Existence of Three Solutions for Difference Equation Involving P-Laplacian

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Abstract

In this paper, using a three critical points theorem established by B. Ricceri and variational methods, we study the existence of solutions of the Dirichlet boundary value problem for p-Laplacian difference equation depending on two parameters λ, µ, and obtain the existence of three solutions under some appropriate assumptions.

Keywords: P-Laplacian; Three solutions; Boundary value problem; Variational methods

1 Introduction

Denote by \( \mathbb{N}, \mathbb{Z} \) and \( \mathbb{R} \) the sets of all natural numbers, integers and real numbers. For \( a, b \in \mathbb{Z}, \mathbb{Z}(a) = \{ a, a+1, \ldots \}, \mathbb{Z}[a, b] = \{ a, a+1, \ldots, b \} \) when \( a \leq b \).

We will consider the following p-Laplacian difference equation with Dirichlet boundary value condition

\[
-\Delta [\phi_p(\Delta u(k-1))] = \lambda(f(k, u(k)) + \mu g(k, u(k))), \quad k \in \mathbb{Z}[1, T],
\]

\[
u(0) = u(T+1) = 0.
\]

(1.1)

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where $T$ is a positive integer, $p$, $\lambda$, $\mu \in \mathbb{R}$ are constants and $p > 1$, $\lambda > 0$, $\Delta$ is defined by $\Delta u(k) = u(k+1) - u(k)$, $\phi_p(s) = |s|^{p-2}s$ is the $p$-Laplacian operator, $f(k, \cdot)$, $g(k, \cdot) \in C(\mathbb{R}, \mathbb{R})$ for all $k \in \mathbb{Z}[1, T]$.

In recent years, the study of difference equations with $p$-Laplacian operator has been an interesting topic because of their applications in many fields. Some results are achieved by using fixed point theorems in cone, we refer to [1, 4, 8, 9]. There also have been a large number of papers that study the nonlinear second discrete equations by using critical point theory, see [2, 5, 7]. And in [3, 6], using different three critical points theorem, the authors have studied existence of three solutions for problem (1.1) when $\mu = 0$, i.e. problem

$$-\Delta[\phi_p(\Delta u(k-1))] = \lambda f(k, u(k)), \quad k \in \mathbb{Z}[1, T],$$
$$u(0) = u(T + 1) = 0.$$ (1.2)

We note that P. Candito and N. Giovannelli [3] established the existence of at least three solutions to (1.2) by pointing out a suitable relationship between the behavior of $F$ with a precise bounded interval of parameters $\lambda$, where $F(\cdot, t) = \int_0^t f(\cdot, s)ds$. And L. Jiang and Z. Zhou [6] obtained the result for existence of at least three bounded solutions to (1.2) under some assumptions to $F$ and the relationship between $F$ with a bounded interval of $\lambda$.

In this paper, our approach is based on a three critical points theorem established by B. Ricceri in [11], and under appropriate assumptions, we admit the existence of an open interval $[-\delta, \delta]$, such that, for every $\mu \in [-\delta, \delta]$, there exists an open interval $\Lambda_\mu \subseteq [0, +\infty)$ and a positive real number $\beta_\mu$ such that for each $\lambda \in \Lambda_\mu$, problem (1.1) admits at least three solutions whose norms in $X$ are less then $\beta_\mu$. Moreover, it is worth noting that the proof of Lemma 2.1 is completely different with respect to the proof of Lemma 2.2 in [6] and in addition, ensure the same result.

2 Preliminary

Firstly, we construct the $T$-dimensional Banach space $X = \{u : \mathbb{Z}[0, T + 1] \to \mathbb{R} | u(0) = u(T+1) = 0\}$, endowed with the norm $\|u\| = \left(\sum_{k=1}^{T+1} |\Delta u(k-1)|^p\right)^{1/p}$.

The following theorems and lemma will be used later.

Theorem 2.1 ([10], [11]). Let $X$ be a separable and reflexive real Banach space; $\Phi : X \to \mathbb{R}$ a continuously Gateaux differential and sequentially weakly lower semicontinuous functional whose Gateaux derivative admits a continuous weakly inverse on $X^*$; $\Psi : X \to \mathbb{R}$ is a continuously Gateaux differentiable functional whose Gateaux derivative is compact $; I \subseteq \mathbb{R}$ an interval. Assume that

$$\lim_{\|u\| \to +\infty} (\Phi(u) + \lambda \Psi(u)) = +\infty$$ (2.1)
for all $\lambda \in I$, and that there exists a continuous concave function $h : I \rightarrow \mathbb{R}$ such that
\[
\sup_{\lambda \in I} \inf_{u \in X} (\Phi(u) + \lambda \Psi(u) + h(\lambda)) < \inf_{u \in X} \sup_{\lambda \in I} (\Phi(u) + \lambda \Psi(u) + h(\lambda)).
\] (2.2)

Then, there exist an open interval $\Lambda \subseteq I$ and a positive real number $\beta$ such that, for each $\lambda \in I$, the equation
\[
\Phi'(u) + \lambda \Psi'(u) = 0
\]
has at least three solutions in $X$ whose norms are less than $\beta$.

**Theorem 2.2 ([10])**. Let $X$ be a nonempty set and $\Phi, J$ two real functionals on $X$. Assume that there are $\gamma > 0$, $u_0, u_1 \in X$, such that
\[
\Phi(u_0) = J(u_0) = 0, \quad \Phi(u_1) > \gamma,
\]
and
\[
\sup_{u \in \Phi^{-1}((-\infty, \gamma])} J(u) < \gamma \frac{J(u_1)}{\Phi(u_1)}.
\] (2.3)

Then, for each $\rho$ satisfying
\[
\sup_{u \in \Phi^{-1}((-\infty, \gamma])} J(u) < \rho < \gamma \frac{J(u_1)}{\Phi(u_1)},
\]
one has
\[
\sup_{\lambda \geq 0} \inf_{u \in X} (\Phi(u) + \lambda (\rho - J(u))) < \inf_{u \in X} \sup_{\lambda \geq 0} (\Phi(u) + \lambda (\rho - J(u))).
\] (2.4)

**Lemma 2.1**. Let $\|u\|_{\infty} = \max_{k \in \mathbb{Z}} |u(k)|$, then for any $u \in X$, the inequality
\[
\|u\|_{\infty} \leq \frac{(T + 1)^{p-1}}{2} \|u\|
\]
holds.

**Proof.** Let $|u(j)| = \max_{k \in \mathbb{Z} \backslash \{1, T\}} |u(k)|$, since $u(0) = u(T + 1) = 0$, we have
\[
|u(j)| = |u(j) - u(j+1) + u(j+1) - \cdots - u(T) + u(T) - u(T+1)|
\leq \sum_{k=j+1}^{T+1} |u(k) - u(k-1)|,
\]
and
\[
|u(j)| = |u(j) - u(j-1) + u(j-1) - \cdots - u(1) + u(1) - u(0)|
\leq \sum_{k=1}^{j} |u(k) - u(k-1)|,
\]
then
\[ 2 | u(j) | \leq \sum_{k=1}^{T+1} | u(k) - u(k-1) | = \sum_{k=1}^{T+1} | \Delta u(k-1) |. \]

Thus using the discrete H"older inequality, one has
\[
| u(j) | < \frac{(T + 1)^{p-1}}{2} \| u \|. 
\]

□

For the convenience of our proof, we define the following three functionals for all \( u \in X \).
\[
\Phi(u) = \frac{1}{p} \sum_{k=1}^{T+1} | \Delta u(k-1) |^p, \quad J_1(u) = \sum_{k=1}^{T} F(k, u(k)), \quad J_2(u) = \sum_{k=1}^{T} G(k, u(k)),
\]

where \( F(k, \xi) = \int_0^\xi f(k, s)ds, \ G(k, \xi) = \int_0^\xi g(k, s)ds \) for any \( \xi \in \mathbb{R} \) and \( k \in \mathbb{Z}[1, T] \).

Obviously, \( \Phi, \ J_1, \ J_2 \in C^1(X, \mathbb{R}) \), and
\[
(\Phi-\lambda(J_1+\mu J_2))'(u)v = -\sum_{k=1}^{T} [\Delta \phi_p(\Delta u(k-1))+\lambda (f(k, u(k)) + \mu g(k, u(k)))]v(k). 
\]

So, solutions of problem (1.1) corresponds to the critical points of \( \Phi - \lambda(J_1 + \mu J_2) \).

3 Main results

\textbf{Theorem 3.1} Suppose there exist four positive constants \( a, \ d, \ \alpha, \ \beta \) such that \( \alpha < p < \beta \) and the following conditions hold for all \( k \in \mathbb{Z}[1, T] \):

\begin{enumerate}
\item[(C1)] \( F(k, d) > 0; \)
\item[(C2)] \( F(k, \xi) = o(\| \xi \|^\beta) \) as \( \xi \to 0; \)
\item[(C3)] \( F(k, \xi), \ |G(k, \xi)| \leq a(1 + |\xi|^\alpha) \) for all \( \xi \in \mathbb{R} \).
\end{enumerate}

Then, there exists \( \delta > 0 \) such that, for each \( \mu \in [-\delta, \delta] \), there exist a positive real number \( \beta_\mu \) and an open interval \( \Lambda_\mu \subset [0, +\infty) \) such that, for each \( \lambda \in \Lambda_\mu \), the problem (1.1) has at least three solutions in \( X \) whose norms are less than \( \beta_\mu \), respectively.
Proof. Let $\Psi(u) = -J(u) = -(J_1(u) + \mu J_2(u))$, then the solutions of the problem (1.1) are equivalent to the solutions of the equation

$$
\Phi'(u) + \lambda \Psi'(u) = 0.
$$

From the definitions of $\Phi$, $J_1$, $J_2$, we know that $\Phi$ is a continuously Gateaux differentiable and sequentially weakly lower semi-continuous functional whose Gateaux derivative admits a continuous inverse on $X^*$, and $\Psi$ is a continuously Gateaux differentiable functional whose Gateaux derivative is compact.

By (C3) and Lemma 2.1, we have

$$
\Phi(u) + \lambda \Psi(u) = \frac{\|u\|^p}{p} - \lambda \left( \frac{T}{\|u\|^\beta} \right) - \mu \left( a(1 + \|u\|^\alpha) \right)
$$

Since $\alpha < p$, we can easily get

$$
\lim_{\|u\| \to +\infty} \Phi(u) + \lambda \Psi(u) = +\infty, \lambda \in [0, +\infty).
$$

Then (2.1) of Theorem 2.1 is satisfied.

If (2.4) of Theorem 2.2 holds, let $h(\lambda) = \rho \lambda$, $\Psi(u) = -J(u)$, then we get the inequality (2.2) and all the assumptions of Theorem 2.1 are satisfied. So we only need to verify the conditions of Theorem 2.2. Let $u_0 = 0$, we can easily get $\Phi(u_0) = \Psi(u_0) = 0$.

Now, Let

$$
u_1(k) = \begin{cases} d, & k \in \mathbb{Z}[1, T], \\ 0, & k \in \{0, T+1\}. \end{cases}
$$

then $\Phi(u_1) = \frac{2d^p}{p}$. For each $\gamma_1$ satisfying $0 < \gamma_1 < \min\left\{ \frac{2d^p}{p}, 1 \right\}$ and choose $\gamma \in (0, \gamma_1)$, we can obtain $\Phi(u_1) > \gamma$.

Next, we will prove that the inequality (2.3) holds.

From (C2), there exist $\eta \in (0, 1]$, $c_3 > 0$, such that

$$
F(k, \xi) \leq c_3 \left| \xi \right|^\beta, \xi \in [-\eta, \eta], k \in \mathbb{Z}[1, T].
$$

(3.1)

Since $\alpha < p < \beta$, if we let

$$
c_4 = \max \left\{ c_3, \sup_{|\xi| > \eta} \frac{a(1 + |\xi|^\alpha)}{|\xi|^\beta} \right\},
$$

then $c_4$ is a finite number. Combining with (C3), we have

$$
F(k, \xi) \leq c_4 \left| \xi \right|^\beta, k \in \mathbb{Z}[1, T], \xi \in \mathbb{R}.
$$

(3.2)
When \( u \in \Phi^{-1}((-\infty, \gamma]) \), that is, \( \|u\| \leq (p\gamma)^{\frac{1}{p}} \), by Lemma 2.1, we have
\[
\|u\|_\infty \leq \frac{(T + 1)^{\frac{p - 1}{p}}}{2} \|u\| \leq \frac{(T + 1)^{\frac{p - 1}{p}}}{2} (p\gamma)^{\frac{1}{p}}.
\] (3.3)

Using the upper estimate in (3.2) and (3.3), we obtain
\[
J_1(u) = \sum_{k=1}^{T} F(k, u(k)) \leq Tc_4 \|u\|_\infty^p \leq Tc_4 \left( \frac{(T + 1)^{\frac{p - 1}{p}}}{2} \right)^{\frac{p}{2}} (p\gamma)^{\beta}. \]

Since \( \beta > p \), one has
\[
\lim_{\gamma \to 0^+} \sup_{u \in \Phi^{-1}((-\infty, \gamma])} \frac{J_1(u)}{\gamma} \leq 0.
\] (3.4)

By (C1) we can get \( J_1(u_1) > 0 \), so when \( \gamma \in (0, \gamma_1) \) is small enough, use (3.4) to obtain
\[
\sup_{u \in \Phi^{-1}((-\infty, \gamma])} J_1(u) \leq \frac{\gamma J_1(u_1)}{2 \Phi(u_1)}. \] (3.5)

Furthermore, from the continuity of \( J_2 \) and (3.3), we know that \( \sup_{u \in \Phi^{-1}((-\infty, \gamma])} |J_2(u)| \) is a finite number, so there exists \( \delta > 0 \) satisfying
\[
\delta \left( \sup_{u \in \Phi^{-1}((-\infty, \gamma])} |J_2(u)| + \gamma \frac{|J_2(u_1)|}{\Phi(u_1)} \right) < \frac{\gamma J_1(u_1)}{2 \Phi(u_1)}. \] (3.6)

Now, when \( \mu \in [-\delta, \delta] \), from (3.5) and (3.6), we have
\[
\sup_{u \in \Phi^{-1}((-\infty, \gamma])} (J_1(u) + \mu J_2(u)) \leq \sup_{u \in \Phi^{-1}((-\infty, \gamma])} J_1(u) + \sup_{u \in \Phi^{-1}((-\infty, \gamma])} \delta |J_2(u)| \]
\[
< \frac{\gamma J_1(u_1)}{2 \Phi(u_1)} + \frac{\gamma J_1(u_1)}{2 \Phi(u_1)} - \gamma \frac{\delta |J_2(u_1)|}{\Phi(u_1)} \]
\[
\leq \frac{\gamma J_1(u_1) + \mu J_2(u_1)}{\Phi(u_1)},
\]
that is,
\[
\sup_{u \in \Phi^{-1}((-\infty, \gamma])} J(u) < \gamma \frac{J(u_1)}{\Phi(u_1)}.
\]

Then, all the assumptions of Theorem 2.2 are satisfied and the proof is complete. \( \square \)

**Example 3.2** We consider problem (1.1) with \( T = 10, \ p = 4, \ k \in \mathbb{Z}[1,10] \), and
\[
f(k, u) = \begin{cases} ku^6, & |u| \leq 1, \\ ku^2 + |u| - 1, & |u| > 1, \end{cases} \quad g(k, u) = \begin{cases} k(u^6 + u^8), & |u| \leq 1, \\ 2ku^2 + |u| - 1, & |u| > 1. \end{cases}
\]
then,

$$F(k, u) = \begin{cases} \frac{1}{7}ku^7, & |u| \leq 1, \\ \frac{1}{3}ku^3 + \frac{1}{2} |u - u|, & |u| > 1. \end{cases}$$

$$G(k, u) = \begin{cases} k(\frac{1}{7}u^7 + \frac{1}{3}u^9), & |u| \leq 1, \\ \frac{2}{3}ku^3 + \frac{1}{2} |u - u|, & |u| > 1. \end{cases}$$

Let $d = 2$, $\beta = 5$, and $\alpha = 3$, then, all the conditions of theorem 3.1 are satisfied and problem (1.1) has at least three solutions when $T$, $p$, $f$, $g$ are defined as above.

References


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