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On a New Generalized Polynomial

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Abstract

Objective of this paper is to define the Laplace formula for theory of probability by using our newly defined Generalized Bernstein type polynomial

$$A_n(f,x) = \sum_{k=0}^{n} f(\frac{k}{n})\, p_k(x;\alpha)$$

where $$p_k(x;\alpha) = \binom{n}{k} x(x+k\alpha)^{k-1}(1-x-k\alpha)^{n-k}$$

Keywords: Bernstein Polynomials, Laplace formula, Generalized Bernstein type polynomials, Sterling formula, Asymptotic Relation

I. Introduction and Results

If $f(x)$ is a function defined on [0, 1], the Bernstein polynomial $B_n^f(x)$ of f is given as

$$B_n^f(x) = \sum_{k=0}^{n} f(k/n)\binom{n}{k}\, x^k(1-x)^{n-k} \qquad (1.1)$$

Anwar [4] define the Bernstein polynomial as

$$A_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_k(x;\alpha) \qquad (1.2)$$

where $p_k(x;\alpha) = \binom{n}{k} x(x+k\alpha)^{k-1}(1-x-k\alpha)^{n-k}$

such that $\sum_{k=0}^{n} p_k(x;\alpha) = 1$ (1.3)

Theorem A: (The Laplace formula for theory of probability):

For fixed $x \in (0,1), \beta > \frac{1}{3}$, the asymptotic relation

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

$$\cong [2\pi(1-x)n]^{-1/2} \exp\left[-\frac{n}{2x(1-x)}\left(\frac{k}{n}-x\right)^2\right] = P_{n,k}(x)$$

holds uniformly for all values of k satisfying the inequality

$$\left|\frac{k}{n}-x\right| \le n^{-\beta} \qquad (1.4)$$

In other words $\lim_{n\to\infty} \frac{p_{n,k}(x)}{P_{n,k}(x)} = 1$ uniformly for all k satisfying (1.4)

The extension of Laplace theorem can be given as, for our newly defined polynomial,

Theorem B: For fixed $x \in (0,1), \beta > \frac{1}{3}$, the asymptotic relation

$$p_k(x;\alpha) = \binom{n}{k} x(x+k\alpha)^{k-1}(1-x-k\alpha)^{n-k}$$

$$\cong [2\pi(1-x)n]^{-1/2} \exp\left[-\frac{n}{2x(1-x)}\left(\frac{k}{n}-x\right)^2\right] = A_{n,k}(x)$$

holds uniformly for all values of k satisfying the inequality

$$\left|\frac{k}{n}-x\right| \le n^{-\beta}$$

II. Proof of theorem: B

Using Sterling formula

$$n! = (2\pi n)^{1/2} n^n e^{-n} H_n \ , \ H_n \to 1 \text{ for } n \to \infty$$

we get

$$p_{n,k}(x) =$$

$$\frac{n^{1/2}}{\sqrt{2\pi k(n-k)}} \cdot \frac{n^n}{k^k (n-k)^{(n-k)}} x(x+\alpha)^{k-1}(1-x-k\alpha)^{n-k}.H_{n,k} \tag{2.1}$$

where $H_{n,k} \to 1 \ as \ n \to \infty, \alpha \to 0$

uniformly for all k satisfying $\left|\frac{k}{n} - x\right| \le n^{-\beta}$

since

$$\left(\frac{n^{1/2}}{\sqrt{2\pi k(n-k)}}\right)^2 = \frac{n}{2\pi k(n-k)} = \frac{n}{2\pi\left(\frac{k}{n}\right)(1-\left(\frac{k}{n}\right))n^2} \approx \frac{1}{2\pi x(1-x)n}$$

$$\frac{n^{1/2}}{\sqrt{2\pi k(n-k)}} \approx \sqrt{\frac{1}{2\pi x(1-x)n}} \tag{2.2}$$

let

$$w = \frac{n^n}{k^k (n-k)^{(n-k)}} x(x+\alpha)^{k-1}(1-x-k\alpha)^{n-k}$$

$$w = \frac{n\, n^{k-1} n^{n-k}}{k^k (n-k)^{(n-k)}} x(x+\alpha)^{k-1}(1-x-k\alpha)^{n-k}$$

$$w = \frac{nn^{k-1}n^{n-k}}{kk^{k-1}(n-k)^{n-k}} x(x+\alpha)^{k-1}(1-x-k\alpha)^{n-k}$$

$$w = \left(\frac{nx}{k}\right) [\frac{n(x+k\alpha)}{k}]^{k-1} [\frac{n(1-x-k\alpha)}{n-k}]^{n-k}$$

$$-\log w = \log\left[1 + x^{-1}\left(\frac{k}{n} - x\right)\right] + (k-1)\log[\frac{k}{n(x+k\alpha)}] + (n-k)\log[\frac{n-k}{n(1-x-k\alpha)}]$$

$$= \log\left[1 + x^{-1}\left(\frac{k}{n} - x\right)\right] + (k-1)\log[1 + (x+k\alpha)^{-1}\{\left(\frac{k}{n} - x\right) - k\alpha\}] + (n-k)\log[1 - (1-x-k\alpha)^{-1}\{\left(\frac{k}{n} - x\right) - k\alpha\}]$$

$$= \log\left[1 + x^{-1}\left(\frac{k}{n} - x\right)\right] + (k-1)\log[1 + (x+k\alpha)^{-1}\{\left(\frac{k}{n} - x\right) - k\alpha\}] + (n-k)\log[1 - (1-x-k\alpha)^{-1}\{\left(\frac{k}{n} - x\right) - k\alpha\}] \quad (2.3)$$

By Taylor's theorem

log(1+u)= $u - \frac{u^2}{2} + \frac{u^3}{3} + \frac{u^4}{4}(1+\theta u)^{-4}$, $0<\theta<1$, $\rho = 1 + \varepsilon u$

log(1+u)= $u - \frac{1}{2}u^2 \ \rho_1$; $\rho_1 = 1 + \varepsilon_1 u$

log(1+u)= $u - \frac{1}{2}u^2 \ \rho_2$; $\rho_2 = 1 + \varepsilon_2 u$

ρ_1 & ρ_2 Σ_2 are bounded as u → 0

neglecting higher power terms, we can get

$$\log\left[1 + x^{-1}\left(\frac{k}{n} - x\right)\right] = x^{-1}\left(\frac{k}{4} - x\right) - \frac{1}{2}x^{-2}(\frac{k}{n} - x)^2 \rho_1 \quad (2.4)$$

$$log\left\{1 + (x+k\alpha)^{-1}\left[\left(\frac{k}{n} - x\right) - k\alpha\right]\right\} = (x+k\alpha)^{-1}\left\{\left(\frac{k}{n} - x\right) - k\alpha\right\} - \frac{1}{2}(x+k\alpha)^{-1}\{\left(\frac{k}{n} - x\right) - k\alpha\}^2 \rho_2 \quad (2.5)$$

$$log\left\{1 - (1-x-k\alpha)^{-1}\left[\left(\frac{k}{n} - x\right) - k\alpha\right]\right\} = -(1-x-k\alpha)^{-1}\left\{\left(\frac{k}{n} - x\right) - k\alpha\right\} - \frac{1}{2}(1-x-k\alpha)^{-2}\{\left(\frac{k}{n} - x\right) - k\alpha\}^2 \rho_3 \quad (2.6)$$

$$1 = \frac{nx}{k}(1 + x^{-1}(\frac{k}{n} - x)) \quad (2.7)$$

$$k - 1 = \left[n(x+k\alpha)[1 + (x+k\alpha)^{-1}\{\left(\frac{k}{n} - x\right) - k\alpha\}\right] - \frac{nx}{k}(1 + x^{-1}(\frac{k}{n} - x)) \quad (2.8)$$

$$n-k=\left[n(1-x-k\alpha)[1-(1-x-k\alpha)^{-1}\{\left(\frac{k}{n}-x\right)-k\alpha\}\right] \tag{2.9}$$

substituting the values from (2.4), (2.5), (2.6), (2.7), (2.8) &(2.9) in, (2.3), we get

$$-\log w=\frac{nx}{k}\left[1+x^{-1}\left(\frac{k}{n}-x\right)\right]\left[x^{-1}\left(\frac{k}{n}-x\right)-\frac{1}{2}x^{-2\left(\frac{k}{n}-x\right)^2}P^1\right]$$
$$+\left[n(x+k\alpha)[1+(x+k\alpha)^{-1}\{\left(\frac{k}{n}-x\right)-k\alpha\}\right]$$
$$-\frac{nx}{k}\left(1+x^{-1}\left(\frac{k}{n}-x\right)\right)+\frac{nx}{k}\left(1+x^{-1}\left(\frac{k}{n}-x\right)\right)$$

$$\left\{(x+k\alpha)^{-1}\left[\left(\frac{k}{n}-x\right)-k\alpha\right]\right\}-\frac{1}{2}(x+k\alpha)^{-2}\left\{\left(\frac{k}{n}-x\right)-k\alpha\right\}^2\rho_1-n(1-x$$
$$-k\alpha)[1-(1-x-k\alpha)^{-1}(\frac{k}{n}-x-k\alpha)$$

$$[(1-x-k\alpha)^{-1}(\left(\frac{k}{n}-x\right)-k\alpha+\frac{1}{2}\left((1-x)-k\alpha\right)^{-2}\left(\left(\frac{k}{n}-x\right)-k\alpha\right)^2\rho_2]$$

On taking absolute value of right hand side of each term it will be less than

$$\text{M n}\left|\frac{k}{n}-x\right|^3\leq Mn.n^{-3\beta}=Mn^{1-3\beta}\quad\text{for}\quad\alpha=\alpha_n=o(\frac{1}{n})$$

$\Rightarrow$ it converges uniformly to zero

Therefore, we have

$$-w\approx\frac{n}{2x(1-x)}(\frac{k}{n}-x)^2\quad\text{for}\quad\alpha=\alpha_n=o(\frac{1}{n})$$

$$w=e^{-\frac{n}{2(x-1-x)}(\frac{k}{n}-x)^2}\quad\text{satisfying}\quad\left|\frac{k}{n}-x\right|\leq n^{-\beta} \tag{2.10}$$

substituting the values in (2.1) from (2.2) and (2.10) , we get

$$p_{n,k}(x) = \sqrt{\frac{1}{2\pi x(1-x)n}}\ e^{-\frac{n}{2(x-1-x)}(\frac{k}{n}-x)^2} \text{ holds}$$

satisfying $\left|\frac{k}{n} - x\right| \le n^{-\beta}$

III. Conclusion

The results of Laplace theorem for the theory of probability have been extended by our newly defined Generalized Polynomials.

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