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Convolutions for Generalized Tribonacci Numbers and Related Results

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Abstract

In this paper, we derive convolution identities for generalized Tribonacci numbers. Some of our results generalize existing relations. Focusing on Tribonacci and Tribonacci-Lucas numbers, we also present two new classes of convolutions that have not been studied before.

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1 Introduction and Preliminaries

Let $G_n = G_n(c_0, c_1, c_2), n \geq 0$, denote a generalized Tribonacci sequence, i.e.

$$G_n = G_{n-1} + G_{n-2} + G_{n-3}, \qquad n \geq 3, \tag{1}$$

with initial values $G_0 = c_0, G_1 = c_1, G_2 = c_2$ not all being zero. It is known that $G_n(c_0, c_1, c_2)$ can be written in the Binet form

$$G_n(c_0, c_1, c_2) = A\alpha^n + B\beta^n + C\gamma^n, \tag{2}$$

[1]Disclaimer: Statements and conclusions made in this article are entirely those of the author. They do not necessarily reflect the views of LBBW.

where α, β and γ are the distinct roots of the equation $x^3 - x^2 - x - 1 = 0$. The coefficients A, B and C depend on the initial values and are determined by the system

$$\begin{aligned} A + B + C &= c_0 \\ A\alpha + B\beta + C\gamma &= c_1 \\ A\alpha^2 + B\beta^2 + C\gamma^2 &= c_2. \end{aligned} \tag{3}$$

Two special cases of G_n will be considered later: $G_n(0,1,1) = T_n$ is the sequence of Tribonacci numbers (sequence A000073 in [10]) and $G_n(3,1,3) = K_n$ are the Tribonacci-Lucas numbers (sequence A001644 in [10]). The first few terms of the sequence $(T_n)_{n\geq 0}$ are

$$0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705,$$

whereas for $(K_n)_{n\geq 0}$ we have

$$3, 1, 3, 7, 11, 21, 39, 71, 131, 241, 443, 815, 1499, 2757, 5071.$$

Their Binet forms are given by

$$T_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \tag{4}$$

and

$$K_n = \alpha^n + \beta^n + \gamma^n, \tag{5}$$

where α, β and γ equal

$$\begin{aligned} \alpha &= \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3}, \\ \beta &= \frac{1 + \omega\sqrt[3]{19 + 3\sqrt{33}} + \omega^2\sqrt[3]{19 - 3\sqrt{33}}}{3}, \\ \gamma &= \frac{1 + \omega^2\sqrt[3]{19 + 3\sqrt{33}} + \omega\sqrt[3]{19 - 3\sqrt{33}}}{3}, \end{aligned}$$

and $\omega = \frac{-1+i\sqrt{3}}{2}$ is a primitive cube root of unity. Some properties of these numbers are studied in [1], [2], [3], [6], [7], [8] and [9].

There is a large amount of studies concerned with convolution identities for important numbers such as Bernoulli, Euler, Cauchy, Fibonacci, Lucas, Balancing and Tribonacci numbers (see the references herein and those given in [6]). For Fibonacci numbers we have for instance

$$\sum_{k=0}^{n} F_k F_{n-k} = \frac{1}{5}((n+1)L_n - 2F_{n+1}), \qquad n \geq 1. \tag{6}$$

More identities of this kind can by found in [4] and [5], among others. One convolution formula for Tribonacci numbers comes from [8] (p. 462)

$$22\sum_{k=0}^{n-2} T_k T_{n-2-k} = 5(n-1)T_n - 2(n-1)T_{n-1} - 4nT_{n-2}, \qquad n \geq 2. \tag{7}$$

The formula, however, apparently contains a typo as becomes obvious by numerical inspection. More convolution identities for Tribonacci numbers with and without binomial coefficients have been derived recently in [6] and [7]. One pretty example is

$$\sum_{k=0}^{n-3} T_k(T_{n-k} + T_{n-2-k} + 2T_{n-3-k}) = (n-2)T_{n-1} - T_{n-2}, \qquad n \geq 3. \tag{8}$$

In this paper, we present further identities of this type for the generalized Tribonacci sequence. The results we obtain are based on some functional relations between the generating functions for these numbers. We also introduce two new classes of convolutions that have not been studied before.

2 Ordinary Generating Functions

First, we give the ordinary generating function for G_n:

Lemma 2.1 *Let $G_n = G_n(c_0, c_1, c_2), n \geq 0$, be defined by (1). Let further $f_{G_n}(x)$ denote the ordinary generating function of G_n,*

$$f_{G_n}(x) = \sum_{n=0}^{\infty} G_n x^n.$$

Then $f_{G_n}(x)$ is given by

$$f_{G_n}(x) = \frac{c_0 + (c_1 - c_0)x + (c_2 - c_1 - c_0)x^2}{1 - x - x^2 - x^3}. \tag{9}$$

PROOF: Calculate

$$f_{G_n}(x) - xf_{G_n}(x) - x^2 f_{G_n}(x) - x^3 f_{G_n}(x) = c_0 + (c_1 - c_0)x + (c_2 - c_1 - c_0)x^2,$$

by (1). □

As particular examples we have

$$f_{T_n}(x) = \frac{x}{1 - x - x^2 - x^3}, \tag{10}$$

as well as

$$f_{K_n}(x) = \frac{3 - 2x - x^2}{1 - x - x^2 - x^3}. \tag{11}$$

Both results are known. The next lemma will help us to derive the generating functions of even and odd-indexed Tribonacci sequences.

Lemma 2.2 *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence $(a_n)_{n\geq 0}$. Then, the generating functions of the sequences $(a_{2n})_{n\geq 0}$ and $(a_{2n+1})_{n\geq 0}$ are given by*

$$\sum_{n=0}^{\infty} a_{2n} x^n = \frac{f(\sqrt{x}) + f(-\sqrt{x})}{2}, \tag{12}$$

and

$$\sum_{n=0}^{\infty} a_{2n+1} x^n = \frac{f(\sqrt{x}) - f(-\sqrt{x})}{2\sqrt{x}}. \tag{13}$$

PROOF: Since

$$\sum_{n=0}^{\infty} a_{2n} x^{2n} = \frac{f(x) + f(-x)}{2}, \tag{14}$$

the first formula follows immediately. The other result is also true, since

$$\sum_{n=0}^{\infty} a_{2n+1} x^{2n+1} = \frac{f(x) - f(-x)}{2}. \tag{15}$$

□

Theorem 2.3 *The generating functions of the sequences G_{2n} and G_{2n+1} are*

$$f_{G_{2n}}(x) = \frac{c_0 + (c_2 - 3c_0)x + (2c_1 - c_2)x^2}{1 - 3x - x^2 - x^3}, \tag{16}$$

and

$$f_{G_{2n+1}}(x) = \frac{c_1 + (c_2 - 2c_1 + c_0)x + (c_2 - c_1 - c_0)x^2}{1 - 3x - x^2 - x^3}. \tag{17}$$

PROOF: Both statements are consequences of Lemma 2.2 applied to (9) and some lengthy algebra. □

The previous Lemma gives the following results as particular examples:

$$f_{T_{2n}}(x) = \frac{x + x^2}{1 - 3x - x^2 - x^3}, \tag{18}$$

$$f_{T_{2n+1}}(x) = \frac{1 - x}{1 - 3x - x^2 - x^3}, \tag{19}$$

$$f_{K_{2n}}(x) = \frac{3 - 6x - x^2}{1 - 3x - x^2 - x^3}, \tag{20}$$

and

$$f_{K_{2n+1}}(x) = \frac{1 + 4x - x^2}{1 - 3x - x^2 - x^3}. \tag{21}$$

Corollary 2.4 *For $n \geq 2$ we have the following identities:*

$$3T_{2n} - 6T_{2n-2} - T_{2n-4} = K_{2n-2} + K_{2n-4}, \tag{22}$$

$$T_{2n+1} + 4T_{2n-1} - T_{2n-3} = K_{2n+1} - K_{2n-1}, \tag{23}$$

$$T_{2n} + 4T_{2n-2} - T_{2n-4} = K_{2n-1} + K_{2n-3}, \tag{24}$$

and

$$3T_{2n+1} - 6T_{2n-1} - T_{2n-3} = K_{2n} - K_{2n-2}. \tag{25}$$

PROOF: From (18) and (20) we get

$$(3 - 6x - x^2) f_{T_{2n}}(x) = (x + x^2) f_{K_{2n}}(x). \tag{26}$$

The LHS equals

$$\begin{aligned} LHS &= (3 - 6x - x^2)\Big(\sum_{n=0}^{\infty} T_{2n} x^n\Big) \\ &= 3x + \sum_{n=2}^{\infty} \Big(3T_{2n} - 6T_{2n-2} - T_{2n-4}\Big) x^n, \end{aligned} \tag{27}$$

whereas the RHS is

$$RHS = 3x + \sum_{n=2}^{\infty} \Big(K_{2n-2} + K_{2n-4}\Big) x^n. \tag{28}$$

Compare the coefficients and the proof of the first identity is done. The other relations are proved similarly using (18)-(21). □

We note that from the Corollary the following formula follows immediately:

$$K_n = 2T_{n+1} - T_{n-1} - T_{n-3}, \quad n \geq 3. \tag{29}$$

3 Convolutions for G_n

Now we are ready to state the first main results of this paper. We start our presentation with an extension of (8) to a generalized Tribonacci sequence.

Theorem 3.1 *For the generalized Tribonacci sequence G_n we have the following convolution identity for $n \geq 4$:*

$$\begin{aligned}
&\sum_{k=0}^{n-4} G_k(A_0 G_{n-k} + A_1 G_{n-1-k} + A_2 G_{n-2-k} + A_3 G_{n-3-k} + A_4 G_{n-4-k}) \\
&= B_0(n+1)G_{n+1} + B_1 n G_n + B_2(n-1)G_{n-1} + B_3(n-2)G_{n-2} + B_4(n-3)G_{n-3} \\
&-(c_2(A_0+A_1) + c_1(A_0+A_2) + c_0(A_0+A_3))G_{n-3} - (c_2 A_0 + c_1 A_1 + c_0 A_2)G_{n-2} \\
&-(c_1 A_0 + c_0 A_1)G_{n-1} - c_0 A_0 G_n, \qquad (30)
\end{aligned}$$

with $A_0 = c_1, A_1 = 2(c_2 - c_1), A_2 = 3c_0 + 2c_1 - c_2, A_3 = 2(c_1 - c_0), A_4 = c_2 - c_1 - c_0$ *and* $B_0 = c_0^2, B_1 = 2c_0(c_1 - c_0), B_2 = (c_1 - c_0)^2 + 2c_0(c_2 - c_1 - c_0), B_3 = 2(c_1 - c_0)(c_2 - c_1 - c_0), B_4 = (c_2 - c_1 - c_0)^2$.

PROOF: Simple calculations show that

$$f_{G_n}^2(x) = \frac{B_0 + B_1 x + B_2 x^2 + B_3 x^3 + B_4 x^4}{(1 - x - x^2 - x^3)^2}, \qquad (31)$$

and

$$f'_{G_n}(x) = \frac{A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4}{(1 - x - x^2 - x^3)^2}, \qquad (32)$$

where the coefficients A_i and $B_i (i = 0, ..., 4)$ are defined as in the Theorem. This yields

$$(B_0 + B_1 x + B_2 x^2 + B_3 x^3 + B_4 x^4) f'_{G_n}(x) = (A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4) f_{G_n}^2(x). \qquad (33)$$

Writing the LHS of the relation in terms of power series gives

$$\begin{aligned}
LHS &= (B_0 + B_1 x + B_2 x^2 + B_3 x^3 + B_4 x^4)\Big(\sum_{n=0}^{\infty}(n+1)G_{n+1}x^n\Big) \\
&= (B_0 G_1)x^0 + (2B_0 G_2 + B_1 G_1)x^1 + (3B_0 G_3 + 2B_1 G_2 + B_2 G_1)x^2 \\
&\quad + (4B_0 G_4 + 3B_1 G_3 + 2B_2 G_2 + B_3 G_1)x^3 + \sum_{n=4}^{\infty}\Big(B_0(n+1)G_{n+1} \\
&\quad + B_1 n G_n + B_2(n-1)G_{n-1} + B_3(n-2)G_{n-2} + B_4(n-3)G_{n-3}\Big)x^n.
\end{aligned}$$

Analogously, the RHS is expressed as

$$\begin{aligned}
RHS &= (A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4)\Big(\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} G_kG_{n-k}\Big)x^n\Big) \\
&= (A_0c_0^2)x^0 + (A_0(c_0c_1 + c_1c_0) + A_1c_0^2)x^1 \\
&\quad +(A_0(c_0c_2 + c_1^2 + c_2c_0) + A_1(2c_0c_1) + A_2c_0^2)x^2 \\
&\quad +(A_0(2c_0G_3 + 2c_1c_2) + A_1(2c_0c_2 + c_1^2) + A_2(2c_0c_1) + A_3(c_0^2))x^3 \\
&\quad +\sum_{n=4}^{\infty}\Big(A_0\sum_{k=0}^{n} G_kG_{n-k} + A_1\sum_{k=0}^{n-1} G_kG_{n-1-k} + A_2\sum_{k=0}^{n-2} G_kG_{n-2-k} \\
&\quad +A_3\sum_{k=0}^{n-3} G_kG_{n-3-k} + A_4\sum_{k=0}^{n-4} G_kG_{n-4-k}\Big)x^n.
\end{aligned}$$

Simplifying and comparing the coefficients of both sides proves the result. □

When $G_n = T_n$, then $A_0 = 1, A_1 = 0, A_2 = 1, A_3 = 2, A_4 = 0, B_0 = 0, B_1 = 0, B_2 = 1, B_3 = B_4 = 0$ and (30) reduces to (8).

When $G_n = K_n$, then $A_0 = 1, A_1 = 4, A_2 = 8, A_3 = -4, A_4 = -1, B_0 = 9, B_1 = -12, B_2 = -2, B_3 = 4, B_4 = 1$ and (30) becomes

$$\begin{aligned}
&\sum_{k=0}^{n-4} K_k(K_{n-k} + 4K_{n-1-k} + 8K_{n-2-k} - 4K_{n-3-k} - K_{n-4-k}) \\
&= -2(n+6)K_n + 2(3n+8)K_{n-1} + 12(n-1)K_{n-2}. \qquad (34)
\end{aligned}$$

Another convolution for G_n is

Theorem 3.2 *For $n \geq 4$*

$$\begin{aligned}
&\sum_{k=0}^{n-4}(k+1)G_{k+1}(B_0G_{n-k} + B_1G_{n-1-k} + B_2G_{n-2-k} + B_3G_{n-3-k} + B_4G_{n-4-k}) \\
&= A_0\sum_{k_1+k_2+k_3=n} G_{k_1}G_{k_2}G_{k_3} + A_1\sum_{k_1+k_2+k_3=n-1} G_{k_1}G_{k_2}G_{k_3} + A_2\sum_{k_1+k_2+k_3=n-2} G_{k_1}G_{k_2}G_{k_3} \\
&+A_3\sum_{k_1+k_2+k_3=n-3} G_{k_1}G_{k_2}G_{k_3} + A_4\sum_{k_1+k_2+k_3=n-4} G_{k_1}G_{k_2}G_{k_3} \\
&-(n-2)G_{n-2}(B_0G_3 + B_1c_2 + B_2c_1 + B_3c_0) - (n-1)G_{n-1}(B_0c_2 + B_1c_1 + B_2c_0) \\
&-nG_n(B_0c_1 + B_1c_0) - (n+1)G_{n+1}B_0c_0, \qquad (35)
\end{aligned}$$

where the A_i and B_i are as in Theorem 3.1.

PROOF: From (33) we obtain

$$(B_0+B_1x+B_2x^2+B_3x^3+B_4x^4)f'_{G_n}(x)f_{G_n}(x) = (A_0+A_1x+A_2x^2+A_3x^3+A_4x^4)f^3_{G_n}(x). \qquad (36)$$

After some steps of algebra we get for the LHS

$$\begin{aligned} LHS &= (B_0c_0c_1)x^0 + (B_0(c_1^2+2c_0c_2) + B_1c_0c_1)x^1 \\ &+(B_0(c_1c_2+2c_1c_2+3G_3c_0) + B_1(c_1^2+2c_0c_2) + B_2c_0c_1)x^2 \\ &+(B_0(4c_1G_3+2c_2^2+4c_0G_4) + B_1(3c_1c_2+3c_0G_3) + B_2(c_1^2+2c_0c_1) + B_3c_0c_1)x^3 \\ &+\sum_{n=4}^{\infty} R_n x^n, \end{aligned}$$

where

$$\begin{aligned} R_n &= \sum_{k=0}^{n-4}(k+1)G_{k+1}(B_0G_{n-k} + B_1G_{n-1-k} + B_2G_{n-2-k} + B_3G_{n-3-k} + B_4G_{n-4-k}) \\ &+(n-2)G_{n-2}(B_0G_3 + B_1c_2 + B_2c_1 + B_3c_0) + (n-1)G_{n-1}(B_0c_2 + B_1c_1 + B_2c_0) \\ &+nG_n(B_0c_1 + B_1c_0) + (n+1)G_{n+1}B_0c_0. \end{aligned} \tag{37}$$

The RHS equals

$$\begin{aligned} RHS &= \Big(A_0 \sum_{k_1+k_2+k_3=0} G_{k_1}G_{k_2}G_{k_3}\Big)x^0 \\ &+\Big(A_0 \sum_{k_1+k_2+k_3=1} G_{k_1}G_{k_2}G_{k_3} + A_1 \sum_{k_1+k_2+k_3=0} G_{k_1}G_{k_2}G_{k_3}\Big)x^1 \\ &+\Big(A_0 \sum_{k_1+k_2+k_3=2} G_{k_1}G_{k_2}G_{k_3} + A_1 \sum_{k_1+k_2+k_3=1} G_{k_1}G_{k_2}G_{k_3} \\ &+A_2 \sum_{k_1+k_2+k_3=0} G_{k_1}G_{k_2}G_{k_3}\Big)x^2 \\ &+\Big(A_0 \sum_{k_1+k_2+k_3=3} G_{k_1}G_{k_2}G_{k_3} + A_1 \sum_{k_1+k_2+k_3=2} G_{k_1}G_{k_2}G_{k_3} \\ &+A_2 \sum_{k_1+k_2+k_3=1} G_{k_1}G_{k_2}G_{k_3} + A_3 \sum_{k_1+k_2+k_3=0} G_{k_1}G_{k_2}G_{k_3}\Big)x^3 \\ &+\sum_{n=4}^{\infty} S_n x^n, \end{aligned} \tag{38}$$

where

$$\begin{aligned} S_n &= A_0 \sum_{k_1+k_2+k_3=n} G_{k_1}G_{k_2}G_{k_3} + A_1 \sum_{k_1+k_2+k_3=n-1} G_{k_1}G_{k_2}G_{k_3} + A_2 \sum_{k_1+k_2+k_3=n-2} G_{k_1}G_{k_2}G_{k_3} \\ &+A_3 \sum_{k_1+k_2+k_3=n-3} G_{k_1}G_{k_2}G_{k_3} + A_4 \sum_{k_1+k_2+k_3=n-4} G_{k_1}G_{k_2}G_{k_3}. \end{aligned} \tag{39}$$

Compaing the coefficients proves the statement. □

When $G_n = T_n$, then with $A_0 = 1, A_1 = 0, A_2 = 1, A_3 = 2, A_4 = 0, B_0 = 0, B_1 = 0, B_2 = 1, B_3 = B_4 = 0$ we get the formula

$$\sum_{k=1}^{n-2} kT_kT_{n-1-k} = \sum_{k_1+k_2+k_3=n} T_{k_1}T_{k_2}T_{k_3} + \sum_{k_1+k_2+k_3=n-2} T_{k_1}T_{k_2}T_{k_3} + 2\sum_{k_1+k_2+k_3=n-3} T_{k_1}T_{k_2}T_{k_3}. \tag{40}$$

When $G_n = K_n$, then with $A_0 = 1, A_1 = 4, A_2 = 8, A_3 = -4, A_4 = -1, B_0 = 9, B_1 = -12, B_2 = -2, B_3 = 4, B_4 = 1$ we get the formula

$$\begin{aligned}&\sum_{k=0}^{n-4}(k+1)K_{k+1}(9K_{n-k} - 12K_{n-1-k} - 2K_{n-2-k} + 4K_{n-3-k} + K_{n-4-k})\\ &= \sum_{k_1+k_2+k_3=n} K_{k_1}K_{k_2}K_{k_3} + 4\sum_{k_1+k_2+k_3=n-1} K_{k_1}K_{k_2}K_{k_3}\\ &+8\sum_{k_1+k_2+k_3=n-2} K_{k_1}K_{k_2}K_{k_3} - 4\sum_{k_1+k_2+k_3=n-1} K_{k_1}K_{k_2}K_{k_3} - \sum_{k_1+k_2+k_3=n-4} K_{k_1}K_{k_2}K_{k_3}\\ &-27K_n - 18(2n+1)K_{n-1} - (64n-47)K_{n-2}.\end{aligned} \tag{41}$$

Theorem 3.3 *For the generalized even-indexed Tribonacci sequence G_{2n} we have the following convolution identity for $n \geq 4$:*

$$\begin{aligned}&\sum_{k=0}^{n-4} G_{2k}\Big(A_0^*G_{2(n-k)} + A_1^*G_{2(n-1-k)} + A_2^*G_{2(n-2-k)} + A_3^*G_{2(n-3-k)} + A_4^*G_{2(n-4-k)}\Big)\\ &= B_0^*(n+1)G_{2(n+1)} + B_1^*nG_{2n} + B_2^*(n-1)G_{2(n-1)}\\ &\quad +B_3^*(n-2)G_{2(n-2)} + B_4^*(n-3)G_{2(n-3)}\\ &\quad -(A_0^*G_6 + A_1^*G_4 + A_2^*G_2 + A_3^*G_0)G_{2(n-3)} - (A_0^*G_4 + A_1^*G_2 + A_2^*G_0)G_{2(n-2)}\\ &\quad -(A_0^*G_2 + A_1^*G_0)G_{2(n-1)} - A_0^*G_0G_{2n},\end{aligned} \tag{42}$$

with $A_0^* = c_2, A_1^* = 2(c_0+2c_1-c_2), A_2^* = 4c_2-6c_1, A_3^* = 2c_2-6c_0, A_4^* = 2c_1-c_2$ and $B_0^* = c_0^2, B_1^* = 2c_0(c_2-3c_0), B_2^* = 2c_0(2c_1-c_2)+(c_2-3c_0)^2, B_3^* = 2(c_2-3c_0)(2c_1-c_2), B_4^* = (2c_1-c_2)^2$.

PROOF: The convolution is essentially a consequence of the relation

$$(B_0^*+B_1^*x+B_2^*x^2+B_3^*x^3+B_4^*x^4)f'_{G_{2n}}(x) = (A_0^*+A_1^*x+A_2^*x^2+A_3^*x^3+A_4^*x^4)f^2_{G_{2n}}(x), \tag{43}$$

where the coefficients A_i^* and B_i^* $(i = 0, ..., 4)$ are defined as in the Theorem. We omit the lengthy details. □

When $G_{2n} = T_{2n}$, then $A_0^* = 1, A_1^* = 2, A_2^* = -2, A_3^* = 2, A_4^* = 1, B_0^* = 0, B_1^* = 0, B_2^* = 1, B_3^* = 2, B_4^* = 1$ and (42) becomes

$$\sum_{k=0}^{n-4} T_{2k}(T_{2(n-k)} + 2T_{2(n-1-k)} - 2T_{2(n-2-k)} + 2T_{2(n-3-k)} + T_{2(n-4-k)}) \\ = (n-2)T_{2(n-1)} + 2(n-5)T_{2(n-2)} + (n-22)T_{2(n-3)}. \tag{44}$$

When $G_{2n} = K_{2n}$, then $A_0^* = 3, A_1^* = 4, A_2^* = 6, A_3^* = -12, A_4^* = -1, B_0^* = 9, B_1^* = -36, B_2^* = 30, B_3^* = 12, B_4^* = 1$ and (42) becomes

$$\sum_{k=0}^{n-4} K_{2k}(3K_{2(n-k)} + 4K_{2(n-1-k)} + 6K_{2(n-2-k)} - 12K_{2(n-3-k)} - K_{2(n-4-k)}) \\ = 9(n+1)K_{2(n+1)} - 9(4n+1)K_{2n} + (30n-51)K_{2(n-1)} \\ +(12n-87)K_{2(n-2)} + (n-146)K_{2(n-3)}. \tag{45}$$

An analogue expression for a convolution containing G_{2n+1} is obtained using similar arguments.

Theorem 3.4 *For the generalized odd-indexed Tribonacci sequence G_{2n+1} the following identity holds true for $n \geq 4$:*

$$\sum_{k=0}^{n-4} G_{2k+1}\Big(\tilde{A}_0 G_{2(n-k)+1} + \tilde{A}_1 G_{2(n-1-k)+1} + \tilde{A}_2 G_{2(n-2-k)+1} \\ + \tilde{A}_3 G_{2(n-3-k)+1} + \tilde{A}_4 G_{2(n-4-k)+1}\Big) \\ = \tilde{B}_0(n+1)G_{2n+3} + \tilde{B}_1 n G_{2n+1} + \tilde{B}_2(n-1)G_{2n-1} \\ +\tilde{B}_3(n-2)G_{2n-3} + \tilde{B}_4(n-3)G_{2n-5} \\ -(\tilde{A}_0 G_7 + \tilde{A}_1 G_5 + \tilde{A}_2 G_3 + \tilde{A}_3 G_1)G_{2n-5} - (\tilde{A}_0 G_5 + \tilde{A}_1 G_3 + \tilde{A}_2 G_1)G_{2n-3} \\ -(\tilde{A}_0 G_3 + \tilde{A}_1 G_1)G_{2n-1} - \tilde{A}_0 G_1 G_{2n+1}, \tag{46}$$

with $\tilde{A}_0 = c_0 + c_1 + c_2, \tilde{A}_1 = 2(c_2 - c_0), \tilde{A}_2 = 2(2c_0 + 2c_1 - c_2), \tilde{A}_3 = 2(c_2 - 2c_1 + c_0), \tilde{A}_4 = c_2 - c_1 - c_0$ and $\tilde{B}_0 = c_1^2, \tilde{B}_1 = 2c_1(c_2 - 2c_1 + c_0), \tilde{B}_2 = 2c_1(c_2 - c_1 - c_0) + (c_2 - 2c_1 + c_0)^2, \tilde{B}_3 = 2(c_2 - 2c_1 + c_0)(c_2 - c_1 - c_0), \tilde{B}_4 = (c_2 - c_1 - c_0)^2$.

PROOF: The result follows straightforwardly from the relation

$$\Big(\tilde{B}_0 + \tilde{B}_1 x + \tilde{B}_2 x^2 + \tilde{B}_3 x^3 + \tilde{B}_4 x^4\Big) f'_{G_{2n+1}}(x) \\ = \Big(\tilde{A}_0 + \tilde{A}_1 x + \tilde{A}_2 x^2 + \tilde{A}_3 x^3 + \tilde{A}_4 x^4\Big) f^2_{G_{2n+1}}(x), \tag{47}$$

where the coefficients $\tilde{A}_i$ and $\tilde{B}_i\,(i = 0, ..., 4)$ are defined as in the Theorem. We omit the lengthy details. □

When $G_{2n+1} = T_{2n+1}$, then $\tilde{A}_0 = 2, \tilde{A}_1 = 2, \tilde{A}_2 = 2, \tilde{A}_3 = -2, \tilde{A}_4 = 0, \tilde{B}_0 = 1, \tilde{B}_1 = -2, \tilde{B}_2 = 1, \tilde{B}_3 = 0, \tilde{B}_4 = 0$ and (46) becomes

$$\sum_{k=0}^{n-4} 2T_{2k+1}\Big(T_{2(n-k)+1} + T_{2(n-1-k)+1} + T_{2(n-2-k)+1} - T_{2(n-3-k)+1}\Big)$$
$$= (n+1)T_{2n+3} - 2(n+1)T_{2n+1} + (n-7)T_{2n-1} - 20T_{2n-3} - 64T_{2n-5}. \tag{48}$$

When $G_{2n+1} = K_{2n+1}$, then $\tilde{A}_0 = 7, \tilde{A}_1 = 0, \tilde{A}_2 = 10, \tilde{A}_3 = 8, \tilde{A}_4 = -1, \tilde{B}_0 = 1, \tilde{B}_1 = 8, \tilde{B}_2 = 14, \tilde{B}_3 = -8, \tilde{B}_4 = 1$ and (46) gives

$$\sum_{k=0}^{n-4} K_{2k+1}\Big(7K_{2(n-k)+1} + 10K_{2(n-2-k)+1} + 8K_{2(n-3-k)+1} - K_{2(n-4-k)+1}\Big)$$
$$= (n+1)K_{2n+3} + (8n-7)K_{2n+1} + (14n-63)K_{2n-1}$$
$$-(8n+141)K_{2n-3} + (n-578)K_{2n-5}. \tag{49}$$

4 Mixed Convolutions

In this section we state some mixed convolution identities. For the sake of a more readable presentation, we only consider the sequences T_n and K_n.

Theorem 4.1 *For $n \geq 3$ we have*

$$\sum_{k=0}^{n-3} T_k(K_{n-k} + K_{n-2-k} + 2K_{n-3-k}) = 3(n-1)T_n - (2n-1)T_{n-1} - (n+4)T_{n-2}. \tag{50}$$

PROOF: From (10) and (11)

$$(3x - 2x^2 - x^3)f'_{T_n}(x) = (1 + x^2 + 2x^3)f_{T_n}(x)f_{K_n}(x). \tag{51}$$

The RHS equals

$$\begin{aligned} RHS &= (1 + x^2 + 2x^3)\Big(\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} T_k K_{n-k}\Big)x^n\Big) \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} T_k K_{n-k}x^n + \sum_{n=2}^{\infty}\sum_{k=0}^{n-2} T_k K_{n-2-k}x^n + \sum_{n=3}^{\infty}\sum_{k=0}^{n-3} 2T_k K_{n-3-k}x^n, \end{aligned}$$

whereas the LHS is

$$\begin{aligned} LHS &= (3x - 2x^2 - x^3)\Big(\sum_{n=1}^{\infty} nT_n x^{n-1}\Big) \\ &= 3x + 4x^2 + \sum_{n=3}^{\infty}\Big(3nT_n - 2(n-1)T_{n-1} - (n-2)T_{n-2}\Big)x^n. \end{aligned}$$

Compare the coefficients and the proof is done. □

Another convolution identity of this type is

Theorem 4.2 *For $n \geq 4$ we have*

$$\sum_{k=0}^{n-4} T_k(K_{n-k} + 4K_{n-1-k} + 8K_{n-2-k} - 4K_{n-3-k} - K_{n-4-k})$$
$$= 3nK_n - 2(n-1)K_{n-1} - (n-2)K_{n-2} - 18T_n + 2T_{n-1} - 16T_{n-2}. \quad (52)$$

PROOF: The expression follows from the relation

$$(3x - 2x^2 - x^3)f'_{K_n}(x) = (1 + 4x + 8x^2 - 4x^3 - x^4)f_{T_n}(x)f_{K_n}(x). \quad (53)$$

□

Two examples for mixed convolutions of even-indexed Tribonacci sequence are stated next.

Theorem 4.3 *For $n \geq 4$ we have*

$$\sum_{k=0}^{n-4} T_{2k}(K_{2(n-k)} + 2K_{2(n-1-k)} - 2K_{2(n-2-k)} + 2K_{2(n-3-k)} + K_{2(n-4-k)})$$
$$= 3(n-1)T_{2n} - 3(n+2)T_{2(n-1)} - (7n-3)T_{2(n-2)} - (n+58)T_{2(n-3)}. \quad (54)$$

PROOF: The expression follows from the relation

$$(3x - 3x^2 - 7x^3 - x^4)f'_{T_{2n}}(x) = (1 + 2x - 2x^2 + 2x^3 + x^4)f_{T_{2n}}(x)f_{K_{2n}}(x). \quad (55)$$

□

Theorem 4.4 *For $n \geq 4$ we have*

$$\sum_{k=0}^{n-4} K_{2k}(3T_{2(n-k)} + 4T_{2(n-1-k)} + 6T_{2(n-2-k)} - 12T_{2(n-3-k)} - T_{2(n-4-k)})$$
$$= 3nK_{2n} - 3nK_{2(n-1)} - (7n+2)K_{2(n-2)} - (n+58)K_{2(n-3)}. \quad (56)$$

PROOF: Use

$$(3x - 3x^2 - 7x^3 - x^4)f'_{K_{2n}}(x) = (3 + 4x + 6x^2 - 12x^3 - x^4)f_{T_{2n}}(x)f_{K_{2n}}(x). \quad (57)$$

□

Formulas for odd-indexed Tribonacci numbers are also obtained straightforwardly from the respective generating functions.

Theorem 4.5 *For $n \geq 3$ it holds that*

$$\sum_{k=0}^{n-3} T_{2k+1}(K_{2(n-k)+1} + K_{2(n-1-k)+1} + K_{2(n-2-k)+1} - K_{2(n-3-k)+1})$$
$$= \frac{1}{2}\Big((n+1)T_{2n+3} + (3n-2)T_{2n+1} - (5n+11)T_{2n-1} + (n-60)T_{2n-3}\Big). \quad (58)$$

PROOF: By (19) and (21)

$$(1+3x-5x^2+x^3)f'_{T_{2n+1}}(x) = (2+2x+2x^2-2x^3)f_{T_{2n+1}}(x)f_{K_{2n+1}}(x), \quad (59)$$

and the expression follows from comparing the coefficients. □

Theorem 4.6 *For $n \geq 4$ it holds that*

$$\sum_{k=0}^{n-4} T_{2k+1}(7K_{2(n-k)+1} + 10K_{2(n-2-k)+1} + 8K_{2(n-3-k)+1} - K_{2(n-4-k)+1})$$
$$= (n+1)K_{2n+3} + 3nK_{2n+1} - 5(n-1)K_{2n-1} + (n-2)K_{2n-3}$$
$$-575T_{2n-5} - 157T_{2n-3} - 49T_{2n-1} - 7T_{2n+1}. \quad (60)$$

PROOF: Again, by (19) and (21)

$$(1+3x-5x^2+x^3)f'_{K_{2n+1}}(x) = (7+10x^2+8x^3-x^4)f_{T_{2n+1}}(x)f_{K_{2n+1}}(x), \quad (61)$$

and the expression follows from comparing the coefficients. □

5 Another Class of Convolution Identities

In this section we list still other convolution relations for the sequences T_n and K_n. The relations exhibit a different structure and are more advanced in the sense that they contain two additional parameters.

Theorem 5.1 *Let a and b be two real numbers with $a \neq 0$. Then for each $n \geq 4$ we have the following identity:*

$$\sum_{k=0}^{n-4} T_k\Big(9a^2T_{n-k} - 6a(2a+b)T_{n-1-k} + ((2a+b)^2 - 6a^2)T_{n-2-k}$$
$$+2a(2a+b)T_{n-3-k} + a^2T_{n-4-k}\Big)$$
$$= \sum_{k=0}^{n-2}(aK_k - bT_k)(aK_{n-2-k} - bT_{n-2-k})$$
$$-(12a^2 + (2a+b)(-4a+b))T_{n-3} + 3a(a+2b)T_{n-2} - 9a^2T_{n-1}. \quad (62)$$

PROOF: Using (10) and (11) we immediately get

$$x^2\Big(af_{K_n}(x) - bf_{T_n}(x)\Big)^2 = \Big(9a^2 - 6a(2a+b)x + ((2a+b)^2 - 6a^2)x^2 + 2a(2a+b)x^3 + a^2x^4\Big) f_{T_n}^2(x). \quad (63)$$

Writing both sides in terms of power series, the expression follows from comparing the coefficients and rearranging. □

When $b = 0$, then the above equation reduces to

$$\sum_{k=0}^{n-4} T_k\Big(9T_{n-k} - 12T_{n-1-k} - 2T_{n-2-k} + 4T_{n-3-k} + T_{n-4-k}\Big) = \sum_{k=0}^{n-2} K_k K_{n-2-k} - 4T_{n-3} + 3T_{n-2} - 9T_{n-1}. \quad (64)$$

When $a = b = 1$ (or $a = b = -1$), then the equation becomes

$$\sum_{k=0}^{n-4} T_k\Big(9T_{n-k} - 18T_{n-1-k} + 3T_{n-2-k} + 6T_{n-3-k} + T_{n-4-k}\Big) = \sum_{k=0}^{n-2}(K_k - T_k)(K_{n-2-k} - T_{n-2-k}) - 3T_{n-3} + 9T_{n-2} - 9T_{n-1}. \quad (65)$$

When $a = 1$ and $b = -1$ (or $a = -1$ and $b = 1$), then we get

$$\sum_{k=0}^{n-4} T_k\Big(9T_{n-k} - 6T_{n-1-k} - 5T_{n-2-k} + 2T_{n-3-k} + T_{n-4-k}\Big) = \sum_{k=0}^{n-2}(K_k + T_k)(K_{n-2-k} + T_{n-2-k}) - 7T_{n-3} - 3T_{n-2} - 9T_{n-1}. \quad (66)$$

The analogue convolution for K_n is

Theorem 5.2 *Let a and b be two real numbers with $b \neq 0$. Then, we have for each $n \geq 4$*

$$\sum_{k=0}^{n-4} K_k\Big(9a^2K_{n-k} - 6a(2a+b)K_{n-1-k} + ((2a+b)^2 - 6a^2)K_{n-2-k} + 2a(2a+b)K_{n-3-k} + a^2K_{n-4-k}\Big) + (37a^2 - 8ab + b^2)K_{n-3} + (9a^2 + 6ab + 3b^2)K_{n-2}$$

$$
\begin{aligned}
&-9a(3a+2b)K_{n-1}+27a^2K_n\\
=&\sum_{k=0}^{n-4}(aK_k-bT_k)\Big(9(aK_{n-k}-bT_{n-k})-12(aK_{n-1-k}-bT_{n-1-k})\\
&-2(aK_{n-2-k}-bT_{n-2-k})+4(aK_{n-3-k}-bT_{n-3-k})+(aK_{n-4-k}-bT_{n-4-k})\Big)\\
&+(37a-4b)(aK_{n-3}-bT_{n-3})+3(3a+b)(aK_{n-2}-bT_{n-2})\\
&-9(3a+b)(aK_{n-1}-bT_{n-1})+27a(aK_n-bT_n). \qquad (67)
\end{aligned}
$$

PROOF: From (10) and (11) we have

$$
\begin{aligned}
&(9-12x-2x^2+4x^3+x^4)\Big(af_{K_n}(x)-bf_{T_n}(x)\Big)^2\\
=&\Big(9a^2-6a(2a+b)x+((2a+b)^2-6a^2)x^2+2a(2a+b)x^3+a^2x^4\Big)f_{K_n}^2(x). \qquad (68)
\end{aligned}
$$

Writing both sides in terms of power series, the expression follows after some lengthy algebra from comparing the coefficients. □

When $a=0$, then (67) turns into (64).
When $a=b=1$ the formula becomes

$$
\begin{aligned}
&\sum_{k=0}^{n-4}K_k\Big(9K_{n-k}-18K_{n-1-k}+3K_{n-2-k}+6K_{n-3-k}+K_{n-4-k}\Big)\\
&\quad+30K_{n-3}+18K_{n-2}-45K_{n-1}+27K_n\\
=&\sum_{k=0}^{n-4}(K_k-T_k)\Big(9(K_{n-k}-T_{n-k})-12(K_{n-1-k}-T_{n-1-k})\\
&-2(K_{n-2-k}-T_{n-2-k})+4(K_{n-3-k}-T_{n-3-k})+(K_{n-4-k}-T_{n-4-k})\Big)\\
&+33(K_{n-3}-T_{n-3})+12(K_{n-2}-T_{n-2})-36(K_{n-1}-T_{n-1})+27(K_n-T_n). \qquad (69)
\end{aligned}
$$

When $a=1$ and $b=-1$ we get

$$
\begin{aligned}
&\sum_{k=0}^{n-4}K_k\Big(9K_{n-k}-6K_{n-1-k}-5K_{n-2-k}+2K_{n-3-k}+K_{n-4-k}\Big)\\
&\quad+46K_{n-3}+6K_{n-2}-9K_{n-1}+27K_n\\
=&\sum_{k=0}^{n-4}(K_k+T_k)\Big(9(K_{n-k}+T_{n-k})-12(K_{n-1-k}+T_{n-1-k})\\
&-2(K_{n-2-k}+T_{n-2-k})+4(K_{n-3-k}+T_{n-3-k})+(K_{n-4-k}+T_{n-4-k})\Big)\\
&+41(K_{n-3}+T_{n-3})+6(K_{n-2}+T_{n-2})-18(K_{n-1}+T_{n-1})+27(K_n+T_n). \qquad (70)
\end{aligned}
$$

It is worth to mention that a mixed convolution of this type can be easily derived from the relation

$$(3x-2x^2-x^3)\Big(af_{K_n}(x)-bf_{T_n}(x)\Big)^2$$
$$=\Big(9a^2-6a(2a+b)x+((2a+b)^2-6a^2)x^2+2a(2a+b)x^3+a^2x^4\Big)f_{T_n}(x)f_{K_n}(x). \tag{71}$$

Also, it is clear that some other results of this nature can be produced more or less routinely from similar relations for the respective generating functions. We conclude the presentation with a final example for the even-indexed sequences T_{2n} and K_{2n}:

Theorem 5.3 *Let a and b be two real numbers with $a \neq 0$. Then for each $n \geq 4$ we have the identity:*

$$\sum_{k=0}^{n-4} T_{2k}\Big(9a^2T_{2(n-k)}-6a(6a+b)T_{2(n-1-k)}+((6a+b)^2-6a(a+b))T_{2(n-2-k)}$$
$$+2(a+b)(6a+b)T_{2(n-3-k)}+(a+b)^2T_{2(n-4-k)}\Big)$$
$$+(3a^2-9ab)T_{2(n-3)}-3abT_{2(n-2)}+9a^2T_{2(n-1)}$$
$$=\sum_{k=0}^{n-4}(aK_{2k}-bT_{2k})\Big((aK_{2(n-2-k)}-bT_{2(n-2-k)})$$
$$+2(aK_{2(n-3-k)}-bT_{2(n-3-k)})+(aK_{2(n-4-k)}-bT_{2(n-4-k)})\Big)$$
$$+a(9a-b)K_{2(n-3)}+3a^2K_{2(n-2)}. \tag{72}$$

PROOF: The expression follows from

$$(x^2+2x^3+x^4)\Big(af_{K_{2n}}(x)-bf_{T_{2n}}(x)\Big)^2=\Big(9a^2-6a(6a+b)x$$
$$+((6a+b)^2-6a(a+b))x^2+2(a+b)(6a+b)x^3+(a+b)^2x^4\Big)f_{T_{2n}}^2(x). \tag{73}$$

Writing both sides in terms of power series, the expression follows from comparing the coefficients and rearranging. □

When $b=0$, then we obtain

$$\sum_{k=0}^{n-4} T_{2k}\Big(9T_{2(n-k)}-36T_{2(n-1-k)}+30T_{2(n-2-k)}+12T_{2(n-3-k)}+T_{2(n-4-k)}\Big)$$
$$+3T_{2(n-3)}+9T_{2(n-1)}$$

$$= \sum_{k=0}^{n-4} K_{2k}\Big(K_{2(n-2-k)} + 2K_{2(n-3-k)} + K_{2(n-4-k)}\Big) + 9K_{2(n-3)} + 3K_{2(n-2)}. \tag{74}$$

When $a = b = 1$, then the result is

$$\begin{aligned}
&\sum_{k=0}^{n-4} T_{2k}\Big(9T_{2(n-k)} - 42T_{2(n-1-k)} + 37T_{2(n-2-k)} + 28T_{2(n-3-k)} + 4T_{2(n-4-k)}\Big)\\
&\qquad -6T_{2(n-3)} - 3T_{2(n-2)} + 9T_{2(n-1)}\\
&= \sum_{k=0}^{n-4}(K_{2k} - T_{2k})\Big((K_{2(n-2-k)} - T_{2(n-2-k)}) + 2(K_{2(n-3-k)} - T_{2(n-3-k)})\\
&\quad +(K_{2(n-4-k)} - T_{2(n-4-k)})\Big) + 8K_{2(n-3)} + 3K_{2(n-2)}.
\end{aligned} \tag{75}$$

When $a = 1$ and $b = -1$, then the formula reduces to

$$\begin{aligned}
&\sum_{k=0}^{n-4} T_{2k}\Big(9T_{2(n-k)} - 30T_{2(n-1-k)} + 25T_{2(n-2-k)}\Big) + 12T_{2(n-3)} + 3T_{2(n-2)} + 9T_{2(n-1)}\\
&= \sum_{k=0}^{n-4}(K_{2k} + T_{2k})\Big((K_{2(n-2-k)} + T_{2(n-2-k)}) + 2(K_{2(n-3-k)} + T_{2(n-3-k)})\\
&\quad +(K_{2(n-4-k)} + T_{2(n-4-k)})\Big) + 10K_{2(n-3)} + 3K_{2(n-2)}.
\end{aligned} \tag{76}$$

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