The $k$-Fibonacci Dual Quaternions

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Abstract

In this paper, $k$-Fibonacci dual quaternions are defined. Also, some algebraic properties of $k$-Fibonacci dual quaternions which are connected with $k$-Fibonacci numbers and Fibonacci numbers are investigated. Furthermore, d’Ocagne’s identity, the Honsberger identity, Binet’s formula, Cassini’s identity and Catalan’s identity for these quaternions are given.

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1. INTRODUCTION

The quaternions constitute an extension of complex numbers into a four-dimensional space and can be considered as four-dimensional vectors, in the same way that complex numbers are considered as two-dimensional vectors. Quaternions were first described by Irish mathematician Hamilton in 1843. Hamilton [10] introduced a set of quaternions which can be represented as

$$H = \{ q = q_0 + i q_1 + j q_2 + k q_3 \mid q_0, q_1, q_2, q_3 \in \mathbb{R} \}$$

where

$$i^2 = j^2 = k^2 = -1, \quad i j = -j i = k, \quad j k = -k j = i, \quad k i = -i k = j.$$
Several authors worked on different quaternions and their generalizations. ([1],[12],[15],[17],[20]).

Now, let’s talk about the work done on Fibonacci quaternion and dual Fibonacci quaternion:

Horadam [11] defined complex Fibonacci and Lucas quaternions as follows
\[
Q_n = F_n + i F_{n+1} + j F_{n+2} + k F_{n+3}
\]
and
\[
K_n = L_n + i L_{n+1} + j L_{n+2} + k L_{n+3}
\]
where
\[
i^2 = j^2 = k^2 = -1, \quad i j = -j i = k, \quad j k = -k j = i, \quad k i = -i k = j.
\]

In 2012, Halıcı [8] gave generating functions and Binet’s formulas for Fibonacci and Lucas quaternions. In 2013, Halıcı [9] defined complex Fibonacci quaternions as follows:
\[
H_{FC} = \{ R_n = C_n + e_1 C_{n+1} + e_2 C_{n+2} + e_3 C_{n+3} \mid C_n = F_n + i F_{n+1}, \quad i^2 = -1 \}
\]
where
\[
e_1^2 = e_2^2 = e_3^2 = e_1 e_2 e_3 = -1, \quad e_1 e_2 = -e_2 e_1 = e_3, e_2 e_3 = -e_3 e_2 = e_1, e_3 e_1 = -e_1 e_3 = e_2, \quad n \geq 1.
\]

Majernik [16] defined dual quaternions as follows:
\[
H_D = \left\{ Q = a + b i + c j + d k \mid a, b, c, d \in \mathbb{R}, \quad i^2 = j^2 = k^2 = i j k = 0, \quad i j = -j i = k, \quad j k = -k j = i, \quad k i = -i k = j \right\}
\]

For more details on dual quaternions, see [3]. In 2015, Yüce and Torunbalcı Aydın [22] defined dual Fibonacci quaternions as follows:
\[
H_D = \{ Q_n = F_n + i F_{n+1} + j F_{n+2} + k F_{n+3} \mid F_n, \quad n - \text{th Fibonacci number} \},
\]
where
\[
i^2 = j^2 = k^2 = i j k = 0, \quad i j = -j i = j k = -k j = k i = -i k = 0.
\]

Ramirez [19] defined the the k-Fibonacci and the k-Lucas quaternions as follows:
\[
D_{k,n} = \{ F_{k,n} + i F_{k,n+1} + j F_{k,n+2} + k F_{k,n+3} \mid F_{k,n}, \quad n - \text{th k-Fibonacci number} \},
\]
and
\[
P_{k,n} = \{ L_{k,n} + i L_{k,n+1} + j L_{k,n+2} + k L_{k,n+3} \mid L_{k,n}, \quad n - \text{th k-Lucas number} \}
\]
where \( i, j, k \) satisfy the multiplication rules (1.2).
In 2015, Polatlı Kızılateş and Kesim [18] defined split k-Fibonacci and split k-Lucas quaternions \((M_{k,n})\) and \((N_{k,n})\) respectively as follows:

\[
M_{k,n} = \{ F_{k,n} + i F_{k,n+1} + j F_{k,n+2} + k F_{k,n+3} \mid F_{k,n}, n - th \text{ k-Fibonacci number} \}
\]

where \(i, j, k\) are split quaternionic units which satisfy the multiplication rules

\[
i^2 = -1, \quad j^2 = k^2 = i j k = 1, \quad i j = -j i = k, \quad j k = -k j = -i, \quad k i = -i k = j.
\]

In 2017, Catarino and Vasco [2] defined dual k-Pell Quaternions and Octanions \((\tilde{R}_{k,n})\) as follows:

\[
\tilde{R}_{k,n} = \{ R_{k,n} + \varepsilon R_{k,n+1} \mid R_{k,n} = P_{k,n} e_0 + P_{k,n+1} e_1 + P_{k,n+2} e_2 + P_{k,n+3} e_3, \quad P_{k,n}, n - th \text{ k-Pell number} \}
\]

where

\[
e_0 = 1, \quad e_i e_j = -e_j e_i, \quad (e_i)^2 = -1, \quad (i, j = 1, 2, 3).
\]

In this paper, the k-Fibonacci dual quaternions and the k-Lucas dual quaternions will be defined respectively, as follows

\[
\mathbb{D}F_{k,n} = \{ \mathbb{D}F_{k,n} = F_{k,n} + i F_{k,n+1} + j F_{k,n+2} + k F_{k,n+3} \mid F_{k,n}, n - th \text{ k-Fibonacci number} \}
\]

and

\[
\mathbb{D}L_{k,n} = \{ \mathbb{D}L_{k,n} = L_{k,n} + i L_{k,n+1} + j L_{k,n+2} + k L_{k,n+3} \mid L_{k,n}, n - th \text{ k-Lucas number} \}
\]

where

\[
i^2 = j^2 = k^2 = i j k = 0, \quad i j = -j i = k, \quad j k = -k j = -i, \quad k i = -i k = 0.
\]

The aim of this work is to present in a unified manner a variety of algebraic properties of both the k-Fibonacci dual quaternions as well as the k-Fibonacci quaternions. In accordance with these definitions, we given some algebraic properties and Binet’s formula for k-Fibonacci dual quaternions. Moreover, some sums formulas and some identities such as d’ocagnes, Honsberger, Cassini’s and Catalan’s identities for these quaternions are given.

### 2. The k-Fibonacci Dual Quaternion

The k-Fibonacci sequence \( \{F_{k,n}\}_{n \in \mathbb{N}} \) [19] is defined as

\[
\begin{align*}
F_{k,0} &= 0, \quad F_{k,1} = 1 \\
F_{k,n+1} &= k F_{k,n} + F_{k,n-1}, \quad n \geq 1
\end{align*}
\]

or

\[
\{F_{k,n}\}_{n \in \mathbb{N}} = \{ 0, 1, k, k^2 + 1, k^3 + 2 k, k^4 + 3 k^2 + 1, ... \}
\]

Here, \(k\) is a positive real number.
In this section, firstly the k-Fibonacci dual quaternions will be defined. The k-Fibonacci dual quaternions are defined by using the k-Fibonacci numbers as follows:

\[ D F_{k,n} = \{ D F_{k,n} = F_{k,n} + i F_{k,n+1} + j F_{k,n+2} + k F_{k,n+3} \mid F_{k,n}, n - th \} \]  

k-Fibonacci number, \( \cdots \) \( ) \],

where

\[ i^2 = j^2 = k^2 = i j k = 0, \ i j = -j i = j k = -k j = k i = -i k = 0. \]

Let \( D F_{k,n}^1 \) and \( D F_{k,n}^2 \) be two k-Fibonacci dual quaternions such that

\[ D F_{k,n}^1 = F_{k,n} + i F_{k,n+1} + j F_{k,n+2} + k F_{k,n+3} \]

and

\[ D F_{k,n}^2 = G_{k,n} + i G_{k,n+1} + j G_{k,n+2} + k G_{k,n+3} \]

Then, the addition and subtraction of two k-Fibonacci dual quaternions are defined in the obvious way,

\[ D F_{k,n}^1 \pm D F_{k,n}^2 = (F_{k,n} + i F_{k,n+1} + j F_{k,n+2} + k F_{k,n+3}) \pm (G_{k,n} + i G_{k,n+1} + j G_{k,n+2} + k G_{k,n+3}) \]

\[ = (F_{k,n} \pm G_{k,n}) + i (F_{k,n+1} \pm G_{k,n+1}) + j (F_{k,n+2} \pm G_{k,n+2}) + k (F_{k,n+3} \pm G_{k,n+3}). \]

Multiplication of two k-Fibonacci dual quaternions is defined by

\[ D F_{k,n}^1 D F_{k,n}^2 = (F_{k,n} + i F_{k,n+1} + j F_{k,n+2} + k F_{k,n+3}) \]

\[ (G_{k,n} + i G_{k,n+1} + j G_{k,n+2} + k G_{k,n+3}) \]

\[ = (F_{k,n} G_{k,n}) + i (F_{k,n} G_{k,n+1} + F_{k,n+1} G_{k,n}) + j (F_{k,n} G_{k,n+2} + F_{k,n+2} G_{k,n}) \]

\[ + k (F_{k,n} G_{k,n+3} + F_{k,n+3} G_{k,n}) \]

\[ = (F_{k,n} G_{k,n}) + F_{k,n} (i G_{k,n+1} + j G_{k,n+2} + k G_{k,n+3}) \]

\[ + (i F_{k,n+1} + j F_{k,n+2} + k F_{k,n+3}) G_{k,n}. \]

The scaler and the vector parts of the k-Fibonacci dual quaternion \( D F_{k,n} \) are denoted by

\[ S_{D F_{k,n}} = F_{k,n} \] and \( V_{D F_{k,n}} = i F_{k,n+1} + j F_{k,n+2} + k F_{k,n+3}. \)

Thus, the k-Fibonacci dual quaternion \( D F_{k,n} \) is given by \( D F_{k,n} = S_{D F_{k,n}} + V_{D F_{k,n}}. \) Then, relation (2.6) is defined by

\[ D F_{k,n}^1 D F_{k,n}^2 = S_{D F_{k,n}^1} S_{D F_{k,n}^2} + S_{D F_{k,n}^1} V_{D F_{k,n}^2} + S_{D F_{k,n}^2} V_{D F_{k,n}^1} + V_{D F_{k,n}^1} V_{D F_{k,n}^2}. \]

The conjugate of k-Fibonacci dual quaternion \( D F_{k,n} \) is denoted by \( \overline{D F_{k,n}} \) and it is

\[ \overline{D F_{k,n}} = F_{k,n} - i F_{k,n+1} - j F_{k,n+2} + k F_{k,n+3}. \]

The norm of the k-Fibonacci dual quaternion \( D F_{k,n} \) is defined as follows

\[ \| D F_{k,n} \|^2 = D F_{k,n} \overline{D F_{k,n}} = (F_{k,n})^2. \]
In the following theorem, some properties related to the k-Fibonacci dual quaternions are given.

**Theorem 2.1.** Let \( F_{k,n} \) and \( \mathbb{D}F_{k,n} \) be the \( n \)-th terms of k-Fibonacci sequence \( (F_{k,n}) \) and the k-Fibonacci dual quaternion \( (\mathbb{D}F_{k,n}) \), respectively. In this case, for \( n \geq 1 \) we can give the following relations:

\[
\mathbb{D}F_{k,n+2} = k \mathbb{D}F_{k,n+1} + \mathbb{D}F_{k,n},
\]

(2.11)

\[
(\mathbb{D}F_{k,n})^2 = 2 F_{k,n} \mathbb{D}F_{k,n} - \mathbb{D}F_{k,n} \mathbb{D}F_{k,n} = 2 F_{k,n} \mathbb{D}F_{k,n} - (F_{k,n})^2,
\]

(2.12)

\[
\mathbb{D}F_{k,n} - i \mathbb{D}F_{k,n+1} - j \mathbb{D}F_{k,n+2} - k \mathbb{D}F_{k,n+3} = F_{k,n}.
\]

(2.13)

**Proof.** (2.11): By the equation (2.2) we get,

\[
\mathbb{D}F_{k,n} + k \mathbb{D}F_{k,n+1} = (F_{k,n} + i F_{k,n+1} + j F_{k,n+2} + k F_{k,n+3})
+ k(F_{k,n+1} + i F_{k,n+2} + j F_{k,n+3} + k F_{k,n+4})
= (F_{k,n} + k F_{k,n+1}) + i (F_{k,n+1} + k F_{k,n+2})
+ j (F_{k,n+2} + k F_{k,n+3}) + k (F_{k,n+3} + k F_{k,n+4})
= F_{k,n+2} + i F_{k,n+3} + j F_{k,n+4} + k F_{k,n+5}
= \mathbb{D}F_{k,n+2}.
\]

(2.12): By the equation (2.2) we get,

\[
(\mathbb{D}F_{k,n})^2 = F_{k,n}^2 + 2i(F_{k,n} F_{k,n+1}) + 2j(F_{k,n} F_{k,n+2}) + 2k(F_{k,n} F_{k,n+3})
= 2 F_{k,n} (F_{k,n} + i F_{k,n+1} + j F_{k,n+2} + k F_{k,n+3}) - F_{k,n}^2
= 2 F_{k,n} \mathbb{D}F_{k,n} - F_{k,n}^2.
\]

(2.13): By the equation (2.2) we get,

\[
\mathbb{D}F_{k,n} - i \mathbb{D}F_{k,n+1} - j \mathbb{D}F_{k,n+2} - k \mathbb{D}F_{k,n+3} = (F_{k,n} + i F_{k,n+1} + j F_{k,n+2} + k F_{k,n+3})
- i(F_{k,n+1} + i F_{k,n+2} + j F_{k,n+3} + k F_{k,n+4})
- j(F_{k,n+2} + i F_{k,n+3} + j F_{k,n+4} + k F_{k,n+5})
- k(F_{k,n+3} + i F_{k,n+4} + j F_{k,n+5} + k F_{k,n+6})
= F_{k,n}.
\]

\[
\square
\]

**Theorem 2.2.** For \( m \geq n + 1 \) the d’Ocagne’s identity for the k-Fibonacci dual quaternions \( \mathbb{D}F_{k,m} \) and \( \mathbb{D}F_{k,n} \) is given by

\[
\mathbb{D}F_{k,m} \mathbb{D}F_{k,n+1} - \mathbb{D}F_{k,m+1} \mathbb{D}F_{k,n} = (-1)^n F_{k,m-n} (\mathbb{D}F_{k,1} + j + k k)
= (-1)^n F_{k,m-n} (\mathbb{D}F_{k,1} + \mathbb{D}F_{k,-1} - 1).
\]

(2.14)
Finally, adding by two sides to the side, we obtain
\[
D F_{k,m} D F_{k,n+1} - D F_{k,m+1} D F_{k,n} = (F_{k,m} F_{k,n+1} - F_{k,m+1} F_{k,n}) + i [(F_{k,m} F_{k,n+2} - F_{k,m+1} F_{k,n+1}) + ((F_{k,m+1} F_{k,n+1} - F_{k,m+2} F_{k,n})]
+ j [(F_{k,m} F_{k,n+3} - F_{k,m+1} F_{k,n+2}) + ((F_{k,m+2} F_{k,n+1} - F_{k,m+3} F_{k,n})]
+ k [(F_{k,m} F_{k,n+4} - F_{k,m+1} F_{k,n+3}) + ((F_{k,m+3} F_{k,n+1} - F_{k,m+4} F_{k,n})]
= (−1)^n F_{k,m−n} [1 + k (1 + k^2) + k (k^3 + 3 k)]
= (−1)^n F_{k,m−n} (D F_{k,1} + j + k).
\]

Here, d’Ocagne’s identity of k-Fibonacci number \( F_{k,m} F_{k,n+1} - F_{k,m+1} F_{k,n} = (−1)^n F_{k,m−n} \) in [5] was used.

**Theorem 2.3.** For \( n, m \geq 0 \) the Honsberger identity for the k-Fibonacci dual quaternions \( D F_{k,n} \) and \( D F_{k,m} \) is given by
\[
D F_{k,n+1} D F_{k,m} + D F_{k,n} D F_{k,m−1} = 2 D F_{k,n+m+1} - F_{k,n+m+1}.
\] (2.15)

**Proof.** (2.15): By using (2.6) and (2.11)
\[
D F_{k,n} D F_{k,m} = F_{k,n} F_{k,m} + i (F_{k,n} F_{k,m+1} + F_{k,n+1} F_{k,m}) + j (F_{k,n} F_{k,m+2} + F_{k,n+2} F_{k,m}) + k (F_{k,n} F_{k,m+3} + F_{k,n+3} F_{k,m})
\]
\[
D F_{k,n+1} D F_{k,m+1} = F_{k,n+1} F_{k,m+1} + i (F_{k,n+1} F_{k,m+2} + F_{k,n+2} F_{k,m+1}) + j (F_{k,n+1} F_{k,m+3} + F_{k,n+3} F_{k,m+1}) + k (F_{k,n+1} F_{k,m+4} + F_{k,n+4} F_{k,m+1}).
\]
Finally, adding by two sides to the side, we obtain
\[
D F_{k,n} D F_{k,m} + D F_{k,n+1} D F_{k,m+1} = (F_{k,n} F_{k,m} + F_{k,n+1} F_{k,m+1}) + i [(F_{k,n} F_{k,m+1} + F_{k,n+1} F_{k,m} + F_{k,n+1} F_{k,m+2} + F_{k,n+2} F_{k,m+1})
+ j [(F_{k,n} F_{k,m+2} + F_{k,n+2} F_{k,m} + H_{k,n+1} F_{k,m+3} + F_{k,n+3} F_{k,m+1})
+ k [(F_{k,n} F_{k,m+3} + F_{k,n+3} F_{k,m} + F_{k,n+1} F_{k,m+4} + F_{k,n+4} F_{k,m+1})
= 2 [F_{k,n+m+1} + i F_{k,n+m+2} + j F_{k,n+m+3} + k F_{k,n+m+4} - F_{k,n+m+1}
= 2 D F_{k,n+m+1} - F_{k,n+m+1}.
\]

Here, the Honsberger identity of k-Fibonacci number \( F_{k,n+1} F_{k,m} + F_{k,n} F_{k,m−1} = F_{k,n+m} \) in [4] was used.

**Theorem 2.4.** Let \( D F_{k,n} \) and \( D L_{k,n} \) be n-th terms of k-Fibonacci dual quaternion \( (D F_{k,n}) \) and k-Lucas dual quaternion \( (D L_{k,n}) \), respectively. The following relation is satisfied
\[
D F_{k,n+1} + D F_{k,n−1} = D L_{k,n}.
\] (2.16)
Theorem 2.5. Let $\overline{D} F_{k,n}$ be conjugation of the k-Fibonacci dual quaternion $(DF_{k,n})$. In this case, we can give the following relations between these quaternions:

\[ DF_{k,n} + \overline{DF}_{k,n} = 2 F_{k,n}, \]  
\[ DF_{k,n}DF_{k,n} + DF_{k,n-1}\overline{DF}_{k,n-1} = F_{k,n}^2 + F_{k,n-1}^2 = F_{k,2n-1}, \]  
\[ 19(DF_{k,n})^2 + (DF_{n-1})^2 = 2 DF_{k,2n-1} - F_{k,2n-1}, \]  
\[ DF_{k,n+1}^2 - DF_{k,n-1}^2 = k (2 DF_{k,2n} - F_{k,2n}). \]

Proof. (2.17): By using (2.9), we get

\[ DF_{k,n} + \overline{DF}_{k,n} = (F_{k,n} + i H_{n+1} + j F_{k,n+2} + k F_{k,n+3}) + (F_{k,n} - i F_{k,n+1} - j F_{k,n+2} - k F_{k,n+3}) = 2 F_{k,n}. \]

(2.18): By using (2.10), we get

\[ DF_{k,n}DF_{k,n} + DF_{k,n-1}\overline{DF}_{k,n-1} = F_{k,n}^2 + F_{k,n-1}^2 = F_{k,2n-1}. \]

where the identity of k-Fibonacci number $F_{k,n}^2 + F_{k,n+1}^2 = F_{k,2n+1}, \ n \geq 0$ \[19\] was used.

(2.19): By using (2.12), we get

\[ DF_{k,n}^2 + DF_{k,n-1}^2 = 2 (DF_{k,n}F_{k,n} + DF_{k,n-1}F_{k,n-1}) - (F_{k,n}^2 + F_{k,n-1}^2) = 2 \left[ (F_{k,n} + F_{k,n+1}) + i (F_{k,n+1} + F_{k,n+2}) + j (F_{k,n+2} + F_{k,n+3}) + k (F_{k,n+3} + F_{k,n+4}) \right] - (F_{k,n}^2 + F_{k,n-1}^2) = 2 (F_{k,2n+1} + i F_{k,2n} + j F_{k,2n+1} + k F_{k,2n+2} - F_{k,2n-1}) = 2 \overline{DF}_{k,2n} - F_{k,2n-1}. \]

where the identity of k-Fibonacci number $F_{k,n}^2 + F_{k,n+1}^2 = F_{k,2n+1}, \ n \geq 0$ \[19\] was used.

(2.20): By using (2.12), we get

\[ DF_{k,n+1}^2 - DF_{k,n-1}^2 = 2 (DF_{k,n+1}F_{k,n+1} - DF_{k,n-1}F_{k,n-1}) - (F_{k,n+1}^2 + F_{k,n-1}^2) = 2 \left[ (F_{k,n+1} + F_{k,n+2}) + i (F_{k,n+2} + F_{k,n+3}) + j (F_{k,n+3} + F_{k,n+4}) + k (F_{k,n+4} + F_{k,n+5}) \right] - (F_{k,n+1}^2 + F_{k,n-1}^2) = 2 k (F_{k,2n} + i F_{k,2n+1} + j F_{k,2n+2} + k F_{k,2n+3}) - F_{k,2n} = k \left[ 2 \overline{DF}_{k,2n} - F_{k,2n} \right]. \]
where the identity of k-Fibonacci number $F^2_{k,n+1} - F^2_{k,n-1} = kF_{k,2n}$, [4] was used.

**Theorem 2.6.** Let $\mathbb{D}F_{k,n}$ be the k-Fibonacci dual quaternion. Then, we have the following identities

\[
\sum_{s=1}^{n} \mathbb{D}F_{k,s} = \frac{1}{k} (\mathbb{D}F_{k,n+1} + \mathbb{D}F_{k,n} - \mathbb{D}F_{k,1} - \mathbb{D}F_{k,0}), \tag{2.21}
\]

\[
\sum_{s=1}^{n} \mathbb{D}F_{k,2s-1} = \frac{1}{k} (\mathbb{D}F_{k,2n} - \mathbb{D}F_{k,0}), \tag{2.22}
\]

\[
\sum_{s=1}^{n} \mathbb{D}F_{k,2s} = \frac{1}{k} (\mathbb{D}F_{k,2n+1} - \mathbb{D}F_{k,1}). \tag{2.23}
\]

**Proof.** (2.21): Since $\sum_{i=1}^{n} F_{k,i} = \frac{1}{k}(F_{k,n+1} + F_{k,n} - 1)$, [?], we get

\[
\sum_{s=1}^{n} \mathbb{D}F_{k,s} = \sum_{s=1}^{n} F_{k,s} + i \sum_{s=1}^{n} F_{k,s+1} + j \sum_{s=1}^{n} F_{k,s+2} + k \sum_{s=1}^{n} F_{k,s+3}
\]

\[
= \frac{1}{k} \left\{ \left[ (F_{k,n+1} + F_{k,n} - 1) + i[F_{k,n+2} + F_{k,n+1} - k - 1] 
+ j[F_{k,n+3} + F_{k,n+2} - (k^2 + 1) - k] 
+ k[F_{k,n+4} + F_{k,n+3} - (k^3 + 2k) - (k^2 + 1)] \right] \right\}
\]

\[
= \frac{1}{k} \left\{ (F_{k,n+1} + iF_{k,n+2} + jF_{k,n+3} + kF_{k,n+4}) 
+ (F_{k,n} + iF_{k,n+1} + jF_{k,n+2} + kF_{k,n+3}) 
- [1 + i(k + 1) + j(k^2 + k + 1) + k(k^3 + k^2 + 2k + 1)] \right\}
\]

\[
= \frac{1}{k} \left\{ (\mathbb{D}F_{k,n+1} + \mathbb{D}F_{k,n} - [F_{k,1} + i(F_{k,2} + F_{k,1}) + j(F_{k,3} + F_{k,2}) + k(F_{k,4} + F_{k,3})] \right\}
\]

\[
= \frac{1}{k} \left( \mathbb{D}F_{k,n+1} + \mathbb{D}F_{k,n} - \mathbb{D}F_{k,1} - \mathbb{D}F_{k,0}. \right)
\]

(2.22): Using $\sum_{i=1}^{n} F_{k,2i-1} = \frac{1}{k}F_{k,2n}$ and $\sum_{i=1}^{n} F_{k,2i} = \frac{1}{k}(F_{2n+1} - 1)$ [4], we get

\[
\sum_{s=1}^{n} \mathbb{D}F_{k,2s-1} = \frac{1}{k} \left\{ (F_{k,2n}) + i(F_{k,2n+1} - 1) + j(F_{k,2n+2} - k) 
+ k[F_{k,2n+3} - (k^2 + 1)] \right\}
\]

\[
= \frac{1}{k} \left\{ (F_{k,2n} + iF_{k,2n+1} + jF_{k,2n+2} + kF_{k,2n+3}) 
- [i + k(j + (k^2 + 1)k)] \right\}
\]

\[
= \frac{1}{k} \left\{ (\mathbb{D}F_{k,2n} - (F_{k,0} + IF_{k,1} + jF_{k,2} + kF_{k,3}) \right\}
\]

\[
= \frac{1}{k} \left( \mathbb{D}F_{k,2n} - \mathbb{D}F_{k,0}. \right)
\]
(2.23): Using \( \sum_{i=1}^{n} F_{k,2i} = \frac{1}{k} (F_{k,2n+1} - 1) \) \cite{[source]} , we obtain

\[
\sum_{s=1}^{n} \mathbb{D} F_{k,2s} = \frac{1}{k} \left\{ (F_{k,2n+1} - 1) + i [F_{k,2n+2} - k] + j [F_{2n+3} - (k^2 + 1)] + k [F_{k,2n+4} - (k^3 + 2k)] \right\} \\
= \frac{1}{k} \left\{ (F_{k,2n+1} + i F_{k,2n+2} + j F_{k,2n+3} + k F_{k,2n+4} ) - [1 + k i + (k^2 + 1) j + (k^3 + 2k) k ] \right\} \\
= \frac{1}{k} (\mathbb{D} F_{k,2n+1} - \mathbb{D} F_{k,1}). \\
\]

\[\Box\]

**Theorem 2.7. (Binet’s Formula).** Let \( \mathbb{D} F_{k,n} \) be the k-Fibonacci dual quaternion. For \( n \geq 1 \), Binet’s formula for these quaternions is as follows:

\[
\mathbb{D} F_{k,n} = \frac{1}{\sqrt{k^2 + 4}} \left( \hat{\alpha} \alpha^n - \hat{\beta} \beta^n \right) \tag{2.24}
\]

where

\[
\hat{\alpha} = 1 + i (k - \beta) + j ((k^2 + 1) - k \beta) + k ((k^3 + 2k) - (k^2 + 1) \beta),
\]

and

\[
\hat{\beta} = -1 + i (\alpha - k) + j [k \alpha - (k^2 + 1)] + k [(k^2 + 1) \alpha - (k^3 + 2k)].
\]

**Proof.** The characteristic equation of recurrence relation \( \mathbb{D} F_{k,n+2} = k \mathbb{D} F_{k,n+1} + \mathbb{D} F_{k,n} \) is

\[t^2 - k t - 1 = 0.\]

The roots of this equation are

\[
\alpha = \frac{k + \sqrt{k^2 + 4}}{2} \quad \text{and} \quad \beta = \frac{k - \sqrt{k^2 + 4}}{2}
\]

where \( \alpha + \beta = k \), \( \alpha - \beta = \sqrt{k^2 + 4} \), \( \alpha \beta = -1 \).

Using recurrence relation and initial values \( \mathbb{D} F_{k,0} = (0, 1, k, k^2 + 1), \mathbb{D} F_{k,1} = (1, k, k^2 + 1, k^3 + 2k) \), the Binet formula for \( \mathbb{D} F_{k,n} \) is

\[
\mathbb{D} F_{k,n} = A \alpha^n + B \beta^n = \frac{1}{\sqrt{k^2 + 4}} \left[ \hat{\alpha} \alpha^n - \hat{\beta} \beta^n \right],
\]

where \( A = \frac{\mathbb{D} F_{k,1} - \beta \mathbb{D} F_{k,0}}{\alpha - \beta} \), \( B = \frac{\alpha \mathbb{D} F_{k,0} - \mathbb{D} F_{k,1}}{\alpha - \beta} \) and

\[
\hat{\alpha} = 1 + i \alpha + j \alpha^2 + k \alpha^3, \quad \hat{\beta} = 1 + i \beta + j \beta^2 + k \beta^3. \]

\[\Box\]

**Theorem 2.8. (Cassini’s Identity).** Let \( \mathbb{D} F_{k,n} \) be the k-Fibonacci dual quaternion. For \( n \geq 1 \), Cassini’s identity for \( \mathbb{D} F_{k,n} \) is as follows:

\[
\mathbb{D} F_{k,n-1} \mathbb{D} F_{k,n+1} - (\mathbb{D} F_{k,n})^2 = (-1)^n (\mathbb{D} F_{k,1} + j + k k) \\
= (-1)^n (\mathbb{D} F_{k,1} + \mathbb{D} F_{k,-1} - 1). \tag{2.25}
\]
Proof. (2.25): By using (2.2) and (2.5), we get
\[
\mathbb{D}F_{k,n-1}\mathbb{D}F_{k,n+1} - (\mathbb{D}F_{k,n})^2 = (F_{k,n-1} + i F_{k,n} + j F_{k,n+1} + k F_{k,n+2})
\]
\[
(F_{k,n+1} + i F_{k,n+2} + j F_{k,n+3} + k F_{k,n+4})
\]
\[
- [F_{k,n} + i F_{k,n+1} + j F_{k,n+2} + k F_{k,n+3}]^2
\]
\[
= [F_{k,n-1}F_{k,n+1} - F_{k,n}^2]
\]
\[
+ i [F_{k,n-1}F_{k,n+2} + F_{k,n}F_{k,n+2} - 2 F_{k,n}F_{k,n+1}]
\]
\[
+ j [F_{k,n-1}F_{k,n+3} - 2 F_{k,n}F_{k,n+2} + F_{k,n+1}^2]
\]
\[
+ k [F_{k,n-1}F_{k,n+4} + F_{k,n+1}F_{k,n+2} - 2 F_{k,n}F_{k,n+3}]
\]
\[
= (-1)^n (\mathbb{D}F_{k,1} + j + k k)
\]
\[
= (-1)^n (\mathbb{D}F_{k,1} + \mathbb{D}F_{k,-1} - 1).
\]

Here, the identity of the k-Fibonacci number \(F_{k,n-1}F_{k,n+1} - F_{k,n}^2 = (-1)^n \) was used.

\[\square\]

**Theorem 2.9. (Catalan’s Identity).** Let \(\mathbb{D}F_{k,n}\) be the k-Fibonacci dual quaternion. For \(n \geq 1\), Catalan’s identity for \(\mathbb{D}F_{k,n}\) is as follows:
\[
\mathbb{D}F_{k,n+r} \mathbb{D}F_{k,n+r+1} - (\mathbb{D}F_{k,n+r})^2 = (-1)^{n+r} (\mathbb{D}F_{k,1} + \mathbb{D}F_{k,-1} - 1). \tag{2.26}
\]

Proof. (2.26): By using (2.16) and (2.17) we get
\[
\mathbb{D}F_{k,n+r} \mathbb{D}F_{k,n+r+1} - (\mathbb{D}F_{k,n+r})^2 = (F_{k,n+r-1} F_{k,n+r+2} - F_{k,n+r} F_{k,n+r+1})
\]
\[
+ j (F_{k,n+r-1} F_{k,n+r+3} + F_{k,n+r+1}^2 - 2 F_{k,n+r} F_{k,n+r+2})
\]
\[
+ k (F_{k,n+r-1} F_{k,n+r+4} + F_{k,n+r+2} F_{k,n+r+1} - 2 F_{k,n+r} F_{k,n+r+3})
\]
\[
= (-1)^{n+r} (1 + k i + (k^2 + 2) j + (k^3 + 3 k) k)
\]
\[
= (-1)^{n+r} [1 + k i + (k^2 + 1) j + (k^3 + 2 k) k + j + k k]
\]
\[
= (-1)^{n+r} (\mathbb{D}F_{k,1} + \mathbb{D}F_{k,-1} - 1).
\]

where we use Catalan’s identity of the k-Fibonacci number \(F_{k,n+r-1}F_{k,n+r+1} - F_{n+r}^2 = (-1)^{n+r} \[4\].

\[\square\]

3. Conclusion

In this study, a number of new results on k-Fibonacci dual quaternions were derived. Quaternions have great importance as they are used in quantum physics, applied mathematics, quantum mechanics, Lie groups, kinematics and differential equations. This study fills the gap in the literature by providing the k-Fibonacci dual quaternion using definitions of the k-Fibonacci number and dual Fibonacci quaternion [22].

**References**


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