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Sums of Tribonacci and Tribonacci-Lucas Numbers

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Abstract

In this short article, we consider finite nonalternating and alternating sums of Tribonacci and Tribonacci-Lucas numbers. By applying an elementary telescoping argument, we obtain new identities for these sums.

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1 Introduction and Preliminaries

For $n \geq 2$, the Tribonacci numbers $(T_n)_{n\geq 0}$ (sequence A000073 in The On-Line Encyclopedia of Integer Sequences [10]) and the Tribonacci-Lucas numbers $(K_n)_{n\geq 0}$ (sequence A001644 in [10]) are defined, respectively, by

$$T_{n+1} = T_n + T_{n-1} + T_{n-2}, \qquad T_0 = 0, T_1 = T_2 = 1, \tag{1}$$

and

$$K_{n+1} = K_n + K_{n-1} + K_{n-2}, \qquad K_0 = 3, K_1 = 1, K_2 = 3. \tag{2}$$

The first few terms of the sequence $(T_n)_{n\geq 0}$ are

$$0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705,$$

[1]Disclaimer: Statements and conclusions made in this article are entirely those of the author. They do not necessarily reflect the views of LBBW.

whereas for $(K_n)_{n\geq 0}$ we have

$$3, 1, 3, 7, 11, 21, 39, 71, 131, 241, 443, 815, 1499, 2757, 5071.$$

Both sequences must be regarded as generalizations of the famous Fibonacci numbers. The name "Tribonacci numbers" was given by Feinberg [5]. Tribonacci as well as Tribonacci-Lucas numbers are members of the general Tribonacci recurrence. The properties of these numbers are studied in many articles. The most recent articles include but are not limited to [1], [2], [3], [4], [7], [8] and [9].
The Binet formulas are given by

$$T_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}, \tag{3}$$

and

$$K_n = \alpha^n + \beta^n + \gamma^n, \tag{4}$$

where α, β and γ are roots of the cubic equation $x^3 - x^2 - x - 1 = 0$, i.e.,

$$\alpha = \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3},$$

$$\beta = \frac{1 + \omega\sqrt[3]{19 + 3\sqrt{33}} + \omega^2\sqrt[3]{19 - 3\sqrt{33}}}{3},$$

$$\gamma = \frac{1 + \omega^2\sqrt[3]{19 + 3\sqrt{33}} + \omega\sqrt[3]{19 - 3\sqrt{33}}}{3},$$

where $\omega = \frac{-1+i\sqrt{3}}{2}$ is a primitive cube root of unity.

In what follows, we will also need to define the Tribonacci numbers and the Tribonacci-Lucas numbers for negative indices. This can be done as follows (see [9]): We set $T_{-n} = B_n$ and $K_{-n} = C_n$ where

$$B_n = -B_{n-1} - B_{n-2} + B_{n-3}, \qquad B_{-1} = 1, B_0 = B_1 = 0, \tag{5}$$

and

$$C_n = -C_{n-1} - C_{n-2} + C_{n-3}, \qquad C_{-1} = 1, C_0 = 3, C_1 = -1. \tag{6}$$

We are interested in finding expressions for the sums $\sum_{k=1}^n T_{mk}$, $\sum_{k=1}^n (-1)^{k-1} T_{mk}$, $\sum_{k=1}^n K_{mk}$, $\sum_{k=1}^n (-1)^{k-1} K_{mk}$, where m is an integer. The first examples of these evaluations, which may be proved by induction on n, are

$$\sum_{k=1}^n T_k = \frac{1}{2}(T_{n+2} + T_n - 1) \tag{7}$$

and

$$\sum_{k=1}^{n} T_{2k} = \frac{1}{2}(T_{2n+1} + T_{2n} - 1). \tag{8}$$

Equation (7) appears as Theorem 2 in [7]. In the same paper, the author also states the following expression for the sum of the $4k$ subscripted Tribonacci numbers ([7], Theorem 5):

$$\sum_{k=1}^{n} T_{4k} = \frac{1}{T_4^2}(T_{4n+4} + 6T_{4n} + T_{4n-4} - T_4). \tag{9}$$

In this note we show that the sum evaluations from above are special cases of more general sum identities that we will present in the next section.

2 Results

We state our results in two separate theorems. We first provide a lemma, which we will use in the proofs of our results.

Lemma 2.1 *The following identities hold for Tribonacci and Tribonacci-Lucas numbers:*

$$T_{k+n} = T_k K_n - T_{k-n} C_n + T_{k-2n}, \tag{10}$$

and

$$K_{k+n} = K_k K_n - K_{k-n} C_n + C_{2n-k}, \tag{11}$$

where the sequence C_n may be expressed as $C_n = \alpha^n \beta^n + \alpha^n \gamma^n + \beta^n \gamma^n$.

PROOF: Both identities follow essentially from the Binet forms (3) and (4). See [9] and [2] for details. □

Now we are ready to state the main results of this paper. Proofs will follow in the next section.

Theorem 2.2 *Let m be a positive integer. Then*

$$\sum_{k=1}^{n} T_{mk} = \frac{T_{m(n+1)} + (1 - C_m)T_{mn} + T_{m(n-1)} - T_m - B_m}{K_m - C_m}, \tag{12}$$

and

$$\sum_{k=1}^{n} (-1)^{k-1} T_{mk} = \frac{(-1)^{n+1}(T_{m(n+1)} + (1 + C_m)T_{mn} - T_{m(n-1)}) + T_m - B_m}{K_m + C_m + 2}. \tag{13}$$

The corresponding identities for the Tribonacci-Lucas numbers are contained in the next theorem:

Theorem 2.3 *Let m be a positive integer. Then*

$$\sum_{k=1}^{n} K_{mk} = \frac{K_{m(n+1)} + (1-C_m)K_{mn} + K_{m(n-1)} + 2C_m - K_m - 3}{K_m - C_m}, \tag{14}$$

and

$$\sum_{k=1}^{n}(-1)^{k-1} K_{mk} = \frac{(-1)^{n+1}(K_{m(n+1)} + (1+C_m)K_{mn} - K_{m(n-1)}) + 2C_m + K_m + 3}{K_m + C_m + 2}. \tag{15}$$

Before providing proofs, we present some explicit evaluations. The identities in equations (7), (8) and (9) follow from the first part of Theorem 2.2 for $m = 1, 2$ and 4, respectively. For $m = 3$ we obtain

$$\sum_{k=1}^{n} T_{3k} = \frac{T_{3n+3} - 4T_{3n} + T_{3n-3} - 1}{2}, \tag{16}$$

which may be stated in the equivalent form

$$\sum_{k=1}^{n} T_{3k} = \frac{T_{3n+2} - T_{3n} - 1}{2}. \tag{17}$$

Moreover, we have

$$\sum_{k=1}^{n}(-1)^{k-1} T_k = \frac{(-1)^{n+1}(T_{n+1} - T_{n-1}) + 1}{2}, \tag{18}$$

$$\sum_{k=1}^{n}(-1)^{k-1} T_{2k} = \frac{(-1)^{n+1}(T_{2n} + T_{2n-1})}{2}, \tag{19}$$

$$\sum_{k=1}^{n} K_k = \frac{K_{n+2} + K_n - 6}{2}, \tag{20}$$

$$\sum_{k=1}^{n}(-1)^{k-1} K_k = \frac{(-1)^{n+1}(K_{n+1} - K_{n-1}) + 2}{2}, \tag{21}$$

$$\sum_{k=1}^{n} K_{2k} = \frac{K_{2n+1} + K_{2n} - 4}{2}, \tag{22}$$

and

$$\sum_{k=1}^{n}(-1)^{k-1} K_{2k} = \frac{(-1)^{n+1}(K_{2n} + K_{2n-1}) + 2}{2}. \tag{23}$$

3 The Proofs

In this section we prove Theorems 2.2 and 2.3. The method of proof is completely elementary. The idea is to combine Lemma 2.1 with the following sum identities:

Proposition 3.1 *Let $f(k)$ be a real sequence and m, n and j be positive integers. Then*

$$\sum_{k=1}^{n}\Big(f(m(k+j)) - f(m(k-j))\Big) = \sum_{k=n+1-j}^{n+j} f(mk) - \sum_{k=1-j}^{j} f(mk), \tag{24}$$

and

$$\sum_{k=1}^{n}(-1)^{k-1}\Big(f(m(k+j)) - f(m(k-j))\Big) = \sum_{k=n+1-j}^{n+j}(-1)^{k+j-1} f(mk) - \sum_{k=1-j}^{j}(-1)^{k+j-1} f(mk). \tag{25}$$

PROOF: These sum relations may be proved straightforwardly by shifting the summation index. See [6] for details and first applications. □

PROOF of the main Theorems:

To prove Theorem 2.2 we start with equation (10). Replacing n by m and k by mk results in

$$T_{m(k+1)} - T_{m(k-1)} = K_m T_{mk} - (1 + C_m)T_{m(k-1)} + T_{m(k-2)}. \tag{26}$$

Set $f(k) = T_k$ and apply (24) with $j = 1$ to get

$$T_{m(n+1)} + T_{mn} - T_m = K_m \sum_{k=1}^{n} T_{mk} - (1 + C_m)\sum_{k=1}^{n} T_{m(k-1)} + \sum_{k=1}^{n} T_{m(k-2)}. \tag{27}$$

Since,

$$\sum_{k=1}^{n} T_{m(k-1)} = \sum_{k=1}^{n} T_{mk} - T_{mn},$$

and

$$\sum_{k=1}^{n} T_{m(k-2)} = \sum_{k=1}^{n} T_{mk} - T_{mn} - T_{m(n-1)} + T_{-m},$$

the first part of Theorem 2.2 follows immediately after rearrangement. To prove the second part, set $f(k) = T_k$ and apply (25) with $j = 1$ to get

$$(-1)^{n+1}T_{m(n+1)} + (-1)^n T_{mn} + T_m =$$

$$K_m \sum_{k=1}^{n}(-1)^{k-1}T_{mk} - (1+C_m)\sum_{k=1}^{n}(-1)^{k-1}T_{m(k-1)} + \sum_{k=1}^{n}(-1)^{k-1}T_{m(k-2)}. \quad (28)$$

Simplifying gives the stated relation.
The proof of Theorem 2.3 is very similar. Combine (11) with (24) and (25) setting $f(k) = K_k$ and $j = 1$. As the proof is straightforwardly completed, we omit the details. □

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