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Bounds for Region Containing All Zeros of a Complex Polynomial

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Abstract

In this paper, the notion of obtaining a bound of region containing all the zeros of a polynomial is carried forward by obtaining new bounds which improves the accuracy of some of the existing bounds for some polynomials with complex coefficients. Examples and comparisons of such polynomials are also given to show that the newly obtained bounds for a given polynomial are not always redundant.

Keywords: location of zeros, polynomials, inequalities, complex coefficients

1. Introduction

Solving polynomial equations is one of the oldest problems in mathematics and plays an important role in other scientific disciplines as well. Determining the location of zeros and critical points is of particular importance in many fields related to applied mathematics such as control theory, signal processing, coding theory, cryptography and combinatorics.
It becomes increasingly difficult to find zeros of higher order polynomials and putting constraints on the region where all zeros lie, significantly reduces the computational load. One of the earliest results in this regard was probably by Gauss, who showed the following:
Given a polynomial,

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \ ; a_k \in \mathbb{R},$$

has no zeros outside the disk $|z| = R$, where

$$R = max_{1\le k\le n}\left(n\sqrt{2}|a_k|\right)^{\frac{1}{k}}$$

Cauchy [1] proved the following theorem which sharpens the bound in which the zeros of the polynomial are located.

Theorem 1.1: Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$, be a complex polynomial. Then all the zeros of $p(z)$ lie in the disc

$$\{z: |z| < \eta\} \subset \{z: |z| < 1 + A\} \text{ where, } A = max_{0\le k\le n-1}|a_k|$$

η is the unique positive root of the real-coefficient equation

$$z^n - |a_{n-1}|z^{n-1} - |a_{n-2}|z^{n-2} - \cdots - |a_1|z - |a_0| = 0$$

The following result provides an annulus containing all the zeros of a polynomial, which is due to Diaz-Barrero [6].

Theorem 1.2: Let $p(z) = \sum_{k=0}^{n} a_k z^k (a_k \neq 0, 0 \le k \le n)$ be a non-constant complex polynomial. Then all its zeros lie in the annulus, $C = \{z: r_1 \le |z| \le r_2\}$, where

$$r_1 = min_{1\le k\le n}\left\{\frac{2^{n-k}3^k F_k C(n,k)}{F_{4n}}\frac{|a_0|}{|a_k|}\right\}^{\frac{1}{k}}$$

$$r_2 = max_{1\le k\le n}\left\{\frac{F_{4n}}{2^{n-k}3^k F_k C(n,k)}\frac{|a_{n-k}|}{|a_n|}\right\}^{\frac{1}{k}}$$

Here F_k is the k^{th} Fibonacci number, defined as follows:
$F_0 = 0, F_1 = 1$ and $F_k = F_{k-1} + F_{k-2}$ for $k \ge 2$.
$C(n,k)$ is the binomial coefficient defined as $C(n,k) = \frac{n!}{k!(n-k)!}$.
The following result also provides an annulus containing all the zeros of a polynomial with complex coefficients which is due to Kim [15].

Theorem 1.2: Let $p(z) = \sum_{k=0}^{n} a_k z^k (a_k \neq 0, 0 \le k \le n)$ be a non-constant complex polynomial. Then all its zeros lie in the annulus, $C = \{z: r_1 \le |z| \le r_2\}$, where

$$r_1 = min_{1\le k\le n}\left\{\frac{C(n,k)}{2^n - 1}\frac{|a_0|}{|a_k|}\right\}^{\frac{1}{k}}$$

$$r_2 = max_{1\le k\le n}\left\{\frac{2^n - 1}{C(n,k)}\frac{|a_{n-k}|}{|a_n|}\right\}^{\frac{1}{k}}$$

$C(n,k)$ is the binomial coefficient defined as $C(n,k) = \frac{n!}{k!(n-k)!}$.
The following theorem and its corollaries are given by Dalal and Govil [2]

Theorem 1.3: Let $A_k > 0$ for $1 \le k \le n$, and be such that $\sum_{k=1}^{n} A_k = 1$.
If $p(z) = \sum_{k=0}^{n} a_k z^k (a_k \ne 0, 0 \le k \le n)$ is a non-constant complex polynomial, then all the zeros of $p(z)$ lie in the annulus, $C = \{z: r_1 \le |z| \le r_2\}$, where

$$r_1 = min_{1\le k\le n} \left\{A_k \frac{|a_0|}{|a_k|}\right\}^{\frac{1}{k}}$$

$$r_2 = max_{1\le k\le n} \left\{\frac{1}{A_k} \frac{|a_{n-k}|}{|a_n|}\right\}^{\frac{1}{k}}.$$

Using this result they also proved the following theorem which involves some special numbers.

Theorem 1.4: Let $p(z) = \sum_{k=0}^{n} a_k z^k (a_k \ne 0, 0 \le k \le n)$ is a non-constant complex polynomial, then all the zeros of $p(z)$ lie in the annulus, $C = \{z: r_1 \le |z| \le r_2\}$, where

$$r_1 = min_{1\le k\le n} \left\{\frac{L_k}{L_{n+2} - 3} \frac{|a_0|}{|a_k|}\right\}^{\frac{1}{k}}$$

$$r_2 = max_{1\le k\le n} \left\{\frac{L_{n+2} - 3}{L_k} \frac{|a_{n-k}|}{|a_n|}\right\}^{\frac{1}{k}}$$

Here, L_k is the k^{th} Lucas number defined by $L_0 = 2, L_1 = 1$ *and* $L_{n+2} = L_n + L_{n+1,}$ *if* $n \ge 0$.
This corollary is obtained by taking $A_k = \frac{L_k}{L_{n+2}-3}$ in theorem 1.3. $\sum_{k=1}^{n} A_k = 1$ [16].

Theorem 1.5: Let $p(z) = \sum_{k=0}^{n} a_k z^k (a_k \ne 0, 0 \le k \le n)$ is a non-constant complex polynomial, then all the zeros of $p(z)$ lie in the annulus, $C = \{z: r_1 \le |z| \le r_2\}$, where

$$r_1 = min_{1\le k\le n} \left\{\frac{C_{k-1}C_{n-k}}{C_n} \frac{|a_0|}{|a_k|}\right\}^{\frac{1}{k}}$$

$$r_2 = max_{1\le k\le n} \left\{\frac{C_n}{C_{k-1}C_{n-k}} \frac{|a_{n-k}|}{|a_n|}\right\}^{\frac{1}{k}}$$

Here, C_n is the k^{th} Catalan number defined as: $C_k = \frac{C(2k,k)}{k+1}$; $C(2k,k) = \binom{2k}{k}$.

Theorem 1.5 is obtained by taking $A_k = \frac{C_{k-1}C_{n-k}}{C_n}$. $\sum_{k=1}^{n} A_k = 1$ [7].

2. Main Results

Two new results are presented to find the annular ring containing all the zeros of a polynomial. Consequently, polynomials of degrees 3, 4 and 5 have been constructed to show that the bound obtained by this corollary improves at least one of the parameters of the region containing all the zeros of a polynomial (r_1, r_2 and the area of the region, given by $\pi(r_2^2 - r_1^2)$) when compared against theorems 1.1 -1.5. Improvement in one or all of these parameters of the region is sufficient to claim that the bound obtained by this result is not redundant.

The main results of this paper are obtained by using various number sequences as A_k. Here, $L(n,k)$ is the unsigned Lah number, which is equivalent to counting the number of ways a set of n elements can be partitioned into k non-empty subsets, defined as

$$L(n,k) = \binom{n-1}{k-1}\frac{n!}{k!}.$$

${}_1F_1(1+n,2,1)$ is the Kummer confluent hypergeometric function and e is Euler constant.

The sum $\sum_{k=1}^{n} L(n,k) = \frac{n!}{e}\,{}_1F_1(n+1,2,1)$ is proven by Feng Qi in [13]. Applying this identity to Theorem 1.3 we easily get the following result.

Theorem 2.1: Let $p(z) = \sum_{k=0}^{n} a_k z^k (a_k \neq 0, 0 \leq k \leq n)$ is a non-constant complex polynomial, then all the zeros of $p(z)$ lie in the annulus, $C = \{z: r_1 \leq |z| \leq r_2\}$, where

$$r_1 = min_{1\leq k\leq n}\left\{\frac{eL(n,k)}{n!\,{}_1F_1(n+1,2,1)}\frac{|a_0|}{|a_k|}\right\}^{\frac{1}{k}}$$

$$r_2 = max_{1\leq k\leq n}\left\{\frac{n!\,{}_1F_1(n+1,2,1)}{eL(n,k)}\frac{|a_{n-k}|}{|a_n|}\right\}^{\frac{1}{k}}.$$

Now, we use the central polygonal numbers or the Lazy Caterer's number (sequence number A000124 in the OEIS [11]). Let $p(k)$ be the k^{th} number in the sequence, then

$$p(k) = 1 + \binom{k+1}{2}.$$

The sum $\sum_{k=1}^{n} p(k) = \frac{(n^3+3n^2+8n)}{6}$ can be easily shown to be true. Again using this identity in Theorem 1.3 we get

Theorem 2.2: Let $p(z) = \sum_{k=0}^{n} a_k z^k (a_k \neq 0, 0 \leq k \leq n)$ is a non-constant complex polynomial, then all the zeros of $p(z)$ lie in the annulus, $C = \{z: r_1 \leq |z| \leq r_2\}$, where

$$r_1 = min_{1\leq k\leq n}\left\{\frac{6\,p(k)}{(n^3+3n^2+8n)}\frac{|a_0|}{|a_k|}\right\}^{\frac{1}{k}}$$

$$r_2 = max_{1\leq k\leq n}\left\{\frac{(n^3+3n^2+8n)}{6\,p(k)}\frac{|a_{n-k}|}{|a_n|}\right\}^{\frac{1}{k}}.$$

Next, we use the Lozanić's triangular array. (Sequence A034851 in the OEIS). The element at n^{th} row and k^{th} column is defined by $T(n,k)$.

$$T(n,k) = \begin{cases} n \text{ is even} \begin{cases} \binom{n}{k}/2 \text{ when } k \text{ is odd} \\ (\binom{n}{k} + \binom{n/2}{k/2})/2 \text{ when } k \text{ is even} \end{cases} \\ n \text{ is odd} \begin{cases} (\binom{n}{k} + \binom{n-1/2}{k-1/2})\Big/_2 \text{ when } k \text{ is odd} \\ (\binom{n}{k} + \binom{n-1/2}{k/2})\Big/_2 \text{ when } k \text{ is even} \end{cases} \end{cases}$$

The sum of all elements of the n^{th} row are given by $2^{n-2} + 2^{\lfloor n/2\rfloor-1}$[14] [5] and thus related ring containing all the zeros of a polynomial can be obtained, and is given below.

Theorem 2.3: Let $p(z) = \sum_{k=0}^{n} a_k z^k (a_k \neq 0, 0 \leq k \leq n)$ is a non-constant complex polynomial, then all the zeros of $p(z)$ lie in the annulus, $C = \{z: r_1 \leq |z| \leq r_2\}$, where

$$r_1 = min_{1\leq k\leq n}\left\{\frac{T(n,k)}{(2^{n-2}+2^{\lfloor n/2\rfloor-1})}\frac{|a_0|}{|a_k|}\right\}^{\frac{1}{k}}$$

$$r_2 = max_{1\leq k\leq n}\left\{\frac{(2^{n-2}+2^{\lfloor n/2\rfloor-1})}{T(n,k)}\frac{|a_{n-k}|}{|a_n|}\right\}^{\frac{1}{k}}.$$

Finally, we use the Floyd's triangular array. [12] It is defined by filling consecutive natural numbers starting from top left and the n^{th} row contains n elements only. The sum of n^{th} row is given by $n(n^2+1)/2$. [10]
Let the element at n^{th} row and k^{th} column be denoted as $F(n,k)$. Then it follows as same as that of previous results that:

Theorem 2.4: Let $p(z) = \sum_{k=0}^{n} a_k z^k (a_k \neq 0, 0 \leq k \leq n)$ is a non-constant complex polynomial, then all the zeros of $p(z)$ lie in the annulus, $C = \{z: r_1 \leq |z| \leq r_2\}$, where

$$r_1 = min_{1\leq k\leq n} \left\{ \frac{2\,F(n,k)}{n(n^2+1)} \frac{|a_0|}{|a_k|} \right\}^{\frac{1}{k}}$$

$$r_2 = max_{1\leq k\leq n} \left\{ \frac{n(n^2+1)}{2\,F(n,k)} \frac{|a_{n-k}|}{|a_n|} \right\}^{\frac{1}{k}}.$$

3. Computation and Comparative Analysis

In this section, we construct examples of polynomials for which our results give sharper bounds for region containing all the zeros of polynomial as compared to theorems 1.1-1.4. Although we only compare the computation with above mentioned theorems, other similar theorems giving bounds for such regions can also be compared. These computations are only to demonstrate that the results of this paper are not redundant and it gives a bound which improves the estimated region for zeros at least some polynomials and hence will be useful to reduce computations when the objective is to find the zeros of polynomials. The final best bound is considered by computing lower and upper bounds using known theorems and corollaries and taking the maximum of lower bound and minimum of upper bound.
The following are the examples.

Example 1: Let $p(z) = 0.7886z^3 + z^2 + z + 1$

Result	r_1	r_2	*Area of Annulus*
Theorem 1.1	0.250	5.072	80.621
Theorem 1.2	0.428	2.958	26.912
Theorem 1.3	0.125	10.144	323.223
Theorem 1.4	0.400	3.170	31.066
Theorem 2.1	0.460	2.546	19.699
Theorem 2.2	0.153	8.242	213.333
Theorem 2.3	0.333	3.804	45.111
Theorem 2.4	0.266	4.755	70.809
Best	Max(r_1)=0.460	Min(r_2)=2.546	19.699
Actual Bound	1.056	1.135	0.543

It is evident from the above table of calculated results, theorem 2.1 gives 26.80% sharper region containing all the zeros of $p(z)$ against the best result among existing results mentioned in section 1. Both parameters r_1 and r_2 are also improved by the theorem 2.1. It is also worthwhile to note that although theorems 2.2-2.4 do not give the best bound, theorem 2.3 improves r_1 against theorem 1.1 and theorem 1.3. Other similar comparisons can be noted from the table.

Example 2: Let $p(z) = 0.2188z^4 + 0.429z^3 + 1.126z^2 + 0.546z + 1$

Result	r_1	r_2	*Area of Annulus*
Theorem 1.1	0.178	20.155	1276.090
Theorem 1.2	0.487	7.352	169.063
Theorem 1.3	0.122	29.410	2719.267
Theorem 1.4	0.355	6.000	112.701
Theorem 2.1	0.500	5.962	110.883
Theorem 2.2	0.152	23.528	1739.008
Theorem 2.3	0.305	11.764	434.478
Theorem 2.4	0.377	9.523	284.456
Best	Max(r_1)=0.500	Min(r_2)=5.962	110.883
Actual Bound	1.163	1.837	6.352

We obtain a sharper bound on both ends of the region containing all the zeros of $p(z)$ and get a 1.613% sharper area for the region against the best bound among section 1 results. The best bound for this example is given by theorem 2.1.
The following example demonstrates that the given result may not always improve all parameters of the region, but may improve only a parameter against limited results only and the best bound from all results can be combined to get an accurate bound for the region of roots of polynomials.

Example 3: Let $p(z) = z^5 + 0.006z^4 + 0.01z^3 + 0.2z^2 + 0.3z + 1$

Result	r_1	r_2	*Area of Annulus*
Theorem 1.1	0.118	1.409	6.196
Theorem 1.2	0.503	1.987	11.606
Theorem 1.3	0.1282	1.1877	4.377
Theorem 1.4	0.7715	1.280	3.280
Theorem 2.1	0.288	3.467	37.501
Theorem 2.2	0.166	1.201	4.444
Theorem 2.3	0.333	1.584	7.542
Theorem 2.4	0.564	2.304	15.677
Best	Max(r_1)=0.7715	Min(r_2)=1.201	2.661
Actual Bound	0.952	1.0607	0.6873

The theorem 2.1 only improves r_1 for given $p(z)$ against theorem 1.1 and theorem 1.3 by 44.06% and 25% respectively. The best result for lower bound among the main results of this paper is given by theorem 2.4 for this example. The best upper bound among all eight results is given by theorem 2.2, improving the previous best result by 6.17%. Combining all results, we get an annular region containing all the zeros of $p(z)$, 18.87% smaller than given by theorem 1.4.
These computations have only been performed against specific results only for demonstrative purposes and can be similarly compared against other existing results which give a bound for region containing all the zeros of a complex coefficient polynomial.

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