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Refinements and Reverses of Young Type Inequalities

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Abstract

Recently some Young type inequalities with Kantorovich constant for square form have been promoted. The purpose of this paper is to give further refinements to them. By using these scalar inequalities, we also obtain some results for operator and matrix.

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1 Introduction

The scalar Young inequality says that

$$a^v b^{1-v} \leq va + (1-v)b, \quad a, b > 0, and\ 0 \leq v \leq 1 \tag{1.1}$$

with equality hold if and only if $a = b$.

Hu [1] gave the following Young type inequalities

$$\begin{cases} v^{2v}(a^v b^{1-v})^2 + v^2(a-b)^2 \leq v^2a^2 + (1-v)^2b^2, & 0 \leq v \leq \frac{1}{2} \\ (1-v)^{2(1-v)}(a^v b^{1-v})^2 + (1-v)^2(a-b)^2 \leq v^2a^2 + (1-v)^2b^2, & \frac{1}{2} \leq v \leq 1. \end{cases} \tag{1.2}$$

Later, Nasiri in [4] improved the inequalities (1.2) with Kantorovich constant

$$\begin{cases} v^2a^2+(1-v)^2b^2 & \geq v^2(a-b)^2+r(b-\sqrt{vab})^2 \\ & +K(\sqrt{\frac{h}{v}},2)^{r'}v^2(a^vb^{1-v})^2, \quad 0\leq v\leq \frac{1}{2} \\ v^2a^2+(1-v)^2b^2 & \geq (1-v)^2(a-b)^2+r(a-\sqrt{(1-v)ab})^2 \\ & +K(\sqrt{(1-v)h},2)^{r'}(1-v)^{2(1-v)}(a^vb^{1-v})^2, \quad \frac{1}{2}\leq v\leq 1. \end{cases} \tag{1.3}$$

where $h=\frac{b}{a}, r=\min\{2v,1-2v\}$ and $r'=\min\{2r,1-2r\}$.
Furthermore, authors in [2] proved that

$$\begin{cases} v^2a^2+(1-v)^2b^2\leq (1-v)^2(a-b)^2+(1-v)^{2(1-v)}(a^vb^{1-v})^2, \quad 0\leq v\leq \frac{1}{2} \\ v^2a^2+(1-v)^2b^2\leq v^2(a-b)^2+v^{2v}(a^vb^{1-v})^2, \quad \frac{1}{2}\leq v\leq 1. \end{cases} \tag{1.4}$$

Which can be regarded as the reverses of Young type inequalities. In this paper, we will present some refinements of (1.3) and (1.4) that have as many terms as we wish.

To state our paper, we adopt the following notations. Like in [5], let $a,b\geq 0$ and $v\in[0,1]$ for $N\in\mathbb{N}$ and $j=1,2,\cdots,N$, let $k_j(v)=[2^{j-1}v], r_j(v)=[2^jv]$ and

$$s_j(v)=(-1)^{r_j(v)}2^{j-1}v+(-1)^{r_j(v)+1}[\frac{r_j(v)+1}{2}]. \tag{1.5}$$

Then define the nonnegative function

$$S_N(v;a,b)=\sum_{j=1}^{N}s_j(v)(\sqrt[2^j]{b^{2^{j-1}-k_j(v)}a^{k_j(v)}}-\sqrt[2^j]{a^{k_j(v)+1}b^{2^{j-1}-k_j(v)-1}})^2. \tag{1.6}$$

Authors [3] proved that

$$K(\sqrt[2^N]{h},2)^{\beta_N(v)}a^vb^{1-v}+S_N(v;a,b)\leq va+(1-v)b \tag{1.7}$$

where $K(h,2)=K(\frac{1}{h},2)=\frac{(h+1)^2}{4h}, h>0, h=\frac{b}{a}, \alpha_N(v)=1+[2^Nv]-2^Nv$ and $\beta_N(v)=\min\{\alpha_N(v),1-\alpha_N(v)\}$ for $N\in\mathbb{N}$.

Throughout the paper, M_n denotes the space of all $n\times n$ complex matrices. It is well-known that the Hilbert-Schmidt norm is unitarily invariant in the sense that $|||UAV|||=|||A|||$ for all unitary matrices $U,V\in M_n$. For $A=[a_{ij}]\in M_n$, the Hilbert-Schmidt norm is defined by:

$$||A||_2=\sqrt{\sum_{i,j=1}^{n}|a_{ij}|^2}.$$

what's more, we defined

$$A\nabla_vB=(1-v)A+vB, \quad v\in[0,1]$$

$$A\sharp_v B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v A^{\frac{1}{2}}, \quad v \in R$$

denoted by $A\nabla B$ and $A\sharp B$ respectively when $v = \frac{1}{2}$.

In this paper, we will give refinements and reverses of Young type inequalities for scalar in Section 2, then we establish two inequalities for operators and Hilbert-Schmidt norm in Section 3 and Section 4 respectively.

2 Young Type Inequalities For Scalar

In this section, we mainly present the improved scalar Young type and its reverse inequalities relating to the Kantorovich constant.

Theorem 2.1

(1) $0 \le v \le \frac{1}{2}$

$$\begin{aligned} & bS_N(2v; va, b) + K(\sqrt[2^N]{\tfrac{h}{v}}, 2)^{\beta_N(2v)} v^{2v}(a^v b^{1-v})^2 + v^2(a-b)^2 \\ \le\ & v^2a^2 + (1-v)^2b^2 \\ \le\ & (1-v)^2(a-b)^2 + K(\sqrt[2^N]{(1-v)h}, 2)^{-\beta_N(2v)}(1-v)^{2(1-v)}(a^v b^{1-v})^2 \\ & -S_N(2v; (1-v)ab, a^2). \end{aligned} \tag{2.1}$$

(2) $\frac{1}{2} \le v \le 1$

$$\begin{aligned} & aS_N(2v-1; a, (1-v)b) + K(\sqrt[2^N]{(1-v)h}, 2)^{\beta_N(2v-1)}(1-v)^{2(1-v)}(a^v b^{1-v})^2 \\ & +(1-v)^2(a-b)^2 \\ \le\ & v^2a^2 + (1-v)^2b^2 \\ \le\ & v^2(a-b)^2 + K(\sqrt[2^N]{\tfrac{h}{v}}, 2)^{-\beta_N(2v-1)} v^{2v}(a^v b^{1-v})^2 - S_N(2v-1; b^2, vab). \end{aligned} \tag{2.2}$$

Proof. By (1.7) we have

$$\begin{aligned} & v^2a^2 + (1-v)^2b^2 - v^2(a-b)^2 \\ & = b[2v(va) + (1-2v)b] \\ & \ge b[K(\sqrt[2^N]{\tfrac{h}{v}}, 2)^{\beta_N(2v)} v^{2v} a^{2v} b^{1-2v} + S_N(2v; va, b)] \\ & = bS_N(2v; va, b) + K(\sqrt[2^N]{\tfrac{h}{v}}, 2)^{\beta_N(2v)} v^{2v}(a^v b^{1-v})^2 \end{aligned}$$

and

$$\begin{aligned} & (1-v)^2(a-b)^2 - v^2a^2 - (1-v)^2b^2 + K(\sqrt[2^N]{(1-v)h}, 2)^{-\beta_N(2v)}(1-v)^{2(1-v)}(a^v b^{1-v})^2 \\ = \ & 2v(1-v)ab + (1-2v)a^2 + (2v-2)ab \\ & +K(\sqrt[2^N]{(1-v)h}, 2)^{-\beta_N(2v)}(1-v)^{2(1-v)}(a^v b^{1-v})^2 \\ \ge \ & K(\sqrt[2^N]{(1-v)h}, 2)^{\beta_N(2v)}(1-v)^{2v}(a^{1-v} b^v)^2 + S_N(2v; (1-v)ab, a^2) + (2v-2)ab \\ & +K(\sqrt[2^N]{(1-v)h}, 2)^{-\beta_N(2v)}(1-v)^{2(1-v)}(a^v b^{1-v})^2 \\ \ge \ & S_N(2v; (1-v)ab, a^2). \end{aligned}$$

So we completed the proof of (2.1).

By (1.7) we have

$$
\begin{aligned}
& v^2a^2+(1-v)^2b^2-(1-v)^2(a-b)^2 \\
&= a[(2v-1)a+2(1-v)(1-v)b] \\
&\geq a[K(\sqrt[2^N]{(1-v)h},2)^{\beta_N(2v-1)}(1-v)^{2(1-v)}a^{2v-1}b^{2(1-v)}+S_N(2v-1;a,(1-v)b)] \\
&= K(\sqrt[2^N]{(1-v)h},2)^{\beta_N(2v-1)}(1-v)^{2(1-v)}(a^vb^{1-v})^2+aS_N(2v-1;a,(1-v)b)
\end{aligned}
$$

and

$$
\begin{aligned}
& v^2(a-b)^2-v^2a^2-(1-v)^2b^2+K(\sqrt[2^N]{\tfrac{h}{v}},2)^{-\beta_N(2v-1)}v^{2v}(a^vb^{1-v})^2 \\
= \;& (2v-1)b^2+(2-2v)vab-2vab+K(\sqrt[2^N]{\tfrac{h}{v}},2)^{-\beta_N(2v-1)}v^{2v}(a^vb^{1-v})^2 \\
\geq \;& K(\sqrt[2^N]{\tfrac{h}{v}},2)^{\beta_N(2v-1)}v^{2-2v}(a^{1-v}b^v)^2+S_N(2v-1;b^2,vab)-2vab \\
& +K(\sqrt[2^N]{\tfrac{h}{v}},2)^{-\beta_N(2v-1)}v^{2v}(a^vb^{1-v})^2 \\
\geq \;& S_N(2v-1;b^2,vab).
\end{aligned}
$$

So we completed the proof of (2.2). □

3 Inequalities For Operators

In this section, we will give some Young type inequalities for operators by the monotonic property of operator functions.

Lemma3.1 Let $X\in B(H)$ be self-adjoint and let f and g be continuous real functions such that $f(t)\geq g(t)$ for all $t\in Sp(X)$ (the Spectrum of X). Then $f(X)\geq g(X)$.

Theorem 3.2 Let $A,B\in B(H)$ be positive invertible operators, I be the identity operator and $v\in[0,1]$. If all positive numbers m,m' and M,M' satisfy the following conditions

$$0<mI\leq A\leq m'I<M'I\leq B\leq MI,$$

then

(1) $0\leq v\leq\frac{1}{2}$,

$$
\begin{aligned}
& \sum_{j=1}^{N} s_j(2v)\Big(v^{\frac{k_j(2v)}{2^{j-1}}}A\sharp_{\frac{k_j(2v)}{2^j}}B+v^{\frac{k_j(2v)+1}{2^{j-1}}}A\sharp_{\frac{k_j(2v)+1}{2^j}}B-2v^{\frac{2k_j(2v)+1}{2^j}}A\sharp_{\frac{2k_j(2v)+1}{2^{j+1}}}B\Big) \\
& \quad +\min_{h'\leq x\leq h}K(\sqrt[2^{N+1}]{xv^2},2)^{\beta_N(2v)}v^{2v}A\sharp_vB+2v^2(A\nabla B-A\sharp B) \\
\leq \;& v^2B+(1-v)^2A \qquad (3.1) \\
\leq \;& 2(1-v)^2(A\nabla B-A\sharp B)+\max_{h'\leq x\leq h}K(\sqrt[2^{N+1}]{\tfrac{x}{(1-v)^2}},2)^{-\beta_N(2v)}(1-v)^{2(1-v)}A\sharp_vB \\
& -\sum_{j=1}^{N} s_j(2v)\Big((1-v)^{\frac{k_j(2v)}{2^{j-1}}}A\sharp_{1-\frac{k_j(2v)}{2^j}}B+(1-v)^{\frac{k_j(2v)+1}{2^{j-1}}}A\sharp_{1-\frac{k_j(2v)+1}{2^j}}B \\
& -2(1-v)^{\frac{2k_j(2v)+1}{2^j}}A\sharp_{1-\frac{2k_j(2v)+1}{2^{j+1}}}B\Big);
\end{aligned}
$$

(2) $\frac{1}{2}\leq v\leq 1$,

$$\begin{aligned}
&\sum_{j=1}^{N} s_j(2v-1)((1-v)^{\frac{2^{j-1}-k_j(2v-1)}{2^{j-1}}} A\sharp_{\frac{2^{j-1}-k_j(2v-1)}{2^j}} B \\
&\quad +(1-v)^{\frac{2^{j-1}-k_j(2v-1)-1}{2^{j-1}}} A\sharp_{\frac{2^{j-1}-k_j(2v-1)-1}{2^j}} B - 2(1-v)^{\frac{2^j-2k_j(2v-1)-1}{2^j}} A\sharp_{\frac{2^j-2k_j(2v-1)-1}{2^{j+1}}} B) \\
&\quad + \min_{h'\le x\le h} K(\sqrt[2^{N+1}]{x(1-v)^2}, 2)^{\beta_N(2v-1)}(1-v)^{2(1-v)} A\sharp_{1-v} B + 2(1-v)^2(A\nabla B - A\sharp B) \\
\le\ & v^2A + (1-v)^2B \qquad (3.2) \\
\le\ & 2v^2(A\nabla B - A\sharp B) + \max_{h'\le x\le h} K(\sqrt[2^{N+1}]{\tfrac{x}{v^2}}, 2)^{-\beta_N(2v-1)} v^{2v} A\sharp_{1-v} B \\
&\quad -\sum_{j=1}^{N} s_j(2v-1)(v^{\frac{2^{j-1}-k_j(2v-1)}{2^{j-1}}} A\sharp_{\frac{2^{j-1}+k_j(2v-1)}{2^j}} B + v^{\frac{2^{j-1}-k_j(2v-1)-1}{2^{j-1}}} A\sharp_{\frac{2^{j-1}+k_j(2v-1)+1}{2^j}} B \\
&\quad -2v^{\frac{2^j-2k_j(2v-1)-1}{2^j}} A\sharp_{\frac{2^j+2k_j(2v-1)+1}{2^{j+1}}} B).
\end{aligned}$$

where $h' = \frac{M'}{m'}$, $h = \frac{M}{m}$, $k_j(v) = [2^j v]$ and $\beta_N(v) = \min\{1+[2^N v] - 2^N v, 2^N v - [2^N v]\}$.

Proof. Let $b = 1,\ a^2 = c$ in (2.1) and expand the summand to get

$$\begin{aligned}
&\sum_{j=1}^{N} s_j(2v)(v^{\frac{k_j(2v)}{2^{j-1}}} c^{\frac{k_j(2v)}{2^j}} + v^{\frac{k_j(2v)+1}{2^{j-1}}} c^{\frac{k_j(2v)+1}{2^j}} - 2v^{\frac{2k_j(2v)+1}{2^j}} c^{\frac{2k_j(2v)+1}{2^{j+1}}}) \\
&\quad +K(\sqrt[2^{N+1}]{cv^2}, 2)^{\beta_N(2v)} v^{2v} c^v + v^2(c+1-2\sqrt{c}) \\
\le\ & v^2c + (1-v)^2 \qquad (3.3) \\
\le\ & (1-v)^2(c+1-2\sqrt{c}) + K(\sqrt[2^{N+1}]{\tfrac{c}{(1-v)^2}}, 2)^{-\beta_N(2v)}(1-v)^{2(1-v)} c^v \\
&\quad -\sum_{j=1}^{N} s_j(2v)((1-v)^{\frac{k_j(2v)}{2^{j-1}}} c^{1-\frac{k_j(2v)}{2^j}} + (1-v)^{\frac{k_j(2v)+1}{2^{j-1}}} c^{1-\frac{k_j(2v)+1}{2^j}} \\
&\quad -2(1-v)^{\frac{2k_j(2v)+1}{2^j}} c^{1-\frac{2k_j(2v)+1}{2^{j+1}}}).
\end{aligned}$$

For $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, by conditions, we can get $I \le h'I = \frac{M'}{m'}I \le X \le hI = \frac{M}{m}I$, and then $Sp(X) \subseteq [h', h] \subseteq (1, +\infty)$. By lemma 3.1 and (3.3), we have

$$\begin{aligned}
&\sum_{j=1}^{N} s_j(2v)(v^{\frac{k_j(2v)}{2^{j-1}}} X^{\frac{k_j(2v)}{2^j}} + v^{\frac{k_j(2v)+1}{2^{j-1}}} X^{\frac{k_j(2v)+1}{2^j}} - 2v^{\frac{2k_j(2v)+1}{2^j}} X^{\frac{2k_j(2v)+1}{2^{j+1}}}) \\
&\quad + \min_{h'\le x\le h} K(\sqrt[2^{N+1}]{xv^2}, 2)^{\beta_N(2v)} v^{2v} X^v + v^2(X + I - 2X^{\frac{1}{2}}) \\
\le\ & v^2X + (1-v)^2I \\
\le\ & (1-v)^2(X + I - 2X^{\frac{1}{2}}) + \max_{h'\le x\le h} K(\sqrt[2^{N+1}]{\tfrac{x}{(1-v)^2}}, 2)^{-\beta_N(2v)}(1-v)^{2(1-v)} X^v \\
&\quad -\sum_{j=1}^{N} s_j(2v)((1-v)^{\frac{k_j(2v)}{2^{j-1}}} X^{1-\frac{k_j(2v)}{2^j}} + (1-v)^{\frac{k_j(2v)+1}{2^{j-1}}} X^{1-\frac{k_j(2v)+1}{2^j}} \\
&\quad -2(1-v)^{\frac{2k_j(2v)+1}{2^j}} X^{1-\frac{2k_j(2v)+1}{2^{j+1}}}).
\end{aligned}$$

Since the Kantorovich constant $K(t, 2) = \frac{(t+1)^2}{4t}$ is an increasing function on $(1, +\infty)$, then

$$\begin{aligned}
&\sum_{j=1}^{N} s_j(2v)(v^{\frac{k_j(2v)}{2^{j-1}}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{k_j(2v)}{2^j}} + v^{\frac{k_j(2v)+1}{2^{j-1}}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{k_j(2v)+1}{2^j}} \\
&\quad -2v^{\frac{2k_j(2v)+1}{2^j}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{2k_j(2v)+1}{2^{j+1}}}) + \min_{h'\le x\le h} K(\sqrt[2^{N+1}]{xv^2}, 2)^{\beta_N(2v)} v^{2v}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v \\
&\quad +v^2((A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) + I - 2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}) \\
\le\ & v^2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) + (1-v)^2 I \qquad (3.4) \\
\le\ & (1-v)^2((A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) + I - 2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}) \\
&\quad + \max_{h'\le x\le h} K(\sqrt[2^{N+1}]{\tfrac{x}{(1-v)^2}}, 2)^{-\beta_N(2v)}(1-v)^{2(1-v)}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v \\
&\quad -\sum_{j=1}^{N} s_j(2v)((1-v)^{\frac{k_j(2v)}{2^{j-1}}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-\frac{k_j(2v)}{2^j}} \\
&\quad +(1-v)^{\frac{k_j(2v)+1}{2^{j-1}}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-\frac{k_j(2v)+1}{2^j}} \\
&\quad -2(1-v)^{\frac{2k_j(2v)+1}{2^j}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-\frac{2k_j(2v)+1}{2^{j+1}}}).
\end{aligned}$$

Then multiplying inequalities (3.4) by $A^{\frac{1}{2}}$ on both sides, we can get the required inequality (3.1).
Using the same technique in (2.2), we can get (3.2). So we completed the proof. □

4 Inequalities for Unitarily Invariant Norm

In this section, we will present Young type inequalities for unitarily invariant norm.
Theorem 4.1 Suppose $A, B, X \in M_n$ such that A, B are two positive definite matrices, then we have
for $0 \le v \le \frac{1}{2}$,

$$\begin{aligned}
&2v(1-v)||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2 + v^2||AX - XB||_2^2 + K_1^{\beta_N(2v)} v^{2v}||A^v XB^{1-v}||_2^2 \\
&+\sum_{j=1}^{N} s_j(2v)||v^{\frac{k_j(2v)}{2^j}} A^{\frac{k_j(2v)}{2^j}} XB^{1-\frac{k_j(2v)}{2^j}} - v^{\frac{k_j(2v)+1}{2^j}} A^{\frac{k_j(2v)+1}{2^j}} XB^{1-\frac{k_j(2v)+1}{2^j}}||_2^2 \\
\le\ & ||vAX + (1-v)XB||_2^2 \qquad (4.1) \\
\le\ & 2v(1-v)||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2 + (1-v)^2||AX - XB||_2^2 + K_2^{-\beta_N(2v)}(1-v)^{2(1-v)}||A^v XB^{1-v}||_2^2 \\
&-\sum_{j=1}^{N} s_j(2v)||(1-v)^{\frac{k_j(2v)}{2^j}} A^{1-\frac{k_j(2v)}{2^j}} XB^{\frac{k_j(2v)}{2^j}} - (1-v)^{\frac{k_j(2v)+1}{2^j}} A^{1-\frac{k_j(2v)+1}{2^j}} XB^{\frac{k_j(2v)+1}{2^j}}||_2^2,
\end{aligned}$$

for $\frac{1}{2} \le v \le 1$,

$$\begin{aligned}
& 2v(1-v)||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2+(1-v)^2||AX-XB||_2^2+K_2^{\beta_N(2v-1)}(1-v)^{2(1-v)}||A^vXB^{1-v}||_2^2 \\
+ & \sum_{j=1}^{N} s_j(2v-1)||(1-v)^{\frac{2^{j-1}-k_j(2v-1)}{2^j}}A^{\frac{2^{j-1}+k_j(2v-1)}{2^j}}XB^{\frac{2^{j-1}-k_j(2v-1)}{2^j}} \\
& -(1-v)^{\frac{2^{j-1}-k_j(2v-1)-1}{2^j}}A^{\frac{2^{j-1}+k_j(2v-1)+1}{2^j}}XB^{\frac{2^{j-1}-k_j(2v-1)-1}{2^j}}||_2^2 \\
\leq & ||vAX+(1-v)XB||_2^2 \qquad (4.2) \\
\leq & 2v(1-v)||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2+v^2||AX-XB||_2^2+K_1^{-\beta_N(2v-1)}v^{2v}||A^vXB^{1-v}||_2^2 \\
& -\sum_{j=1}^{N} s_j(2v-1)||v^{\frac{2^{j-1}-k_j(2v-1)}{2^j}}A^{\frac{2^{j-1}-k_j(2v-1)}{2^j}}XB^{\frac{2^{j-1}+k_j(2v-1)}{2^j}} \\
& -v^{\frac{2^{j-1}-k_j(2v-1)-1}{2^j}}A^{\frac{2^{j-1}-k_j(2v-1)-1}{2^j}}XB^{\frac{2^{j-1}+k_j(2v-1)+1}{2^j}}||_2^2.
\end{aligned}$$

where $K_1=\min\{K(\sqrt[2^N]{\frac{t_{il}}{v}},2), \quad 1\leq i,l\leq n\}$, $K_2=\min\{K(\sqrt[2^N]{(1-v)t_{il}},2), \quad 1\leq i,l\leq n\}$ for $t_{il}=\frac{\nu_l}{\lambda_i}$, $\beta_N(v)=\min\{\alpha_N(v),1-\alpha_N(v)\}$, $\alpha_N(v)=1+[2^Nv]-2^Nv$, λ_i are eigenvalues of A and ν_l are eigenvalues of B.

Proof. Since A and B are positive definite, it follows by spectral theorem that there exist unitary matrices $U,V\in M_n$ such that

$$A=U\Lambda_1U^*, B=V\Lambda_2V^*,$$

where $\Lambda_1=diag(\lambda_1,\lambda_2,\cdots,\lambda_n), \Lambda_2=diag(\nu_1,\nu_2,\cdots,\nu_n), \lambda_i,\nu_i>0, i=1,2,\cdots,n$. Let $Y=U^*XV=[y_{il}]$, then

$$vAX+(1-v)XB=U[v\Lambda_1Y+(1-v)Y\Lambda_2]V^*=U[v\lambda_i+(1-v)\nu_l)y_{il}]V^*,$$

$$AX-XB=U[(\lambda_i-\nu_l)y_{il}]V^*, \quad A^vXB^{1-v}=U[(\lambda_i^v\nu_l^{1-v})y_{il}]V^*$$

and

$$\begin{aligned}
& v^{\frac{k_j(2v)}{2^j}}A^{\frac{k_j(2v)}{2^j}}XB^{1-\frac{k_j(2v)}{2^j}}-v^{\frac{k_j(2v)+1}{2^j}}A^{\frac{k_j(2v)+1}{2^j}}XB^{1-\frac{k_j(2v)+1}{2^j}} \\
= & U[(v^{\frac{k_j(2v)}{2^j}}\lambda_i^{\frac{k_j(2v)}{2^j}}\nu_l^{1-\frac{k_j(2v)}{2^j}}-v^{\frac{k_j(2v)+1}{2^j}}\lambda_i^{\frac{k_j(2v)+1}{2^j}}\nu_l^{1-\frac{k_j(2v)+1}{2^j}})y_{il}]V^*.
\end{aligned}$$

$$\begin{aligned}
& (1-v)^{\frac{k_j(2v)}{2^j}}A^{1-\frac{k_j(2v)}{2^j}}XB^{\frac{k_j(2v)}{2^j}}-(1-v)^{\frac{k_j(2v)+1}{2^j}}A^{1-\frac{k_j(2v)+1}{2^j}}XB^{\frac{k_j(2v)+1}{2^j}} \\
= & U[((1-v)^{\frac{k_j(2v)}{2^j}}\lambda_i^{1-\frac{k_j(2v)}{2^j}}\nu_l^{\frac{k_j(2v)}{2^j}}-(1-v)^{\frac{k_j(2v)+1}{2^j}}\lambda_i^{1-\frac{k_j(2v)+1}{2^j}}\nu_l^{\frac{k_j(2v)+1}{2^j}})y_{il}]V^*.
\end{aligned}$$

Now, by (2.1) and the unitary invariance of the Hilbert-Schmidt norm, we have

$$
\begin{aligned}
&2v(1-v)||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2+v^2||AX-XB||_2^2+K_1^{\beta_N(2v)}v^{2v}||A^vXB^{1-v}||_2^2\\
&+\sum_{j=1}^{N}s_j(2v)||v^{\frac{k_j(2v)}{2^j}}A^{\frac{k_j(2v)}{2^j}}XB^{1-\frac{k_j(2v)}{2^j}}-v^{\frac{k_j(2v)+1}{2^j}}A^{\frac{k_j(2v)+1}{2^j}}XB^{1-\frac{k_j(2v)+1}{2^j}}||_2^2\\
=&\sum_{i,l=1}^{n}\{2v(1-v)(\lambda_i^{\frac{1}{2}}\nu_l^{\frac{1}{2}})^2+v^2(\lambda_i-\nu_l)^2+\min K(\sqrt[2^N]{\tfrac{t_{il}}{v}},2)^{\beta_N(2v)}v^{2v}(\lambda_i^v\nu_l^{1-v})^2\\
&+\sum_{j=1}^{N}s_j(2v)(v^{\frac{k_j(2v)}{2^j}}\lambda_i^{\frac{k_j(2v)}{2^j}}\nu_l^{1-\frac{k_j(2v)}{2^j}}-v^{\frac{k_j(2v)+1}{2^j}}\lambda_i^{\frac{k_j(2v)+1}{2^j}}\nu_l^{1-\frac{k_j(2v)+1}{2^j}})^2\}|y_{il}|^2\\
\le&\sum_{i,l=1}^{n}(v\lambda_i+(1-v)\nu_l)^2|y_{il}|^2\\
=&||vAX+(1-v)XB||_2^2\\
\le&\sum_{i,l=1}^{n}\{2v(1-v)(\lambda_i^{\frac{1}{2}}\nu_l^{\frac{1}{2}})^2+(1-v)^2(\lambda_i-\nu_l)^2\\
&+\max K(\sqrt[2^N]{(1-v)t_{il}},2)^{-\beta_N(2v)}(1-v)^{2(1-v)}(\lambda_i^v\nu_l^{1-v})^2\\
&-\sum_{j=1}^{N}s_j(2v)((1-v)^{\frac{k_j(2v)}{2^j}}\lambda_i^{1-\frac{k_j(2v)}{2^j}}\nu_l^{\frac{k_j(2v)}{2^j}}-(1-v)^{\frac{k_j(2v)+1}{2^j}}\lambda_i^{1-\frac{k_j(2v)+1}{2^j}}\nu_l^{\frac{k_j(2v)+1}{2^j}})^2\}|y_{il}|^2\\
=&2v(1-v)||A^{\frac{1}{2}}XB^{\frac{1}{2}}||_2^2+(1-v)^2||AX-XB||_2^2+K_2^{-\beta_N(2v)}(1-v)^{2(1-v)}||A^vXB^{1-v}||_2^2\\
&-\sum_{j=1}^{N}s_j(2v)||(1-v)^{\frac{k_j(2v)}{2^j}}A^{1-\frac{k_j(2v)}{2^j}}XB^{\frac{k_j(2v)}{2^j}}-(1-v)^{\frac{k_j(2v)+1}{2^j}}A^{1-\frac{k_j(2v)+1}{2^j}}XB^{\frac{k_j(2v)+1}{2^j}}||_2^2.
\end{aligned}
$$

Using the same technique in (2.2), we can get (4.2). So we completed the proof. □

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