseteq U$ and $IF\beta - cl(U) \cap K = 0_X$.

(5) For every pair of intuitionistic fuzzy disjoint $\beta$ closed sets $H$ and $K$ of $X$ there exists an intuitionistic fuzzy $\beta$-open sets $U$ and $V$ of $X$ such that $H \subseteq U, K \subseteq V$ and $IF\beta - cl(U) \cap IF\beta - cl(V) = 0_X$.

Proof

(1) $\Rightarrow$ (2) Let $U$ and $V$ be two intuitionistic fuzzy open sets in an $\beta$-normal space $X$ such that $U \cup V = 1_X$. Then $U^c, V^c$ are intuitionistic fuzzy disjoint closed sets. Since $X$ is an intuitionistic fuzzy $\beta$-normal space there exist intuitionistic fuzzy disjoint $\beta$-open sets $U_1$ and $V_1$ such that $U^c \subseteq U_1$ and $V^c \subseteq V_1$. Let $A = U_1^c, B = V_1^c$. Then $A$ and $B$ are intuitionistic fuzzy $\beta$-closed sets such that $A \subseteq U, B \subseteq V$ and $A \cup B = 1_X$.

(2) $\Rightarrow$ (3) Let $H$ be intuitionistic fuzzy closed set and $K$ be an intuitionistic fuzzy open set containing $H$. Then $H^c$ and $K$ are intuitionistic fuzzy open sets such that $H^c \cup K = 1_X$. Then by (2) there exist an intuitionistic fuzzy $\beta$-closed sets $M_1$ and $M_2$ such that $M_1 \subseteq H^c$ and $M_2 \subseteq K$ and $M_1 \cup M_2 = 1_X$. Thus, we obtain $H \subseteq M_1^c, K^c \subseteq M_2^c$ and $M_1^c \cap M_2^c = 0_X$. Let $U = M_1^c$ and $V = M_2^c$. Then $U$ and $V$ are intuitionistic fuzzy disjoint $\beta$-open sets such that $H \subseteq U \subseteq V^c \subseteq K$. As $V^c$ an intuitionistic fuzzy $\beta$-closed set, we have $H \subseteq U \subseteq \beta - cl(U) \subseteq K$.

(3) $\Rightarrow$ (4). Let $H$ and $K$ be disjoint IF$\beta$-closed set of $X$. Then $H \subseteq K^c$ where $K^c$ is IF$\beta$-open. By the part (c), there exists a IF-open subset $U$ of $X$ such that $H \subseteq U \subseteq \beta - cl(U) \subseteq K^c$. Thus, $IF\beta cl(U) \cap K = 0_X$.

(4) $\Rightarrow$ (5). Let $H$ and $K$ be any disjoint IF$\beta$-closed set of $X$. Then by the part (4), there exists a IF IF$\beta$-open set $U$ containing $H$ such that $IF\beta cl(U) \cap K = 0_X$. Since $IF\beta cl(U)$ is an IF$\beta$-closed, then it is IFIF$\beta$-closed. Thus $IF\beta cl(U)$ and $K$ are disjoint IF$\beta$-closed sets of $X$. Again by the part (d), there exists a IF$\beta$-open set $V$ in $X$ such that $K \subseteq V$ and $IF\beta cl(U) \cap IF\beta cl(V) = 0_X$.

(5) $\Rightarrow$ (1) Let $H$ and $K$ be any disjoint IF$\beta$-closed sets of $X$. Then by the part (e), there exist IFIF$\beta$-open sets $U$ and $V$ such that $H \subseteq U, K \subseteq V$ and $IF\beta cl(U) \cap IF\beta cl(V) = 0_X$. Therefore, we obtain that $U \cap V = 0_X$. Hence $X$ is IF$\beta$-normal.
Definition 3.3. An IF function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be:

(i) intuitionistic fuzzy pre-continuous function if \( f^{-1}(B) \in \text{IFPO}(X) \) for every \( B \in \sigma \).[13]

(ii) intuitionistic fuzzy \( \beta \)-continuous function if \( f^{-1}(B) \in \text{IF}\beta O(X) \) for every \( B \in \sigma \).[13]

(iii) intuitionistic fuzzy \( \beta \) open function (IF\( \beta \)O function for short) if \( f(A) \) is an IF\( \beta \)OS in \( Y \) for each IFOS \( A \) in \( X \).[12]

Definition 3.4. A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is called

(1) IF pre \( \beta \)-open if \( f(U) \in \text{IF}\beta O(Y) \) for each \( U \in \text{IF}\beta O(X) \).[17]

(2) IF pre \( \beta \)-closed if \( f(U) \in \text{IF}\beta C(Y) \) for each \( U \in \text{IF}\beta C(X) \).[17].

(3) IF almost \( \beta \)-irresolute if for each IF point \((x(\alpha, \beta)) \) in \( X \) and each IF\( \beta \)-neighbourhood \( V \) of \( f(x) \), \( \text{\( \beta - cl(f^{-1}(V)) \}) \) is an IF\( \beta \)-neighbourhood of \( x(\alpha, \beta) \).

Theorem 3.5 A surjective function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is an intuitionistic fuzzy pre-\( \beta \)-open continuous almost \( \beta \) irresolute function from an intuitionistic fuzzy \( \beta \) normal space \((X, \tau) \) onto \((Y, \sigma) \). Then \((Y, \sigma) \) is an intuitionistic fuzzy \( \beta \) -normal space.

proof. Let \( A \) be an intuitionistic fuzzy closed set of \( Y \) and \( B \) an intuitionistic fuzzy open set of \( Y \) containing \( A \). Then since \( f \) is continuous \( f^{-1}(A) \) and \( f^{-1}(B) \) are intuitionistic fuzzy closed (resp. open) in \( X \) such that \( f^{-1}(A) \subseteq f^{-1}B \). Since \( X \) is an intuitionistic fuzzy \( \beta \)-normal there exits an intuitionistic fuzzy \( \beta \) open set \( U \) in \( X \) such that \( f^{-1}(A) \subseteq U \subseteq \beta - cl(U) \subseteq f^{-1}(B) \). by Theorem (3.2) \( (f^{-1}(A)) \subseteq f(U) \subseteq f(\beta - cl(U)) \subseteq f(f^{-1}(B)). \) Since \( f \) is an intuitionisyc fuzzy pre \( \beta \)-open almost \( \beta \) irresolute- surjection function,we obtain \( A \subseteq f(U) \subseteq \beta - cl(f(U)) \subseteq B \). Then again by Theorem (3.2). The space \((Y, \sigma) \) is intuitionistic fuzzy \( \beta \) normal space.

Theorem 3.6 A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is an intuitionistic fuzzy pre-\( \beta \)-closed function if and only if for each intuitionistic fuzzy set \( A \) in \( Y \) and for each intuitionistic fuzzy \( \beta \) open set \( U \) in \( X \) containing \( f^{-1}(A) \) there exist an intuitionistic fuzzy \( \beta \) open set \( V \) of \( Y \) containing \( A \) such that \( f^{-1}(V) \subseteq U \).

Theorem 3.7 Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be an intuitionistic fuzzy pre-\( \beta \)-closed continuous function from an intuitionistic fuzzy \( \beta \) normal space \( X \) onto
a space $Y$, then $Y$ is an intuitionistic fuzzy $\beta$ normal space.

**Proof.** Let $M_1$ and $M_2$ are intuitionistic fuzzy disjoint closed sets in $Y$. $f^{-1}(M_1)$ and $f^{-1}(M_2)$ are intuitionistic fuzzy closed sets in $X$. Since $X$ is an intuitionistic fuzzy $\beta$-normal space, there exist disjoint intuitionistic fuzzy $\beta$ open sets $U$ and $V$ such that $f^{-1}(M_1) \subseteq U$ and $f^{-1}(M_2) \subseteq V$. By Theorem 3.6 there exist an intuitionistic fuzzy $\beta$ open sets $A$ and $B$ such that $M_1 \subseteq A$ and $M_2 \subseteq B$, $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Also $A$ and $B$ are disjoint. Thus $Y$ is an intuitionistic fuzzy $\beta$ normal space.

**Definition 3.8.** An intuitionistic fuzzy function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called $\alpha$-closed if $f(U)$ is an IF$\alpha$-closed set in $Y$ for each closed set $U$ in $X$.

**Theorem 3.9** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be an intuitionistic fuzzy $\alpha$ closed continuous surjection and $X$ is an intuitionistic fuzzy normal, then $Y$ is $\beta$ normal space.

**Proof.** Let $A$ and $B$ be an intuitionistic fuzzy disjoint closed sets in $Y$. Since $f$ is continuous then $f^{-1}(A)$ and $f^{-1}(B)$ are intuitionistic fuzzy disjoint closed sets in $X$. As $X$ is an intuitionistic fuzzy normal, there exist intuitionistic fuzzy disjoint open sets $U$ and $V$ in $X$ such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Then there are intuitionistic fuzzy disjoint open sets $G$ and $H$ in $Y$ such that $A \subseteq G$ and $B \subseteq H$. Since every intuitionistic fuzzy $\alpha$ open set is $\beta$ open, $G$ and $H$ are intuitionistic fuzzy disjoint $\beta$ open sets containing $A$ and $B$, respectively. Therefore $Y$ is an intuitionistic fuzzy $\beta$ normal.

4. Intuitionistic Fuzzy $\beta^*$-Normal

**Definition 4.1** An intuitionistic fuzzy topological space $(X, \tau)$ is said to be $\beta^*$-normal space (or in short IF$\beta^*$ - N) if for every pair of disjoint intuitionistic fuzzy closed sets $A$ and $B$, there exist two disjoint intuitionistic fuzzy $\beta^*$ open sets(IF$\beta^*$ OSs)$U$ and $V$ such that $A \subseteq U, B \subseteq V$.

**Theorem 4.2** Let $(X, \tau)$ be an intuitionistic fuzzy topological space the following are equivalent:

1. $X$ is an intuitionistic fuzzy $\beta^*$ normal.

2. For every pair of intuitionistic fuzzy open sets $U$ and $V$ whose union is $1_X$, there exist intuitionistic fuzzy $\beta^*$-closed sets $A$ and $B$ such that $A \subseteq U$, $B \subseteq V$ and $A \cup B = 1_X$. 
(3) For every intuitionistic fuzzy closed set $H$ and every intuitionistic fuzzy open set $K$ containing $H$, there exists an intuitionistic fuzzy $\beta^*$-open set $U$ such that $H \subseteq U \subseteq \beta^* - \text{cl}(U) \subseteq K$.

**Proof**

(1) $\Rightarrow$ (2) Let $U$ and $V$ be two intuitionistic fuzzy open sets in an $\beta^*$-normal space $X$ such that $A \cup B = 1_X^c$. Then $U^c, V^c$ are intuitionistic fuzzy disjoint closed sets. Since $X$ is an intuitionistic fuzzy $\beta^*$-normal space there exist disjoint intuitionistic fuzzy $\beta^*$-open sets $U_1$ and $V_1$ such that $U_1^c \subseteq U$ and $V_1^c \subseteq V$. Let $A = U_1^c, B = V_1^c$. Then $A$ and $B$ are intuitionistic fuzzy $\beta^*$-closed sets such that $A \subseteq U, B \subseteq V$ and $A \cup B = 1_X^c$.

(2) $\Rightarrow$ (3) Let $H$ be an intuitionistic fuzzy closed set and $K$ be an intuitionistic open set containing $H$. Then $H^c$ and $K$ are intuitionistic fuzzy open sets such that $H^c \cup K = 1_X^c$. Then by (2) there exist an intuitionistic fuzzy $\beta^*$-closed sets $M_1$ and $M_2$ such that $M_1 \subseteq H^c$ and $M_2 \subseteq K$ and $M_1 \cup M_2 = 1_X^c$, then $H \subseteq M_1^c, K \subseteq M_2^c$ and $M_1^c \cap M_2^c = 0_X^c$. Let $U = M_1^c$ and $V = M_2^c$. Then $U$ and $V$ are intuitionistic fuzzy disjoint $\beta^*$-open sets such that $H \subseteq U \subseteq V^c \subseteq K$. As $V^c$ an intuitionistic fuzzy $\beta^*$-closed set, we have $H \subseteq U \subseteq \beta^* - \text{cl}(U) \subseteq K$.

(3) $\Rightarrow$ (1) Let $H_1$ and $H_2$ be two intuitionistic fuzzy closed set of $X$. Let $k = H_2^c$, then $H_2 \cap K = 0_X^c$. $H_1 \subseteq K$ where $K$ is an intuitionistic fuzzy open set. Then by (3) there exist an intuitionistic fuzzy $\beta^*$ open set $U$ in $X$ such that $H \subseteq U \subseteq \beta - \text{cl}(U) \subseteq K$. It follows that $H_1 \subseteq \beta \text{cl}(U) = V$, say, then $V$ is an intuitionistic fuzzy $\beta^*$ open set and $U \cap V = 0_X^c$. Hence $H_1$ and $H_2$ are separated by an intuitionistic fuzzy $\beta^*$ open sets $U$ and $V$. Therefore $X$ is intuitionistic fuzzy $\beta^*$ normal space.

**Definition 4.3** An intuitionistic fuzzy topological space $(X, \tau)$ is said to be

-intuitionistic fuzzy $\beta^*$\textit{generalized -normal} (or in short IF$\beta^*$g - N) if for every pair of disjoint intuitionistic fuzzy $\beta^*$\textit{g} - closed sets $A$ and $B$, there exist two disjoint intuitionistic fuzzy $\beta^*$ open sets(IF$\beta^*$OSs) $U$ and $V$ such that $A \subseteq U, B \subseteq V$.

**Theorem 4.4.** For an intuitionistic fuzzy topological space $(X, \tau)$, the following are equivalent:

(1) $X$ is an IF $\beta^*$g -normal
(2) For any pair of intuitionistic fuzzy disjoint $\beta^*$g open sets $U$ and $V$ of $X$
whose union is $1^\sim_X$, there exist disjoint IF $\beta^*$-closed sets $A$ and $B$ of $X$ such that $A \subseteq U$ and $B \subseteq V$ and $A \cup B = 1^\sim_X$.

(3) For each IF $\beta^*g$-closed set $H$ and an IF $\beta^*g$-open set $K$ containing $H$, there exists an IF $\beta^*$-open set $U$ such that $H \subseteq U \subseteq IF\beta^* - cl(U) \subseteq K$.

(4) For any pair of $\beta^*g$-closed sets $H$ and $K$ of $X$ there exists a IF $\beta^*$-open set $U$ of $X$ such that $H \subseteq U$ and $IF\beta^* - cl(U) \cap K = 0^\sim_X$.

(5) For any pair of disjoint IF $\beta^*g$-closed sets $H$ and $K$ of $X$ there exists an IF $\beta^*$-open sets $U$ and $V$ of $X$ such that $H \subseteq U, K \subseteq V$ and $IF\beta^* - cl(U) \cap IF\beta^* - cl(V) = 0^\sim_X$.

**Proof.**

(1) $\Rightarrow$ (2) Let $U$ and $V$ be two intuitionistic fuzzy $\beta^*g$ open sets in an IF $\beta^*$-normal space $X$ such that $A \cup B = 1^\sim_X$. Then $U^c, V^c$ are intuitionistic fuzzy disjoint $\beta^*g$ closed sets. Since $X$ is an intuitionistic fuzzy $\beta^*$-normal space there exist intuitionistic fuzzy disjoint $\beta^*$-open sets $U_1$ and $V_1$ such that $U^c \subseteq U_1$ and $V^c \subseteq V_1$. Let $A = U_1^c, B = V_1^c$. Then $A$ and $B$ are intuitionistic fuzzy $\beta^*$-closed sets $A \subseteq U, B \subseteq V$ and $A \cup B = 1^\sim_X$.

(2) $\Rightarrow$ (3) Let $H$ be intuitionistic fuzzy $\beta^*g$ closed set and $K$ be an intuitionistic fuzzy $\beta^*g$ open set containing $H$. Then $H^c$ and $K$ are intuitionistic fuzzy $\beta^*g$ open sets such that $H^c \cap K = 1^\sim_X$. By (2) there exist an intuitionistic fuzzy $\beta^*$-closed sets $M_1$ and $M_2$ such that $M_1 \subseteq H^c$ and $M_2 \subseteq K$ and $M_1 \cup M_2 = 1^\sim_X$. Thus, we obtain $H \subseteq M_1^c, K^c \subseteq M_2^c$ and $M_1^c \cap M_2^c = 0^\sim_X$. Let $U = M_1^c$ and $V = M_2^c$. Then $U$ and $V$ are intuitionistic fuzzy disjoint $\beta^*$-open sets such that $H \subseteq U \subseteq V^c \subseteq K$. As $V^c$ an intuitionistic fuzzy $\beta^*$-closed set, we have $H \subseteq U \subseteq \beta^* - cl(U) \subseteq K$.

(3) $\Rightarrow$ (4). Let $H$ and $K$ be disjoint IF $\beta^*g$-closed set of $X$. Then $H \subseteq K^c$ where $K^c$ is IF $\beta^*g$-open. By the part (c), there exists a IF $\beta^*$-open subset $U$ of $X$ such that $H \subseteq U \subseteq \beta^* - cl(U) \subseteq K^c$. Thus, $IF\beta^* - cl(U) \cap K = 0^\sim_X$.

(4) $\Rightarrow$ (5). Let $H$ and $K$ be any disjoint IF $\beta^*g$-closed subset of $X$. Then by the part (4), there exists a IF $\beta^*$-open set $U$ containing $H$ such that $IF\beta^* - cl(U) \cap K = 0^\sim_X$. Since $IF\beta^* - cl(U)$ is an IF $\beta^*g$-closed, then it is IF $\beta^*g$-closed Thus $IF\beta^* - cl(U)$ and $K$ are disjoint IF $\beta^*$-closed sets of $X$. Again by the part (d), there exists a IF $\beta^*$-open set $V$ in $X$ such that $K \subseteq V$ and $IF\beta^*c1(U) \cap IF\beta^*c1(V) = 0^\sim_X$.

(5) $\Rightarrow$ (1) Let $H$ and $K$ be any disjoint IF $\beta^*g$-closed sets of $X$. Then by the part (e), there exist IF $\beta^*$-open sets $U$ and $V$ such that $H \subseteq U, K \subseteq V$ and $IF\beta^*c1(U) \cap IF\beta^*c1(V) = 0^\sim_X$. Therefore, we obtain that $U \cap V = 0^\sim_X$. Hence $X$ is IF $\beta^*g$-normal.
5. **Intuitionistic Fuzzy π Generalized β*-Normal Spaces.**

In this section, we introduce the notion of IFπgβ*-normal space and study some of its properties.

**Definition 5.1** An IF topological space $X$ is said to be IFπgβ*-normal if for every pair of disjoint IFπgβ*-closed subsets $A$ and $B$ of $X$, there exist disjoint IFβ*-open sets $U, V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$.

**Theorem 5.2.** For an intuitionistic fuzzy topological space $(X, \tau)$, the following are equivalent:

1. $X$ is πgβ*-normal
2. for any pair of intuitionistic fuzzy disjoint πgβ*-open sets $U$ and $V$ of $X$, there exist disjoint gβ*-closed sets $A$ and $B$ of $X$ such that $A \subseteq U$ and $B \subseteq V$ and $U \cup V = X$.
3. for each IFπgβ*-closed set $A$ and an IFπgβ*-open set $B$ containing $A$, there exists an IFβ*-open set $U$ such that $A \subseteq U \subseteq IFβ* - cl(U) \subseteq B$.
4. for any pair of intuitionistic fuzzy disjoint πgβ*-closed sets $A$ and $B$ of $X$ there exists a IFβ*-open set $U$ of $X$ such that $A \subseteq U$ and $IFβ* - cl(U) \cap B = 0_X$.
5. for any pair of intuitionistic fuzzy disjoint πgβ*-closed sets $A$ and $B$ of $X$ there exists a IFβ*-open sets $U$ and $V$ of $X$ such that $A \subseteq U, B \subseteq V$ and $IFβ* - cl(U) \cap IFβ* - cl(V) = 0_X$.

**Proof.** (1) $\Rightarrow$ (2) Let $U$ and $V$ be two intuitionistic fuzzy πgβ*-open sets in an IF πgβ*-normal space $X$ such that $A \cup B = 1_X$. Then $U^c, V^c$ are intuitionistic fuzzy disjoint πgβ*-closed sets. Since $X$ is an intuitionistic fuzzy πgβ*-normal space there exist intuitionistic fuzzy disjoint β*-open sets $U_1$ and $V_1$ such that $U^c \subseteq U_1$ and $V^c \subseteq V_1$. Let $A = U_1^c, B = V_1^c$. Then $A$ and $B$ are intuitionistic fuzzy β*-closed sets $A \subseteq U, B \subseteq V$ and $A \cup B = 1_X$.

(2) $\Rightarrow$ (3) Let $H$ be intuitionistic fuzzy πgβ*-closed set and $K$ be an intuitionistic fuzzy πgβ*-open set containing $H$. Then $H^c$ and $K$ are intuitionistic fuzzy πgβ*-open sets such that $H^c \cup K = 1_X$. Then by (2) there exist an intuitionistic fuzzy β*-closed sets $M_1$ and $M_2$ such that $M_1 \subseteq H^c$ and $M_2 \subseteq K$ and $M_1 \cup M_2 = 1_X$. Thus, we obtain $H \subseteq M_1^c, K \subseteq M_2^c$ and $M_1^c \cap M_2^c = 0_X$. Let $U = M_1^c$ and $V = M_2^c$. Then $U$ and $V$ are intuitionistic fuzzy disjoint β*-open sets such that $H \subseteq U \subseteq V^c \subseteq K$. As $V^c$ an intuitionistic fuzzy β*-closed set, we have $H \subseteq U \subseteq \beta* - cl(U) \subseteq K$.

(3) $\Rightarrow$ (4). Let $H$ and $K$ be disjoint IFπgβ*-closed set of $X$. Then $H \subseteq K^c$ where $K^c$ is IFβ*-open. By the part (c), there exists a IFβ*-open subset $U$ of
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X such that $H \subseteq U \subseteq \beta^* - cl(U) \subseteq K^c$. Thus, $IF\beta^* - c1(U) \cap K = 0_X$.

(4) $\Rightarrow$ (5). Let H and K be any disjoint IF$\pi g\beta^*$ -closed subset of X. Then by the part (4), there exists a IF$\beta^*$-open set U containing H such that $IF\beta^* - c1(U) \cap K = 0_X$. Since $IF\beta^* - c1(U)$ is an IF$\pi g\beta^*$ - closed, then it is IF$\pi g\beta^*$ -closed Thus $IF\beta^* - c1(U)$ and K are disjoint IF$\pi g\beta^*$-closed sets of X. Again by the part (d), there exists a $IF\beta^* - c1(U)$ in X such that $K \subseteq V$ and $IF\beta^* c1(U) \cap IF\beta^* c1(V) = 0_X$.

(5) $\Rightarrow$ (1) Let H and K be any disjoint IF$\pi g\beta^*$ -closed sets of X. Then by the part (e), there exist IF$\beta^*$-open sets U and V such that $H \subseteq U, K \subseteq V$. and $IF\beta^* c1(U) \cap IF\beta^* c1(V) = 0_X$. Therefore, we obtain that $U \cap V = 0_X$. Hence X is IF$\pi g\beta^*$ -normal.

**Definition 5.3.** An IF function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be an IF $\pi g\beta^*$ $\alpha$ -continuous function if $f^{-1}(F)$ is F $\pi g\beta^*$ $\alpha$ - closed set in in X for every IF $\alpha$ closed set $F$ in Y.

**Lemma 5.4** (a) The image of IF $\beta^*$-open subset under an IF-open continuous function is IF$\beta^*$-open.

(b) The inverse image of IF$\beta^*$-open subset under an open continuous function is IF$\beta^*$-open subset.

**Lemma 5.5** The image of IF regular open subset under open and closed continuous function is IF regular open subset.

**Lemma 5.6** The image of a IF$\beta^*$-open subset under IF-open and IF-closed continuous function is IF$\beta^*$-open.

**Theorem 5.7** If $f : X \rightarrow Y$ be an IF-open and IF-closed continuous bijection function and A be a IF $\pi g\beta^*$ -closed set in Y, then $f^{-1}(A)$ is IF $\pi g\beta^*$-closed set in X.

**Proof**. Let A be an $\pi g\beta^*$ -closed set in Y and U be any IF$\pi$ -open set of X such that $f^{-1}(A) \subseteq U$. Then by Lemma 5.6, we have $f(U)$ is IF$\pi$ open set of Y such that $A \subseteq f(U)$. Since A is an IF $\pi g\beta^*$ -closed set of Y and $f(U)$ is IF$\pi$- open set in Y. Thus IF$\beta^* cl(A) \subseteq U$. By Lemma 5.4 we obtain that $f^{-1}(A) \subseteq f^{-1}(IF\beta^* - c1(A)) \subseteq U$, where $f^{-1}(IF\beta^* c1(A))$ is $\beta^*$- closed in X. This implies that $IF\beta^* cl(f^{-1}(A)) \subseteq U$. Therefore $f^{-1}(A)$ is IF $\pi g\beta^*$-closed set in X.
Theorem 5.8 If \( f : X \to Y \) be an IF-open and IF-closed continuous bi-
jection function and \( X \) be a IF \( \pi g \beta^* \)-normal space , then \( Y \) is IF \( \pi g \beta^* \)-normal space .

Proof. Let \( A \) and \( B \) be any disjoint \( \pi g \beta^* \)-closed set in \( Y \). Then by The-
orem 5.7, \( f^{-1}(A) \) and \( f^{-1}(B) \) are disjoint of IF \( \pi g \beta^* \)-closed set in \( X \). By IF \( \pi g \beta^* \) - normality of \( X \), there exist IF \( \beta^* \)-open subsets \( U \) and \( V \) of \( X \) such
that \( f^{-1}(A) \subseteq U \), \( f^{-1}(B) \subseteq V \) and \( U \cap V = \emptyset_X \). By assumption, we have
\( A \subseteq f(U) \), \( B \subseteq f(V) \) and \( f(U) \cap f(V) = \emptyset_X \). By Lemma 5.4, \( f(U) \) and \( f(V) \) are disjoint IF\( \beta^* \)-open set of \( Y \) such that \( A \subseteq f(U) \) and \( B \subseteq f(V) \). Hence, \( Y \) is IF \( \pi g \beta^* \)-normal space.

6. IF \( \pi g \beta^* \)-normality in subspaces.

Lemma 6.1 If \( M \) be an IF open subspace of a space \( X \) and \( A \subseteq M \), then
\( IF \beta^* cl_X(A) = IF \beta^* cl_X(A) \cap M \).

Lemma 6.2 [12]. If \( M \) be an IF open subspace of a space \( X \) and \( A \subseteq M \), then
\( IF int_M(cl_M(A)) = IF int_X(cl_X(A) \cap M) \).

Lemma 6.3 [12]. If \( M \) be a IF\( \pi \)-open subspace of a space \( X \) and \( U \) be an
IF \( \pi \) open subset of \( X \), then \( U \cap M \) is an IF\( \pi \) open set in \( M \).

Lemma 6.4. If \( A \) is both IF\( \pi \)-open and IF \( \pi g \beta^* \)-closed subset of a space
\( X \), then \( A \) is an IF \( \beta^* \)-closed set in \( X \).

Proof. Since \( A \) is both IF\( \pi \)-open and IF \( \pi g \beta^* \)-closed subset of a space \( X \), and
since \( A \subseteq A \), then IF \( \beta^* cl(A) \subseteq A \), but \( A \subseteq IF \beta^* cl(A) \). Then \( A = IF \beta^* cl(A) \).
Hence \( A \) is an IF \( \beta^* \)-closed set in \( X \).

Corollary 6.5. If \( A \) is both IF\( \pi \)-open and IF \( \pi g \beta^* \)-closed subset of a space
\( X \), then \( A \) is an IF \( \beta^* \) regular-closed set in \( X \).

Theorem 6.6. Let \( M \) be an IF\( \pi \)-open subspace of a space \( X \) and \( A \subseteq M \).
If \( M \) is an IF \( \pi g \beta^* \)-closed subset of a space \( X \) and \( A \) is an IF \( \pi g \beta^* \)-closed
subset of \( M \). Then \( A \) is an IF \( \pi g \beta^* \)-closed subset of \( X \).

Proof. Suppose that \( M \) is an IF \( \pi g \beta^* \)-closed in \( X \) and \( A \) is \( \pi g \beta^* \)- closed
set in \( M \). Let \( U \) be any IF\( \pi \)-open set in \( X \) such that \( A \subseteq U \). Then by Lemma
6.3, we have \( A \subseteq M \cap U \), where \( M \cap U \) is an IF \( \pi \)-open set in \( M \). Since \( A \)
is IF \( \pi g \beta^* \)-closed in \( M \), thus \( IF \beta^* cl_M(A) \subseteq M \cap U \). Then by Lemma 6.1,
\( IF \beta^* cl_X(A) \cap M \subseteq M \cap U \). By Lemma 6.4, we obtain that \( IF \beta^* cl_X(M) = M \).
Thus, IF $\beta^* cl_X(A) \subseteq IF\beta^* cl_X(M) = M$. So, $IF\beta^* cl_X(A) \cap M = IF\beta^* cl_X(A)$.
Hence, $IF\beta^* cl_X(M) \subseteq M \cap U$ Thus,$IF\beta^* cl_X(A) \subseteq U$. Therefore, A is an IF $\pi g\beta^*$-closed set in X.

**Lemma 6.7** Let M be an intuitionistic fuzzy closed domain subspace of a space X. If U is an IF $\beta^*$ open set in X, then $U \cap M$ is an IF $\beta^*$ open set in M.

**Theorem 6.8** An intuitionistic fuzzy $\pi g\beta^*$-closed and IF $\pi$ open subspace of an intuitionistic fuzzy $\pi g\beta^*$-normal space is an intuitionistic fuzzy $\pi g\beta^*$ normal.

**Proof.** Suppose that M is an IF $\pi g\beta^*$-closed and IF $\pi$ open subspace of an intuitionistic fuzzy $\pi g\beta^*$-normal space X. Let A and B be any intuitionistic fuzzy disjoint $\pi g\beta^*$-closed subsets of M. Then by Theorem 7.6, we have A and B are intuitionistic fuzzy disjoint $\pi g\beta^*$ closed sets in X. By intuitionistic fuzzy $\pi g\beta^*$ normality of X, there exist intuitionistic fuzzy $\beta^*$ open subsets $U$ and $V$ of X such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset_X$. By Corollary 6.5 and Lemma 6.4, we obtain that $U \cap M$ and $V \cap M$ are intuitionistic fuzzy disjoint $\beta^*$ open sets in M such that $A \subset U \cap M$ and $B \subset V \cap M$. Hence, M is an intuitionistic fuzzy $\pi g\beta^*$ normal subspace of intuitionistic fuzzy $\pi g\beta^*$ normal space X.

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**References**


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