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Existence of Three Solutions for Difference Equation Involving P-Laplacian[1]

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Abstract

In this paper, using a three critical points theorem established by B. Ricceri and variational methods, we study the existence of solutions of the Dirichlet boundary value problem for p-Laplacian difference equation depending on two parameters λ, μ, and obtain the existence of three solutions under some appropriate assumptions.

Keywords: P-Laplacian; Three solutions; Boundary value problem; Variational methods

1 Introduction

Denote by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ the sets of all natural numbers, integers and real numbers. For $a,\ b \in \mathbb{Z}$, $\mathbb{Z}(a) = \{a,\ a+1,\ \cdots\}$, $\mathbb{Z}[a,\ b] = \{a,\ a+1,\ \cdots,\ b\}$ when $a \le b$.

We will consider the following p-Laplacian difference equation with Dirichlet boundary value condition

$$\begin{aligned} -\triangle[\phi_p(\triangle u(k-1))] &= \lambda(f(k,u(k)) + \mu g(k,u(k))),\ k \in \mathbb{Z}[1,T], \\ u(0) &= u(T+1) = 0. \end{aligned} \tag{1.1}$$

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where T is a positive integer, $p,\ \lambda,\ \mu \in \mathbb{R}$ are constants and $p > 1,\ \lambda > 0$, $\triangle$ is defined by $\triangle u(k) = u(k+1) - u(k)$, $\phi_p(s) =\mid s \mid^{p-2} s$ is the p-Laplacian operator, $f(k,\ \cdot),\ g(k,\ \cdot) \in C(\mathbb{R},\ \mathbb{R})$ for all $k \in \mathbb{Z}[1,T]$.

In recent years, the study of difference equations with p-Laplacian operator has been an interesting topic because of their applications in many fields. Some results are achieved by using fixed point theorems in cone, we refer to [1, 4, 8, 9]. There also have been a large number of papers that study the nonlinear second discrete equations by using critical point theory, see[2, 5, 7]. And in [3, 6], using different three critical points theorem, the authors have studied existence of three solutions for problem (1.1) when $\mu = 0$, i.e. problem

$$\begin{array}{c} -\triangle[\phi_p(\triangle u(k-1))] = \lambda f(k, u(k)),\ k \in \mathbb{Z}[1,T], \\ u(0) = u(T+1) = 0. \end{array} \tag{1.2}$$

We note that P. Candito and N. Giovannelli [3] established the existence of at least three solutions to (1.2) by pointing out a suitable relationship between the behavior of F with a precise bounded interval of parameters λ, where $F(\cdot,\ t) = \int_0^t f(\cdot, s)ds$. And L. Jiang and Z. Zhou [6] obtained the result for existence of at least three bounded solutions to (1.2) under some assumptions to F and the relationship between F with a bounded interval of λ.

In this paper, our approach is based on a three critical points theorem established by B.Ricceri in [11], and under appropriate assumptions, we admit the existence of an interval $[-\delta,\ \delta]$, such that, for every $\mu \in [-\delta,\ \delta]$, there exists an open interval $\Lambda_\mu \subseteq [0,\ +\infty)$ and a positive real number β_μ such that for each $\lambda \in \Lambda_\mu$, problem (1.1) admits at least three solutions whose norms in X are less then β_μ. Moreover, it is worth noting that the proof of Lemma 2.1 is completely different with respect to the proof of Lemma 2.2 in [6] and in addition, ensure the same result.

2 Preliminary

Firstly, we construct the T-dimensional Banach space $X = \{u :\ \mathbb{Z}[0, T+1] \rightarrow \mathbb{R} \mid u(0) = u(T+1) = 0\}$, endowed with the norm $\parallel u \parallel = \left(\sum_{k=1}^{T+1} \mid \triangle u(k-1) \mid^p\right)^{1/p}$.

The following theorems and lemma will be used later.

Theorem 2.1 *([10], [11]). Let X be a separable and reflexive real Banach space; $\Phi : X \longrightarrow \mathbb{R}$ a continuously Gateaux differential and sequentially weakly lower semicontinuous functional whose Gateaux derivative admits a continuous inverse on X^*; $\Psi : X \longrightarrow \mathbb{R}$ is a continuously Gateaux differentiable functional whose Gateaux derivative is compact;$I \subseteq \mathbb{R}$ an interval. Assume that*

$$\lim_{\|u\| \to +\infty} (\Phi(u) + \lambda\Psi(u)) = +\infty \tag{2.1}$$

for all $\lambda \in I$*, and that there exists a continuous concave function* $h : I \longrightarrow \mathbb{R}$ *such that*

$$\sup_{\lambda\in I}\inf_{u\in X}(\Phi(u) + \lambda\Psi(u) + h(\lambda)) < \inf_{u\in X}\sup_{\lambda\in I}(\Phi(u) + \lambda\Psi(u) + h(\lambda)). \tag{2.2}$$

Then, there exist an open interval $\Lambda \subseteq I$ *and a positive real number* β *such that, for each* $\lambda \in \Lambda$*, the equation*

$$\Phi'(u) + \lambda\Psi'(u) = 0$$

has at least three solutions in X *whose norms are less than* β*.*

Theorem 2.2 *([10]). Let* X *be a nonempty set and* Φ*,* J *two real functionals on* X*. Assume that there are* $\gamma > 0,\ u_0,\ u_1 \in X$*, such that*

$$\Phi(u_0) = J(u_0) = 0, \quad \Phi(u_1) > \gamma,$$

$$\sup_{u\in\Phi^{-1}((-\infty,\ \gamma])} J(u) < \gamma\frac{J(u_1)}{\Phi(u_1)}. \tag{2.3}$$

Then, for each ρ *satisfying*

$$\sup_{u\in\Phi^{-1}((-\infty,\ \gamma])} J(u) < \rho < \gamma\frac{J(u_1)}{\Phi(u_1)},$$

one has

$$\sup_{\lambda\geq 0}\inf_{u\in X}(\Phi(u) + \lambda(\rho - J(u))) < \inf_{u\in X}\sup_{\lambda\geq 0}(\Phi(u) + \lambda(\rho - J(u))). \tag{2.4}$$

Lemma 2.1 *Let* $\| u \|_\infty = \max_{k\in\mathbb{Z}[1,T]} | u(k) |$*, then for any* $u \in X$*, the inequality*

$$\| u \|_\infty \leq \frac{(T+1)^{\frac{p-1}{p}}}{2} \| u \|$$

holds.

Proof. Let $| u(j) | = \max_{k\in\mathbb{Z}(1,T)} | u(k) |$, since $u(0) = u(T+1) = 0$, we have

$$\begin{aligned} | u(j) | &= | u(j) - u(j+1) + u(j+1) - \cdots - u(T) + u(T) - u(T+1) | \\ &\leq \sum_{k=j+1}^{T+1} | u(k) - u(k-1) |, \end{aligned}$$

and

$$\begin{aligned} | u(j) | &= | u(j) - u(j-1) + u(j-1) - \cdots - u(1) + u(1) - u(0) | \\ &\leq \sum_{k=1}^{j} | u(k) - u(k-1) |, \end{aligned}$$

then

$$2 \mid u(j) \mid \leq \sum_{k=1}^{T+1} \mid u(k) - u(k-1) \mid = \sum_{k=1}^{T+1} \mid \triangle u(k-1) \mid .$$

Thus using the discrete Holder inequality, one has

$$\mid u(j) \mid < \frac{(T+1)^{\frac{p-1}{p}}}{2} \parallel u \parallel .$$

□

For the convenience of our proof, we define the following three functionals for all $u \in X$.

$$\Phi(u) = \frac{1}{p}\sum_{k=1}^{T+1} \mid \triangle u(k-1) \mid^p, \; J_1(u) = \sum_{k=1}^{T} F(k, u(k)), \; J_2(u) = \sum_{k=1}^{T} G(k, u(k)),$$

where $F(k,\ \xi) = \int_0^\xi f(k,s)ds$, $G(k,\xi) = \int_0^\xi g(k,s)ds$ for any $\xi \in \mathbb{R}$ and $k \in \mathbb{Z}[1,\ T]$.

Obviously, $\Phi,\ J_1,\ J_2 \in C^1(X,\mathbb{R})$, and

$$(\Phi-\lambda(J_1+\mu J_2))'(u)v = -\sum_{k=1}^{T}[\triangle\phi_p(\triangle u(k-1))+\lambda\left(f(k,u(k)) + \mu g(k,u(k))\right)]v(k).$$

So, solutions of problem (1.1) corresponds to the critical points of $\Phi - \lambda(J_1 + \mu J_2)$.

3 Main results

Theorem 3.1 *Suppose there exist four positive constants $a,\ d,\ \alpha,\ \beta$ such that $\alpha < p < \beta$ and the following conditions hold for all $k \in \mathbb{Z}[1,T]$:*

(C1) $F(k,\ d) > 0$;

(C2) $F(k,\ \xi) = o(\mid \xi \mid^\beta)$ *as* $\xi \to 0$;

(C3) $F(k,\xi), \ \mid G(k,\xi) \mid \leq a(1+ \mid \xi \mid^\alpha)$ *for all* $\xi \in \mathbb{R}$.

Then, there exists $\delta > 0$ such that, for each $\mu \in [-\delta,\ \delta]$, there exist a positive real number β_μ and an open interval $\Lambda_\mu \subset [0,+\infty)$ such that, for each $\lambda \in \Lambda_\mu$, the problem (1.1) has at least three solutions in X whose norms are less than β_μ, respectively.

Proof. Let $\Psi(u) = -J(u) = -(J_1(u) + \mu J_2(u))$, then the solutions of the problem (1.1) are equivalent to the solutions of the equation

$$\Phi'(u) + \lambda\Psi'(u) = 0.$$

From the definitions of $\Phi,\ J_1,\ J_2$, we know that Φ is a continuously Gateaux differentiable and sequentially weakly lower semi-continuous functional whose Gateaux derivative admits a continuous inverse on X^*, and Ψ is a continuously Gateaux differentiable functional whose Gateaux derivative is compact.

By $(C3)$ and Lemma 2.1, we have

$$\begin{aligned}\Phi(u) + \lambda\Psi(u) &= \tfrac{\|u\|^p}{p} - \lambda(\sum_{k=1}^{T} F(k, u(k)) + \mu \sum_{k=1}^{T} G(k, u(k)))\\ &\geq \tfrac{\|u\|^p}{p} - \lambda\left(Ta(1+ \parallel u \parallel_\infty^\alpha) - T \mid \mu \mid a(1+ \parallel u \parallel_\infty^\alpha)\right)\\ &= \tfrac{\|u\|^p}{p} - \lambda(1- \mid \mu \mid)Ta\tfrac{(T+1)^{\frac{p-1}{p}}}{2} \parallel u \parallel^\alpha - \lambda(1+ \mid \mu \mid)Ta.\end{aligned}$$

Since $\alpha < p$, we can easily get

$$\lim_{\|u\|\to+\infty} \Phi(u) + \lambda\Psi(u) = +\infty,\ \lambda \in [0, +\infty).$$

Then (2.1) of Theorem 2.1 is satisfied.

If (2.4) of Theorem 2.2 holds, let $h(\lambda) = \rho\lambda,\ \Psi(u) = -J(u)$, then we get the inequality (2.2) and all the assumptions of Theorem 2.1 are satisfied. So we only need to verify the conditions of Theorem 2.2. Let $u_0 = 0$, we can easily get $\Phi(u_0) = \Psi(u_0) = 0$.

Now, Let

$$u_1(k) = \begin{cases} d, & k \in \mathbb{Z}[1, T],\\ 0, & k \in \{0, T+1\}.\end{cases}$$

then $\Phi(u_1) = \frac{2d^p}{p}$. For each γ_1 satisfying $0 < \gamma_1 < \min\{\frac{2d^p}{p}, 1\}$ and choose $\gamma \in (0,\ \gamma_1)$, we can obtain $\Phi(u_1) > \gamma$.

Next, we will prove that the inequality (2.3) holds.

From $(C2)$, there exist $\eta \in (0, 1],\ c_3 > 0$, such that

$$F(k,\ \xi) \leq c_3 \mid \xi \mid^\beta,\ \xi \in [-\eta,\ \eta],\ k \in \mathbb{Z}[1, T]. \tag{3.1}$$

Since $\alpha < p < \beta$, if we let

$$c_4 = \max\left\{c_3,\ \sup_{|\xi|>\eta} \frac{a(1+ \mid \xi \mid^\alpha)}{\mid \xi \mid^\beta}\right\},$$

then c_4 is a finite number. Combining with $(C3)$, we have

$$F(k,\ \xi) \leq c_4 \mid \xi \mid^\beta,\ k \in \mathbb{Z}[1, T],\ \xi \in \mathbb{R}. \tag{3.2}$$

When $u \in \Phi^{-1}((-\infty,\ \gamma])$, that is, $\| u \| \leq (p\gamma)^{\frac{1}{p}}$, by Lemma 2.1, we have

$$\| u \|_\infty \leq \frac{(T+1)^{\frac{p-1}{p}}}{2} \| u \| \leq \frac{(T+1)^{\frac{p-1}{p}}(p\gamma)^{\frac{1}{p}}}{2}. \tag{3.3}$$

using the upper estimate in (3.2) and (3.3), we obtain

$$J_1(u) = \sum_{k=1}^{T} F(k, u(k)) \leq Tc_4 \| u \|_\infty^\beta \leq Tc_4 \left(\frac{(T+1)^{\frac{p-1}{p}}}{2} \right)^\beta (p\gamma)^{\frac{\beta}{p}}.$$

Since $\beta > p$, one has

$$\lim_{\gamma \to 0^+} \frac{\sup_{u \in \Phi^{-1}((-\infty,\ \gamma])} J_1(u)}{\gamma} \leq 0. \tag{3.4}$$

By (C1) we can get $J_1(u_1) > 0$, so when $\gamma \in (0,\ \gamma_1)$ is small enough, use (3.4) to obtain

$$\sup_{u \in \Phi^{-1}((-\infty,\ \gamma])} J_1(u) \leq \frac{\gamma}{2} \frac{J_1(u_1)}{\Phi(u_1)}. \tag{3.5}$$

Furthermore, from the continuity of J_2 and (3.3), we know that $\sup_{u \in \Phi^{-1}((-\infty,\ \gamma])} | J_2(u) |$ is a finite number, so there exists $\delta > 0$ satisfying

$$\delta \left(\sup_{u \in \Phi^{-1}((-\infty,\ \gamma])} | J_2(u) | + \gamma \frac{| J_2(u_1) |}{\Phi(u_1)} \right) < \frac{\gamma}{2} \frac{J_1(u_1)}{\Phi(u_1)}. \tag{3.6}$$

Now, when $\mu \in [-\delta,\ \delta]$, from (3.5) and (3.6), we have

$$\begin{aligned} \sup_{u \in \Phi^{-1}((-\infty,\ \gamma])} (J_1(u) + \mu J_2(u)) &\leq \sup_{u \in \Phi^{-1}((-\infty,\ \gamma])} J_1(u) + \sup_{u \in \Phi^{-1}((-\infty,\ \gamma]} \delta \mid J_2(u) \mid \\ &< \tfrac{\gamma}{2} \tfrac{J_1(u_1)}{\Phi(u_1)} + \tfrac{\gamma}{2} \tfrac{J_1(u_1)}{\Phi(u_1)} - \gamma \tfrac{\delta | J_2(u_1) |}{\Phi(u_1)} \\ &\leq \gamma \tfrac{J_1(u_1) + \mu J_2(u_1)}{\Phi(u_1)}, \end{aligned}$$

that is,

$$\sup_{u \in \Phi^{-1}((-\infty,\ \gamma])} J(u) < \gamma \frac{J(u_1)}{\Phi(u_1)}.$$

Then, all the assumptions of Theorem 2.2 are satisfied and the proof is complete. □

Example 3.2 *We consider problem* (1.1) *with* $T = 10,\ p = 4,\ k \in \mathbb{Z}[1, 10]$, *and*

$$f(k,\ u) = \begin{cases} ku^6, & | u | \leq 1, \\ ku^2 + | u | - 1, & | u | > 1. \end{cases}, \quad g(k,\ u) = \begin{cases} k(u^6 + u^8), & | u | \leq 1, \\ 2ku^2 + | u | - 1, & | u | > 1. \end{cases}$$

then,

$$F(k,\ u) = \begin{cases} \frac{1}{7}ku^7, & \mid u \mid \leq 1, \\ \frac{1}{3}ku^3 + \frac{1}{2} \mid u \mid u - u, & \mid u \mid > 1. \end{cases},$$

$$G(k,\ u) = \begin{cases} k(\frac{1}{7}u^7 + \frac{1}{9}u^9), & \mid u \mid \leq 1, \\ \frac{2}{3}ku^3 + \frac{1}{2} \mid u \mid u - u, & \mid u \mid > 1. \end{cases}$$

Let $d = 2,\ \beta = 5,$ *and* $\alpha = 3$, *then, all the conditions of theorem* 3.1 *are satisfied and problem* (1.1) *has at least three solutions when* $T,\ p,\ f,\ g$ *are defined as above.*

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