Fixed Point Theorems through Pseudo Contractive Mappings

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Abstract

In 1972, Gatika and Kirk [2] proved some fixed point theorems for Lipschitzian pseudo-contractive mappings in Banach space. In this research article, we generalize those fixed point theorems in 2-Banach space.

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Introduction

The notion of 2-metric space was introduced by Gahler [1] in 1963 as a generalization of area function for Euclidean triangles. Many fixed point theorems were established by various authors like Rhoades [5], etc. A point $x \in X$ is said to be a fixed point of a self-map $f : X \to X$ if $f(x) = x$, where $X$ is a non-empty set. Theorems concerning fixed points of self-maps are known as fixed point theorems. Most of the fixed point theorems were proved for contraction mappings. It is well known that every contraction on a metric space is continuous. The converse is not necessarily true. The identity mapping on $[0, 1]$ simply serves the counter example.
In this present work, some of the fixed point theorems of Gatika and Kirk are extended to a more generalized 2-Banach space setting. In what follows $X$ stands for a 2-Banach space.

1. Preliminaries

This section is devoted to some basic definitions which are needed for the further study of this Article.

**Definition- 1.1** Let $X$ be a real linear space and $\|\cdot\|$ be a non negative real valued function defined on $X$ satisfying the following conditions:

(i) $\|x, y\| = 0$ iff $x$ and $y$ are linearly dependent.

(ii) $\|x, y\| = \|y, x\|$ for all $x, y \in X$.

(iii) $\|x, ay\| = |a|\|x, y\|$, $a$ being real, for all $x, y \in X$.

(iv) $\|x, y + z\| \leq \|x, y\| + \|y, z\|$, for all $x, y, z \in X$.

then $\|\cdot\|$ is called a 2-norm and the pair $(X, \|\cdot\|)$ is called a linear 2-normed space.

**Definition- 1.2**: A linear 2-normed space is said to be complete if every Cauchy sequence is convergent to an element of $X$. A complete 2-normed space $X$ is called 2-Banach spaces.

**Definition- 1.3**: Let $X$ be a 2-Banach space and $C$ be non empty bounded closed and convex subset of $X$. A mapping $T : C \to X$ is said to be non expansive if $\|T(x) - T(y), a\| < \|x - y, a\|$ for all $x, y \in C$.

**Definition- 1.4**: Let $X$ be a 2-Banach space and $D$ be a subset of $X$. A mapping $T : D \to X$ is said to be pseudo-contractive if $\|u - v, a\| \leq \|(1 + r)(u - v) - r(T(u) - T(v)), a\|$ for all $u, v, a \in D$ and for each $r > 0$.

2. Fixed Point Theorems for Pseudo-Contractive Mappings

In this section we prove fixed point theorems for pseudo-contractive mappings.

The following Theorem-2.1 is a generalization of Gatika and Kirk [2] in 2-Banach space setting.

**Theorem- 2.1**: Let $X$ be a uniformly convex 2-Banach space and $B$ be a closed sphere in $X$. Let $U$ be a Lipschitzian pseudo contractive mapping from $B$ to $X$ such that $U$ also maps boundary of $B$ into $B$. Then $U$ has a fixed point in $B$. 
**Proof.** We may assume that $B$ is a sphere with centre at origin and radius $\rho$ without loss of generality.
Let $\partial B$ be denote the boundary of $B$.

Since $U$ is a pseudo- contractive mapping from $B$ to $X$,
\[
\|u-v, a\| \leq \| (1+r) (u-v) - r (U(u)-U(v)), a \| \tag{1}
\]

for all $u,v,a \in B$ and for each $r > 0$

Let $\lambda = \frac{r}{1+r} \Rightarrow 0 < \lambda < 1$

Then (1) is equivalent to
\[
(1-\lambda)\|u-v, a\| \leq \| (u-v) - \lambda (U(u)-U(v)), a \| \tag{2}
\]

Let $T_\lambda = I - \lambda U$, then (2) implies
\[
(1-\lambda)\|u-v, a\| \leq \| T_\lambda (u) - T_\lambda (v), a \| \text{ for all } u,v,a \in B \tag{3}
\]

Since $U$ is a Lipschitzian, there is a constant $M$ such that
\[
\|U(u)-U(v), a\| < M \|u-v, a\| \text{ where } 0 \leq M < 1 \tag{4}
\]

Since $0 < \lambda < 1$ and $M < 1$, $\lambda M < 1$.

Let $U_\lambda = \lambda U$

Then $\|U_\lambda (u)-U_\lambda (v), a\| \leq \lambda \|U(u)-U(v), a\| < \lambda M \|u-v, a\|$

$\Rightarrow \|U_\lambda (u)-U_\lambda (v), a\| < \lambda M \|u-v, a\| \text{ where } \lambda M < 1 \tag{5}$

So $U_\lambda$ is strictly contractive.

Since $\|U(x), a\| \leq \rho$ for $x,a \in B$, $\|U_\lambda (x), a\| \leq \lambda \rho$ for $x,a \in B$.

Let $y^* \in B, = \{ x \in X : \|x, a\| \leq (1-\lambda) \rho \}$

Define $V_\lambda (x) = U_\lambda (x) + y^*$

If $x \in \partial B$, then $\| V_\lambda (x), a \| = \| U_\lambda (x) + y^* \| \leq \| U_\lambda (x), a \| + \| y^*, a \| \leq \lambda \rho + (1-\lambda) \rho = \rho$

$\Rightarrow \| V_\lambda (x), a \| \leq \rho$

$\Rightarrow V_\lambda$ maps the boundary of $B$ in to $B$

Define $F = \frac{(I+V_\lambda)}{2}$

Then $F$ maps $B$ in to $B$
By (5), \[ \| V_\lambda(x) - V_\lambda(y) \| = \|(U_\lambda(x) + y^*) - (U_\lambda(y) + y^*)\| = \|U_\lambda(x) - U_\lambda(y)\| < \lambda M \| x - y \| \]
\[ \Rightarrow \| V_\lambda(x) - V_\lambda(y) \| < \lambda M \| x - y \| \]
\[ \Rightarrow V_\lambda \text{ is Strictly contractive} \]
\[ \Rightarrow F \text{ is Strictly contractive} \]

Hence by the application of contraction principle, there exists a point \( x^* \in B \) such that \( F(x^*) = x^* = V(x^*) \).

Since \( V(x^*) = x^* \) and \( U_\lambda = \lambda U \), \( V_\lambda(x^*) = U_\lambda(x^*) + y^* = \lambda U(x^*) + y^* = x^* \)
\[ \Rightarrow y^* = x^* - \lambda U(x^*) \]
\[ \Rightarrow y^* = (I - \lambda U)(x^*) \]
\[ \Rightarrow y^* = T_\lambda(x^*) \in T_\lambda(B) \quad \Rightarrow B_1 \subseteq T_\lambda(B) \]
\[ \Rightarrow T_\lambda^{-1}(B_1) \subseteq B \]

So \((1-\lambda)T_\lambda^{-1}\) is a mapping of \( B_1 \) in to \( B \).

By (3) \((1-\lambda)T_\lambda^{-1}\) is a non expansive.

Hence by Kirk’s theorem, there exists a point \( z \) in \( B_1 \) such that \((1-\lambda)T_\lambda^{-1}(z) = z \)
\[ \Rightarrow T_\lambda^{-1}(z) = \frac{z}{1-\lambda} \]
Let \( \frac{z}{1-\lambda} = z^* \)

Then \( z^* \in B \) and \( T_\lambda^{-1}(z) = z^* \)
\[ \Rightarrow T_\lambda(z^*) = z \]
\[ \Rightarrow (I - \lambda U)(z^*) = z \]
\[ \Rightarrow (I - \lambda U)(z^*) = (1-\lambda)(z^*) \]
\[ \Rightarrow z^* - \lambda U(z^*) = z^* - \lambda z^* \]
\[ \Rightarrow U(z^*) = z^* \]
Hence \( U \) has a fixed point in \( B \).

**Theorem-2.1:** Let \( X \) be a 2-Banach space, \( G \) be an bounded subset of \( X \) with \( 0 \in G \) and \( U \) be a Lipschitzian pseudo contractive mapping from \( G \) to \( X \) satisfying:

1. \( U(x) \neq \lambda x \) if \( x \in \partial G \) and \( x \in \partial G \)


(2) \((I-U)G\) is closed
Then \(U\) has a fixed point in \(G\).

**Proof.** Let \(0 < r < 1\) be chosen so that \(rU\) is a contraction mapping.
Let \(G\) be an open bounded subset of \(X\) with \(0 \in G\)
Define mappings \(S\) and \(T\) from \(G\) to \(X\) as follows
\[ S = (1-r)I \quad \text{and} \quad T = I - rU \]
Then \(T\) is one to one, \(T(G)\) is open and \(\partial T(G) = T(\partial G)\)

\[ \Rightarrow T(G) = cl(T(G)) \]

Since \(U\) is pseudo contractive mapping,
\[ \|x - y, a\| \leq \|(1+r)(x-y) - r(U(x) - U(y)), a\| \quad \text{for all } x, y, a \in G \quad \text{and} \quad \text{for each } r > 0 \]
\[ \leq \|x - U(x) - (y - U(y)), a\| + r \|x - y, a\| \]
\[ \Rightarrow (1-r)\|x - y, a\| \leq \|T(x) - T(y), a\| \]
\[ \Rightarrow \|(1-r)(x-y), a\| \leq \|T(x) - T(y), a\| \]
\[ \Rightarrow \|S(x) - S(y), a\| \leq \|T(x) - T(y), a\| \quad \text{for all } x, y, a \in G \quad (6) \]

Define \(H : B \rightarrow X\) by \(H(z) = ST^{-1}(z)\)

Let \(z_1, z_2 \in B\)
Then by (6), we have
\[ \|H(z_1) - H(z_2), a\| = \|ST^{-1}(z_1) - ST^{-1}(z_2), a\| \]
\[ \leq \|TT^{-1}(z_1) - TT^{-1}(z_2), a\| \]
\[ \leq \|z_1 - z_2, a\| \]
\[ \Rightarrow \|H(z_1) - H(z_2), a\| \leq \|z_1 - z_2, a\| \]
Hence \(H\) is non expansive on \(B\).
We claim that \((I-H)B\) is closed

Suppose \(z_n - H(z_n) \rightarrow y\) for \(z_n \in B\)
Then \(z_n - (1-r)T^{-1}(z_n) \rightarrow y\)
\[ \Rightarrow \frac{z_n}{1-r} - T^{-1}(z_n) \rightarrow \frac{y}{1-r} \]

Let \( z = \frac{y}{1-r} \) and \( x_n = T^{-1}(z_n) \)

Then \[ \frac{r[x_n - U(x_n)]}{1-r} = \frac{x_n - U(x_n)}{1-r} - x_n \]

\[ = \frac{T(x_n)}{1-r} - x_n \]

\[ = \frac{z_n}{1-r} - T^{-1}(z_n) \rightarrow z \]

\[ \Rightarrow \frac{r[x_n - U(x_n)]}{1-r} \rightarrow z \text{ where } z = \frac{y}{1-r} \]

\[ \Rightarrow x_n - U(x_n) \rightarrow (1-r)\frac{z}{r} \]

Since \( x_n - U(x_n) \rightarrow (1-r)\frac{z}{r} \) and \( (I-U)G \) is closed, there exists a point \( x \) in \( G \) such that

\[ (1-r)\frac{z}{r} = x - U(x) \]

\[ \Rightarrow (1-r)z = rx - rU(x) \]

\[ \Rightarrow (1-r)z = [x - (1-r)x] - rU(x) \]

\[ = x - rU(x) - (1-r)x \]

\[ = T(x) - (1-r)x \]

\[ \Rightarrow (1-r)z = T(x) - (1-r)x \]

\[ \Rightarrow z = \frac{T(x)}{1-r} - x \Rightarrow \frac{T(x)}{1-r} - x = z \]

Put \( T(x) = w \), then we have

\[ \Rightarrow \frac{w}{1-r} = T^{-1}(w) = z \]
\[ w - (1-r) T^{-1}(w) = (1-r) z \]
\[ w - (1-r) T^{-1}(w) = y \]
\[ w - H(w) = y \]

If \( z_n - H(z_n) \rightarrow y \) for \( z_n \in B \), then \( w - H(w) = y \) for some \( z_n \in B \)

Hence \((I - H) B\) is closed

Now we show that \( H \) satisfies (6) on \( B \).
Let \( x \in \partial B \)

Let \( H(x) = \lambda x \) for \( \lambda > 1 \)

Then \( (1-r) T^{-1}(x) = \lambda x \)

\[ T^{-1}(x) = \frac{\lambda}{1-r} x \]

Since \( T(\partial G) = \partial (TG) \) and \( T^{-1}(x) = \frac{\lambda}{1-r} x, \frac{\lambda}{1-r} x \in \partial G \)

Then we have \( x = T \left( \frac{\lambda x}{1-r} \right) \)

\[ x = (I - rU) \left( \frac{\lambda x}{1-r} \right) \]

\[ x = \frac{\lambda x}{1-r} - rU \left( \frac{\lambda x}{1-r} \right) \]

\[ U \left( \frac{\lambda x}{1-r} \right) = \frac{(\lambda + r - 1)x}{r} = \frac{(\lambda + r - 1)}{\lambda r} \frac{\lambda x}{1-r} \]

Let \( z = \frac{\lambda x}{1-r} \)

Then \( z \in \partial G \) and \( U(z) = \mu z \) where \( \mu = \frac{(\lambda + r - 1)}{\lambda r} \)
But \( \mu - 1 = \frac{(\lambda - 1)(1-r)}{\lambda r} > 0 \)

This is contradiction to our hypothesis (1) for \( U \) on \( \partial G \).

\[ \Rightarrow H(x) \neq \lambda x \text{ for } \lambda > 1 \]

Hence by Petryshyn’s theorem there exists a point \( y \) in \( B \) such that \( H(y) = \lambda y \).

Let \( z^* = T^{-1}(y) \)

Then \( z^* \in G \) and \( S(z^*) = ST^{-1}(y) = H(y) = y = T(z^*) \)

\[ \Rightarrow S(z^*) = T(z^*) \]

\[ \Rightarrow (1-r)I(z^*) = (I-rU)(z^*) \]

\[ \Rightarrow z^* - rz^* = z^* - rU(z^*) \]

\[ \Rightarrow U(z^*) = z^* \]

Hence \( U \) has a fixed point in \( G \).

References


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