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Best Proximity Points for Tricyclic Contractions in a (S) Convex Metric Spaces

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Abstract

In this paper, we introduce the notion of (S) convex structure, thereby, we acquire a best proximity point theorem for tricyclic contractions in the framework of convex metric spaces.

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1 Introduction

Let A and B be nonempty closed subsets of a complete metric space and let $T : A \cup B \longrightarrow A \cup B$ be a cyclic mapping, that is, $T(A) \subseteq B$ and $T(B) \subseteq A$, for which there exists $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x \in A$ and $y \in B$. In [2], it was shown that $A \cap B$ is nonempty and contains the unique fixed point of T. A good many different generalizations of this situation were studied under the assumption of $A \cap B = \emptyset$. In the same context, the class of cyclic contractions was introduced by Eldred and Veeramani [3].

Definition 1.1 *Let A and B be nonempty subsets of a metric space (X,d), we denote by $dist\,(A,B)$ the distance between the subsets A and B, a mapping $T : A\cup B \longrightarrow A\cup B$ is said to be cyclic contraction if T is cyclic and*

$$d\,(Tx,Ty) \leq kd\,(x,y) + (1-k)\,dist\,(A,B)$$

for some $k \in (0,1)$ and for all $x \in A$ and $y \in B$.

Under this weaker supposition over T, the existence, uniqueness and convergence of the iterates to the best proximity point, *i.e.* a point $x \in A\cup B$ such that $d\,(x,Tx) = dist\,(A,B)$, were obtained in several frameworks. First, the problem was studied in [3] for uniformly convex Banach spaces and the following result was obtained.

Theorem 1.2 *Let A and B be nonempty, closed and convex subsets of a uniformly convex Banach space X and let $T : A\cup B \longrightarrow A\cup B$ be a cyclic contraction map. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then there exists a unique $x \in A$ such that $x_{2n} \longrightarrow x$ and $\|x - Tx\| = dist\,(A,B)$.*

Afterwards, in [4] the problem was investigated from a distinct and more general angle.

Very recently, the current authors introduced the class of tricyclic contractions [1] and proved the existence of the best proximity point for such mappings in the setting of reflexive Banach spaces. The purpose of this paper is to establish a best proximity point theorem for tricyclic contractions in a special kind of metric spaces, namely (S) convex metric spaces.

2 Preliminaries

Let A, B and C be three nonempty subsets of a metric space (X,d), we shall adopt the following notations throughout this paper:

$$D : X^3 \longrightarrow [0,+\infty)\,,\ (x,y,z) \longmapsto D\,(x,y,z) = d\,(x,y) + d\,(y,z) + d\,(z,x)\,,$$

$$\delta\,(A,B,C) = \inf\{D\,(x,y,z) : x \in A,\ y \in B \text{ and } z \in C\}\,,$$
$$\Delta\,(A,B,C) = \sup\{D\,(x,y,z) : x \in A,\ y \in B \text{ and } z \in C\}\,,$$
$$\Delta_{(x,y)}\,(C) = \sup\{D\,(x,y,z) : z \in C\} \text{ for all } x \in A \text{ and } y \in B.$$

Next we introduce the concepts of tricyclic contractions and best proximity point thereof.

Definition 2.1 [1] *Let A, B and C be nonempty subsets of a metric space (X,d). A mapping $T : A\cup B\cup C \longrightarrow A\cup B\cup C$ is said to be tricyclic contraction if*

1. *T is a tricyclic map, i.e., $T(A) \subseteq B$, $T(B) \subseteq C$ and $T(C) \subseteq A$.*
2. *$D(Tx, Ty, Tz) \leq kD(x, y, z) + (1-k)\,\delta(A, B, C)$, for some $k \in (0,1)$ and for all $(x, y, z) \in A \times B \times C$.*

Definition 2.2 [1] *Let A, B and C be nonempty subsets of a metric space (X, d), let $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ be a tricyclic mapping. A point $x \in A \cup B \cup C$ is said to be a best proximity point for T if*

$$D\left(x, Tx, T^2x\right) = \delta(A, B, C).$$

Under the suitable assumptions, existence and convergence of iterates to a best proximity point for a tricyclic contraction have lately been studied in the setting of metric spaces [1]. Nevertheless, the main theorem was established in the particular framework of normed linear spaces.

Theorem 2.3 *Let A, B and C be nonempty, closed, bounded and convex subsets of a reflexive Banach space X, let $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ be a tricyclic contraction map. Then T has a best proximity point.*

The aim of this paper is to extend the previous result on the existence of best proximity point of tricyclic contractions to a special kind of metric spaces, namely (S) convex metric spaces, those are firstly introduced in this paper.

In [5], the concept of convexity in metric spaces was introduced as follows.

Definition 2.4 *Let (X, d) be a metric space, a mapping $W : X \times X \times I \longrightarrow X$ is said to be a convex structure on X provided that*

$$d(u, W(x, y; \lambda)) \leq \lambda d(u, x) + (1-\lambda)\, d(u, y),$$

for all $u, x, y \in X$ and $\lambda \in I := [0,1]$.

A metric space (X, d) along with a convex structure W is called a *convex metric space* and is denoted by (X, d, W). We shall denote by $[x, y]$ the set $\{W(x, y; \alpha) : \alpha \in I\}$. A subset C of a convex metric space is said to be convex if $[x, y] \subseteq C$ whenever $x, y \in C$. The closed and convex hull of a set A will be denoted by $\overline{co}(A)$ and is defined by

$$\overline{co}(A) := \bigcap \{C : C \text{ is a closed and convex subset of } X \text{ that contains } A\}.$$

It was shown in [5] that an arbitrary intersection of convex subsets is convex and so is any closed ball, and for every $x, y \in X$ and $\alpha \in I$,

$$d(x, W(x, y; \alpha)) = (1-\alpha)\, W(x, y; \alpha) \text{ and } d(y, W(x, y; \alpha)) = \alpha W(x, y; \alpha).$$

Note that a normed linear space and each of its convex subsets are convex metric spaces with $W(x, y; \lambda) = \lambda x + (1-\lambda)\, y$.

Definition 2.5 [5] *A convex metric space is said to have the property* (C) *if every bounded decreasing net of nonempty, closed and convex subsets of* X *has a nonempty intersection.*

For instance, every weakly compact convex subset of a Banach space has the property (C). Note that convex metric spaces with the property (C) generalize the notion of reflexivity from Banach to metric spaces.

Definition 2.6 [1] *Let* a *and* b *be two points of a convex metric space* (X, d, W) *and* $r > 0$*, the closed ball (resp.opened) of focuses* a *and* b*, and of ray* r*, is defined by*

$$B(a, b, r) = \{x \in X : D(a, b, x) \leq (resp. <) r\}.$$

The ball of foci a and b can be empty, $d(a, b) \leq \frac{r}{2}$ is a necessary and sufficient condition for $B(a, b, r)$ to be nonempty. Let be $x \in B(a, b, r)$, then $D(a, b, x) \leq r$, which implies $x \in B(a, r) \cap B(b, r)$, consequently $B(a, b, r)$ is bounded and it is closed for the mapping D is continued. Furthermore, $B(a, b, r)$ is convex, let be $x, y \in B(a, b, r)$ and $\alpha \in [0, 1]$, we have

$$\begin{aligned} D(W(x, y; \alpha), a, b) &= d(W(x, y; \alpha), a) + d(W(x, y; \alpha), b) + d(a, b) \\ &\leq \alpha d(x, a) + (1 - \alpha) d(y, a) + \alpha d(x, b) + (1 - \alpha) d(y, b) + d(a, b) \\ &= \lambda D(x, a, b) + (1 - \lambda) D(y, a, b) \leq r. \end{aligned}$$

We now turn to a concept introduced by Talman [6],

Definition 2.7 [6] *Let* (X, d, W) *be a convex metric space,* W *is said to be a strict convex structure if it has the property that whenever* $w \in X$ *and there is* $(x, y; \lambda) \in X \times X \times I$ *for which*

$$d(u, w) \leq \lambda d(u, x) + (1 - \lambda) d(u, y), \quad \text{for all } u \in X,$$

then $w = W(x, y; \lambda)$*. If* W *is a a strict convex structure on* (X, d)*, then* (X, d, W) *is called a strictly convex metric space.*

Notice that the use of the term "strictly convex" does not comply with the standard usage for Banach spaces. For instance, the plane equipped with the norm $\|(x_1, x_2)\| = |x_1| + |x_2|$ is strictly convex in the sense of [6], but not in the standard sense. One can easily see that in a strictly convex metric space, the following property holds,

$$W(x, y; \lambda) = W(y, x; 1 - \lambda), \text{ forall } (x, y; \lambda) \in X \times X \times I.$$

Lemma 2.8 [6] *Let* (X,d,W) *be a strictly convex metric space, then for every* $(x,y;\lambda_1,\lambda_2)\in X^2\times I^2$, *we have*

$$W\left(W\left(x,y;\lambda_1\right),y,\lambda_2\right)=W\left(x,y,\lambda_1\lambda_2\right).$$

The concept of metric convexity is inherited from the natural one, thus, the convex structure in convex metric spaces satisfies some of the properties of the usual convex structure of linear normed spaces. Yet, it doesn't seem to verify the following property: Let X be a linear normed space and $W\left(x,y;\lambda\right)=\lambda x+\left(1-\lambda\right)y$. Let $x_1=W\left(a,b;\alpha_1\right)$ and $x_2=W\left(a,b;\alpha_2\right)$ where $\alpha_1>\alpha_2$ and $a,b\in X$, we have

$$\begin{aligned} W\left(a,x_2,\frac{\alpha_1-\alpha_2}{1-\alpha_2}\right) &= \frac{\alpha_1-\alpha_2}{1-\alpha_2}a+\frac{1-\alpha_1}{1-\alpha_2}\left(\alpha_2 a+\left(1-\alpha_2\right)b\right)\\ &= \alpha_1 a+\left(1-\alpha_1\right)b=x_1.\end{aligned}$$

$$\text{And}\qquad \begin{aligned} W\left(x_1,b,\frac{\alpha_2}{\alpha_1}\right) &= \frac{\alpha_2}{\alpha_1}\left(\alpha_1 a+\left(1-\alpha_1\right)b\right)+\frac{\alpha_1-\alpha_2}{\alpha_1}b\\ &= \alpha_2 a+\left(1-\alpha_2\right)b=x_2.\end{aligned}$$

Which leads to the following definition.

Definition 2.9 *Let* (X,d,W) *be a convex metric space.* W *is to be called a* (S) *convex structure on* X *provided that whenever* $x,y\in X$ *such that* $x=W\left(a,b,\alpha\right)$ *and* $y=W\left(a,b,\beta\right)$ *where* $\alpha>\beta$ *and* $a,b\in X$, *we have*

$$x=W\left(a,y,\frac{\alpha-\beta}{1-\beta}\right),\ \text{and}\ y=W\left(x,b,\frac{\beta}{\alpha}\right).$$

If W *is a* (S) *convex structure on* X, *then*(X,d,W) *is called a* (S) *convex metric space.*

Proposition 2.10 *Every strictly convex metric space* (X,d,W) *is a* (S) *convex metric space.*

Proof: Let $x,y\in X$ such that $x=W\left(a,b,\alpha\right)$ and $y=W\left(a,b,\beta\right)$ where $\alpha>\beta$ and $a,b\in X$, then

$$\begin{aligned} W\left(a,y,\frac{\alpha-\beta}{1-\beta}\right) &= W\left(W\left(a,b,\beta\right),a,\frac{1-\alpha}{1-\beta}\right)\\ &= W\left(W\left(b,a,1-\beta\right),a,\frac{1-\alpha}{1-\beta}\right)\\ &= W\left(b,a,1-\alpha\right)=x.\end{aligned}$$

Similarly, $y=W\left(x,b,\frac{\beta}{\alpha}\right)$.

The next examples ensure that the (S) property is not fulfilled by every convex metric space.

Example 2.11 *Let $\mathbb{R}^2$ be equipped with the Euclidean scalar product defined by $\langle (x_1,x_2),(y_1,y_2)\rangle = x_1y_1 + x_2y_2$ and let X be a nonempty closed subset of $\{x\in\mathbb{R}^2 : \|x\| = 1\}$, where $\|(x_1,x_2)\| = \sqrt{\langle (x_1,x_2),(x_1,x_2)\rangle}$, such that if $x,y\in X$ and $\alpha+\beta = 1$, then $\frac{\alpha x+\beta y}{\|\alpha x+\beta y\|}\in X$ and $diam(X)\le\frac{\sqrt{2}}{2}$ where $diam(X) = \sup\{\|x-y\| : x,y\in X\}$. Let $d(x,y) := \cos^{-1}(\langle x,y\rangle)$ for each $x,y\in X$. We define a convex structure $W : X\times X\times I\longrightarrow X$ with $W(x,y;\lambda) := \frac{\alpha x+(1-\alpha)y}{\|\alpha x+(1-\alpha)y\|}$. Then (X,d,W) is a complete convex metric space which has the (C) property, for more information consult [7]. However, W is not a (S) convex structure. Consider $a = (0,1)$ and $b = \left(\frac{3}{5},\frac{4}{5}\right)$, we have $d(a,b) = \|a-b\| = \frac{\sqrt{10}}{5}\le\frac{\sqrt{2}}{2}$. Let $x_1 = W\left(a,b;\frac{1}{2}\right) = \left(\frac{\sqrt{10}}{10},3\frac{\sqrt{10}}{10}\right)$ and $x_2 = W\left(a,b;\frac{1}{3}\right) = \left(\frac{6}{\sqrt{205}},\frac{13}{\sqrt{205}}\right)$, then $W\left(a,x_2,\frac{1}{4}\right) = \left(\frac{18}{\sqrt{2050+78\sqrt{205}}},\frac{39+\sqrt{205}}{\sqrt{2050+78\sqrt{205}}}\right)\neq x_1$.*

Example 2.12 [8] *Let $X := [-1,1]$ and define a metric on X by $d(x,y) = \max\{|x|,|y|\}$ if $x\neq y$. Define $W : X\times X\times I\longrightarrow X$ with $W(x,y;\lambda) = \lambda\min\{|x|,|y|\}$, for all $x,y\in X$ and $\lambda\in[0,1]$. Then W is a convex structure on X which dosen't satisfy the (S) property. Indeed, let $x_1 = W\left(-1,1;\frac{1}{2}\right) = \frac{1}{2}$ and $x_2 = W\left(-1,1;\frac{1}{3}\right) = \frac{1}{3}$, then $W\left(-1,\frac{1}{3};\frac{1}{4}\right) = \frac{1}{4}\min\left\{|-1|,\left|\frac{1}{3}\right|\right\} = \frac{1}{12}\neq x_1$.*

Here, we give an example of a (S) convex metric space.

Example 2.13 *Let I be the unit interval and X be the family of closed intervals $[a_i,b_i]$ such that $0\le a_i\le b_i\le 1$. For $I_i = [a_i,b_i]$, $I_j = [a_j,b_j]$ and λ $(0\le\lambda\le 1)$, define a mapping W by*

$$W(I_i,I_j,\lambda) = [\lambda a_i + (1-\lambda)a_j, \lambda b_i + (1-\lambda)b_j],$$

If d is the Hausdorff distance, then (X,d,W) is a convex metric space (see [5]). Moreover, W is a (S) convex structure on X. In this order, let $I_1 = W(I_i,I_j,\alpha_1)$ and $I_2 = W(I_i,I_j,\alpha_2)$ where $\alpha_1 > \alpha_2$, then

$$\begin{aligned} I_1 &= [\alpha_1 a_i + (1-\alpha_1)a_j, \alpha_1 b_i + (1-\alpha_1)b_j], \\ I_2 &= [\alpha_2 a_i + (1-\alpha_2)a_j, \alpha_2 b_i + (1-\alpha_2)b_j]. \end{aligned}$$

$$W\left(I_i,I_2,\frac{\alpha_1-\alpha_2}{1-\alpha_2}\right) = W\left([a_i,b_i],[\alpha_2 a_i + (1-\alpha_2)a_j, \alpha_2 b_i + (1-\alpha_2)b_j],\frac{\alpha_1-\alpha_2}{1-\alpha_2}\right),$$

we have

$$\begin{array}{c} \frac{\alpha_1-\alpha_2}{1-\alpha_2}a_i + \frac{1-\alpha_1}{1-\alpha_2}(\alpha_2 a_i + (1-\alpha_2)a_j) = \alpha_1 a_i + (1-\alpha_1)a_j, \\ \frac{\alpha_1-\alpha_2}{1-\alpha_2}b_i + \frac{1-\alpha_1}{1-\alpha_2}(\alpha_2 b_i + (1-\alpha_2)b_j) = \alpha_1 b_i + (1-\alpha_1)b_j. \end{array}$$

Hence

$$W\left(I_i,I_2,\frac{\alpha_1-\alpha_2}{1-\alpha_2}\right) = [\alpha_1 a_i + (1-\alpha_1)a_j, \alpha_1 b_i + (1-\alpha_1)b_j] = I_1.$$

Similarly, we prove that $W\left(I_1,I_j,\frac{\alpha_2}{\alpha_1}\right) = I_2$.

3 Main Results

We begin our main results with the following observation which is simple but very useful.

Proposition 3.1 *Let (X, d, W) be a (S) convex metric space and let $x, y \in X$ such that $x = W(a, b, \alpha)$, $y = W(a, b, \beta)$ and $z = W(a, b, \gamma)$ where $a, b \in X$, and $\alpha > \beta > \gamma$, then $d(x, y) = (\alpha - \beta) d(a, b)$ and $y = W\left(x, z, \frac{\beta-\gamma}{\alpha-\gamma}\right)$.*

Proof: By the (S) property, we have $x = W\left(a, y, \frac{\alpha-\beta}{1-\beta}\right)$, therefore

$$\begin{aligned} d(x, y) &= d\left(W\left(a, y, \frac{\alpha-\beta}{1-\beta}\right), y\right) = \frac{\alpha-\beta}{1-\beta} d(a, y) \\ &= \frac{\alpha-\beta}{1-\beta} d(a, W(a, b, \beta)) = (\alpha - \beta) d(a, b). \end{aligned}$$

Since $\alpha > \gamma$ and $\beta > \gamma$, then

$$x = W\left(a, z, \frac{\alpha-\gamma}{1-\gamma}\right) \text{ and } y = W\left(a, z, \frac{\beta-\gamma}{1-\gamma}\right)$$

Put $\alpha' = \frac{\alpha-\gamma}{1-\gamma}$ and $\beta' = \frac{\beta-\gamma}{1-\gamma}$, since $\alpha' > \beta'$, we have $y = W\left(x, z, \frac{\beta'}{\alpha'}\right) = W\left(x, z, \frac{\beta-\gamma}{\alpha-\gamma}\right)$.

Let (X, d_X, W_X) and (X, d_Y, W_Y) be a two convex metric spaces, the mapping $d_p : (X \times Y) \times (X \times Y) \longrightarrow [0, \infty)$ defined by

$$d_p((x_1, y_1), (x_2, y_2)) = \begin{cases} (d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p)^{\frac{1}{p}}, 1 \le p < \infty; \\ \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}, p = \infty. \end{cases}$$

is a distance on the Cartesian product $X \times Y$.

Lemma 3.2 [9] *The mapping $W_{X\times Y} : (X \times Y) \times (X \times Y) \times I \longrightarrow [0, \infty)$ defined as follows*

$$W_{X\times Y}((x_1, y_1), (x_2, y_2), \lambda) = (W_X(x_1, x_2, \lambda), W_Y(y_1, y_2, \lambda)).$$

is a convex stucture on the product metric space $(X \times Y, d_1)$.

In the light of the previous lemma, we can describe convex subsets of convex product metric spaces.

Definition 3.3 *A subset E of the convex product metric space $(X \times Y, d_1, W_{X\times Y})$ is convex if $W_{X\times Y}((x_1, y_1), (x_2, y_2), \lambda) \in E$ for all $(x_1, y_1), (x_2, y_2) \in E$ and $\lambda \in I$.*

We now state our first main result.

Theorem 3.4 *Let A, B and C be nonempty, closed, bounded and convex subsets of a (S) convex metric space (X, d, W) which has the (C) property, suppose A, B and C are disjoint subsets of $[a, b]$ where $a, b \in X$, let $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ be a tricyclic contraction map. Then T has a best proximity point.*

Proof: We denote by Σ the set of all nonempty, bounded, closed and convex triads $(E, F, G) \subseteq (A, B, C)$ such that T is tricyclic on $E \cup F \cup G$. Then Σ is nonempty for $(A, B, C) \in \Sigma$ and is partially ordered by reverse inclusion, that is,

$$(E_1, F_1, G_1) \leq (E_2, F_2, G_2) \Longleftrightarrow (E_2, F_2, G_2) \subseteq (E_1, F_1, G_1).$$

Let $(E_i, F_i, G_i)_{i \in I}$ be an increasing chain of Σ, since X has the property (C), $\underset{i \in I}{\cap} E_i$, $\underset{i \in I}{\cap} F_i$ and $\underset{i \in I}{\cap} G_i$ are nonempty, bounded, closed and convex. We have

$$(E_i, F_i, G_i) \leq \left(\underset{i \in I}{\cap} E_i, \ \ \underset{i \in I}{\cap} F_i, \ \ \underset{i \in I}{\cap} G_i \right), \text{ for all } i \in I.$$

Every increasing chain in Σ is bounded above, so by using Zorn's lemma we obtain a maximal element, say $(H, I, J) \in \Sigma$. We have

$$\left(\overline{co}\left(T\left(J\right)\right), \overline{co}\left(T\left(H\right)\right), \overline{co}\left(T\left(I\right)\right)\right) \subseteq (H, I, J)$$

Hence

$$T\left(\overline{co}\left(T\left(J\right)\right)\right) \subseteq T\left(H\right) \subseteq \overline{co}\left(T\left(H\right)\right).$$

Thus T is tricyclic on $\overline{co}\left(T\left(J\right)\right) \cup \overline{co}\left(T\left(H\right)\right) \cup \overline{co}\left(T\left(I\right)\right)$, it now follows from the maximality of (H, I, J) that

$$\overline{co}\left(T\left(J\right)\right) = H, \ \overline{co}\left(T\left(H\right)\right) = I \text{ and } \overline{co}\left(T\left(I\right)\right) = J.$$

Let $x \in H$, $y \in I$, for all $z \in J$, we have

$$\begin{aligned} D\left(Tx, Ty, Tz\right) &\leq kD\left(x, y, z\right) + (1-k)\,\delta\left(A, B, C\right) \\ &\leq k\Delta\left(H, I, J\right) + (1-k)\,\delta\left(A, B, C\right) := \Lambda. \end{aligned}$$

Then

$$Tz \in B\left(Tx, Ty, \Lambda\right), \text{ for all } z \in J.$$

Therefore $T\left(J\right) \subset B\left(Tx, Ty, \Lambda\right)$ which is nonempty, closed, bounded and convex. Thus

$$H = \overline{co}\left(T\left(J\right)\right) \subseteq B\left(Tx, Ty, \Lambda\right).$$

So

$$\Delta_{(Tx,Ty)}\left(H\right) \leq \Lambda.$$

Put

$$\begin{aligned} E &= \left\{(x,y)\in H\times I:\Delta_{(x,y)}\left(J\right)\leq\Lambda\right\},\\ F &= \left\{(y',z)\in I\times J:\Delta_{(y,z)}\left(H\right)\leq\Lambda\right\},\\ G &= \left\{(z',x')\in J\times H:\Delta_{(z,x)}\left(I\right)\leq\Lambda\right\}. \end{aligned}$$

Note that E, F and G are nonempty, bounded and closed, indeed

$$E=\bigcap_{z\in J}\Psi_z^{-1}\left([0,\Lambda]\right)\quad\text{where }\Psi_z:H\times I\longrightarrow\mathbb{R}_+;\ \ (x,y)\longmapsto D\left(x,y,z\right).$$

Besides, E, F and G are convex, in this order, let $(x_1,y_1),(x_2,y_2)\in E$, $z\in J$ and $\lambda\in[0,1]$, we have

$$\begin{aligned} D\left(W\left(x_1,x_2,\lambda\right),W\left(y_1,y_2,\lambda\right),z\right) &= d\left(W\left(x_1,x_2,\lambda\right),z\right)+d\left(W\left(y_1,y_2,\lambda\right),z\right)\\ &\quad +d\left(W\left(x_1,x_2,\lambda\right),W\left(y_1,y_2,\lambda\right)\right).\\ &\leq \lambda d\left(x_1,z\right)+\left(1-\lambda\right)d\left(x_2,z\right)+\lambda d\left(y_1,z\right)\\ &\quad +\left(1-\lambda\right)d\left(y_2,z\right)+d\left(W\left(x_1,x_2,\lambda\right),W\left(y_1,y_2,\lambda\right)\right). \end{aligned}$$

On the other hand

$$\begin{aligned} d\left(W\left(x_1,x_2,\lambda\right),W\left(y_1,y_2,\lambda\right)\right) &\leq \lambda d\left(x_1,W\left(y_1,y_2,\lambda\right)\right)+\left(1-\lambda\right)d\left(x_2,W\left(y_1,y_2,\lambda\right)\right)\\ &\leq \lambda^2 d\left(x_1,y_1\right)+\lambda\left(1-\lambda\right)\left[d\left(x_1,y_2\right)+d\left(x_2,y_1\right)\right]\\ &\quad +\left(1-\lambda\right)^2 d\left(x_2,y_2\right). \end{aligned}$$

Since $x_1,x_2,y_1,y_2\in[a,b]$,there exist $\alpha_1,\alpha_2,\beta_1$ and $\beta_2\in[0,1]$ such that

$$\begin{aligned} x_1 &= W\left(a,b,\alpha_1\right)\ \text{ and } x_2=W\left(a,b,\alpha_2\right),\\ y_1 &= W\left(a,b,\beta_1\right)\ \text{ and } y_2=W\left(a,b,\beta_2\right). \end{aligned}$$

Thus

$$\begin{aligned} d\left(x_1,y_1\right) &= \left|\alpha_1-\beta_1\right|d\left(a,b\right),\quad d\left(x_1,y_2\right)=\left|\alpha_1-\beta_2\right|d\left(a,b\right),\\ d\left(x_2,y_1\right) &= \left|\alpha_2-\beta_1\right|d\left(a,b\right),\quad d\left(x_2,y_2\right)=\left|\alpha_2-\beta_2\right|d\left(a,b\right). \end{aligned}$$

For the reason that A and B are disjoint, the cases $\alpha_i<\beta_i,\beta_j<\alpha_j$, $\beta_i<\alpha_i,\alpha_j<\beta_j$, $i,j=1,2$, aren't possible. Indeed, suppose $\alpha_1<\beta_1<\alpha_2$, then $y_1=W\left(x_2,x_1,\frac{\alpha_2-\alpha_1}{\alpha_2-\beta_1}\right)$, that means $y_1\in[x_2,x_1]$, since H is convex, then $[x_2,x_1]\subseteq H$, thus $y_1\in H$ which is a contradiction. Therefore, we either have $\alpha_1,\alpha_2>\beta_1,\beta_2$ or $\alpha_1,\alpha_2<\beta_1,\beta_2$. Hence

$$d\left(x_1,y_2\right)+d\left(x_2,y_1\right)=d\left(x_1,y_1\right)+d\left(x_2,y_2\right).$$

Consequently

$$\begin{aligned} d\left(W\left(x_1,x_2,\lambda\right),W\left(y_1,y_2,\lambda\right)\right) &\le \lambda^2 d\left(x_1,y_1\right)+\lambda\left(1-\lambda\right)\left[d\left(x_1,y_1\right)+d\left(x_2,y_2\right)\right] \\ &\quad +\left(1-\lambda\right)^2 d\left(x_2,y_2\right) \\ &= \lambda d\left(x_1,y_1\right)+\left(1-\lambda\right)d\left(x_2,y_2\right). \end{aligned}$$

Then

$$\begin{aligned} D\left(W\left(x_1,x_2,\lambda\right),W\left(y_1,y_2,\lambda\right),z\right) &\le \lambda d\left(x_1,z\right)+\left(1-\lambda\right)d\left(x_2,z\right)+\lambda d\left(y_1,z\right) \\ &\quad +\left(1-\lambda\right)d\left(y_2,z\right)+\lambda d\left(x_1,y_1\right)+\left(1-\lambda\right)d\left(x_2,y_2\right) \\ &= \lambda D\left(x_1,y_1,z\right)+\left(1-\lambda\right)D\left(x_2,y_2,z\right) \\ &\le \max\left\{D\left(x_1,y_1,z\right),D\left(x_2,y_2,z\right)\right\}\le\Lambda \end{aligned}$$

That means $\left(W\left(x_1,x_2,\lambda\right),W\left(y_1,y_2,\lambda\right)\right)=W_{X\times X}\left(\left(x_1,y_1\right),\left(x_2,y_2\right),\lambda\right)\in E$, for all $\left(x_1,y_1\right),\left(x_2,y_2\right)\in E$ and $\lambda\in[0,1]$. Define

$$\widetilde{T}\ :(A\times B)\cup(B\times C)\cup(C\times A)\longrightarrow(A\times B)\cup(B\times C)\cup(C\times A)$$

by

$$\widetilde{T}\left(x,y\right)=\left(Tx,Ty\right).$$

Since T is tricyclic on $A\cup B\cup C$, $\widetilde{T}$ is tricyclic on $(A\times B)\cup(B\times C)\cup(C\times A)$. Let $(x,y)\in H\times I$, $\widetilde{T}\left(x,y\right)=\left(Tx,Ty\right)\in F$, then $\widetilde{T}\left(H\times I\right)\subset F$. So $\widetilde{T}$ is tricyclic on $E\cup F\cup G$. Furthermore $(H\times I,I\times J,J\times H)$ is maximal in

$$\widetilde{\Sigma}=\left\{\begin{array}{c} \left(\left(E\times F\right),\left(F\times G\right),\left(G\times E\right)\right)\subset\left(\left(A\times B\right),\left(B\times C\right),\left(C\times A\right)\right) \\ \left(E\times F\right),\left(F\times G\right)\text{ and }\left(G\times E\right)\text{ are non}-\text{empty, bounded, closed} \\ \text{and convex with }\widetilde{T}\text{ is tricyclic on }\left(E\times F\right)\cup\left(F\times G\right)\cup\left(G\times E\right)\ . \end{array}\right\}.$$

$\widetilde{\Sigma}$ is partially ordered by

$$\left(\left(E_1\times F_1\right),\left(F_1\times G_1\right),\left(G_1\times E_1\right)\right)\ \widetilde{\leq}\ \left(\left(E_2\times F_2\right),\left(F_2\times G_2\right),\left(G_2\times E_2\right)\right)\Longleftrightarrow$$

$$\left(\left(E_2\times F_2\right),\left(F_2\times G_2\right),\left(G_2\times E_2\right)\right)\subset\left(\left(E_1\times F_1\right),\left(F_1\times G_1\right),\left(G_1\times E_1\right)\right).$$

So,

$$E=H\times I,\ \ F=I\times J\text{ and }G=J\times H.$$

Inductively,

$$\begin{aligned} \Delta_{(x,y)}\left(J\right) &\le \Lambda,\ \forall\left(x,y\right)\in H\times I &&\Longrightarrow \\ \Delta_{(x,y)}\left(J\right)-k\Delta\left(H,I,J\right) &\le \left(1-k\right)\delta\left(A,B,C\right),\forall\left(x,y\right)\in H\times I &&\Longrightarrow \\ \left(1-k\right)\Delta\left(H,I,J\right) &\le \left(1-k\right)\delta\left(A,B,C\right) &&\Longrightarrow \\ \Delta\left(H,I,J\right) &\le \delta\left(A,B,C\right). \end{aligned}$$

For all $(p,q,r)\in(H,I,J)$, we have

$$\delta\left(A,B,C\right)\le D\left(p,Tp,T^2p\right),D\left(T^2q,q,Tq\right),D\left(Tr,T^2r,r\right)\le\Delta\left(H,I,J\right)=\delta\left(A,B,C\right),$$

which finshes the proof of the theorem.

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