Sums of Tribonacci and Tribonacci-Lucas Numbers

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Abstract

In this short article, we consider finite nonalternating and alternating sums of Tribonacci and Tribonacci-Lucas numbers. By applying an elementary telescoping argument, we obtain new identities for these sums.

Mathematics Subject Classification: 11B39, 11B37

Keywords: Tribonacci number, Tribonacci-Lucas number, Sum

1 Introduction and Preliminaries

For \( n \geq 2 \), the Tribonacci numbers \((T_n)_{n \geq 0}\) (sequence A000073 in The On-Line Encyclopedia of Integer Sequences [10]) and the Tribonacci-Lucas numbers \((K_n)_{n \geq 0}\) (sequence A001644 in [10]) are defined, respectively, by

\[
T_{n+1} = T_n + T_{n-1} + T_{n-2}, \quad T_0 = 0, T_1 = T_2 = 1, \quad (1)
\]

and

\[
K_{n+1} = K_n + K_{n-1} + K_{n-2}, \quad K_0 = 3, K_1 = 1, K_2 = 3. \quad (2)
\]

The first few terms of the sequence \((T_n)_{n \geq 0}\) are

\[0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705,\]

\[\cdots\]

\[\]
whereas for \((K_n)_{n \geq 0}\) we have

\[3, 1, 3, 7, 11, 21, 39, 71, 131, 241, 443, 815, 1499, 2757, 5071.\]

Both sequences must be regarded as generalizations of the famous Fibonacci numbers. The name "Tribonacci numbers" was given by Feinberg [5]. Tribonacci as well as Tribonacci-Lucas numbers are members of the general Tribonacci recurrence. The properties of these numbers are studied in many articles. The most recent articles include but are not limited to [1], [2], [3], [4], [7], [8] and [9].

The Binet formulas are given by

\[T_n = \frac{\alpha^{n+1}}{\alpha - \beta} + \frac{\beta^{n+1}}{(\beta - \alpha)(\beta - \gamma)} + \frac{\gamma^{n+1}}{(\gamma - \alpha)(\gamma - \beta)},\]  

(3)

and

\[K_n = \alpha^n + \beta^n + \gamma^n,\]  

(4)

where \(\alpha, \beta\) and \(\gamma\) are roots of the cubic equation \(x^3 - x^2 - x - 1 = 0\), i.e.,

\[\alpha = \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3},\]

\[\beta = \frac{1 + \omega \sqrt[3]{19 + 3\sqrt{33}} + \omega^2 \sqrt[3]{19 - 3\sqrt{33}}}{3},\]

\[\gamma = \frac{1 + \omega^2 \sqrt[3]{19 + 3\sqrt{33}} + \omega \sqrt[3]{19 - 3\sqrt{33}}}{3},\]

where \(\omega = \frac{1+i\sqrt{3}}{2}\) is a primitive cube root of unity.

In what follows, we will also need to define the Tribonacci numbers and the Tribonacci-Lucas numbers for negative indices. This can be done as follows (see [9]): We set \(T_{-n} = B_n\) and \(K_{-n} = C_n\) where

\[B_n = -B_{n-1} - B_{n-2} + B_{n-3}, \quad B_{-1} = 1, B_0 = B_1 = 0,\]  

(5)

and

\[C_n = -C_{n-1} - C_{n-2} + C_{n-3}, \quad C_{-1} = 1, C_0 = 3, C_1 = -1.\]  

(6)

We are interested in finding expressions for the sums \(\sum_{k=1}^{n} T_{mk}, \sum_{k=1}^{n} (-1)^{k-1} T_{mk}, \sum_{k=1}^{n} K_{mk}, \sum_{k=1}^{n} (-1)^{k-1} K_{mk}\), where \(m\) is an integer. The first examples of these evaluations, which may be proved by induction on \(n\), are

\[\sum_{k=1}^{n} T_k = \frac{1}{2}(T_{n+2} + T_n - 1)\]  

(7)
Sums of Tribonacci and Tribonacci-Lucas numbers

and

\[ \sum_{k=1}^{n} T_{2k} = \frac{1}{2}(T_{2n+1} + T_{2n} - 1). \] (8)

Equation (7) appears as Theorem 2 in [7]. In the same paper, the author also states the following expression for the sum of the 4\(k\) subscripted Tribonacci numbers ([7], Theorem 5):

\[ \sum_{k=1}^{n} T_{4k} = \frac{1}{T_4}(T_{4n+4} + 6T_{4n} + T_{4n-4} - T_4). \] (9)

In this note we show that the sum evaluations from above are special cases of more general sum identities that we will present in the next section.

2 Results

We state our results in two separate theorems. We first provide a lemma, which we will use in the proofs of our results.

**Lemma 2.1** The following identities hold for Tribonacci and Tribonacci-Lucas numbers:

\[ T_{k+n} = T_k K_n - T_{k-n} C_n + T_{k-2n}, \] (10)

and

\[ K_{k+n} = K_k K_n - K_{k-n} C_n + C_{2n-k}, \] (11)

where the sequence \( C_n \) may be expressed as \( C_n = \alpha^n \beta^n + \alpha^n \gamma^n + \beta^n \gamma^n \).

PROOF: Both identities follow essentially from the Binet forms (3) and (4). See [9] and [2] for details. \( \square \)

Now we are ready to state the main results of this paper. Proofs will follow in the next section.

**Theorem 2.2** Let \( m \) be a positive integer. Then

\[ \sum_{k=1}^{n} T_{mk} = \frac{T_{m(n+1)} + (1 - C_m)T_{mn} + T_{m(n-1)} - T_m - B_m}{K_m - C_m}, \] (12)

and

\[ \sum_{k=1}^{n} (-1)^{k-1} T_{mk} = \frac{(-1)^{n+1}(T_{m(n+1)} + (1 + C_m)T_{mn} - T_{m(n-1)}) + T_m - B_m}{K_m + C_m + 2}. \] (13)
The corresponding identities for the Tribonacci-Lucas numbers are contained in the next theorem:

**Theorem 2.3** Let \( m \) be a positive integer. Then

\[
\sum_{k=1}^{n} K_{mk} = \frac{K_{m(n+1)} + (1 - C_m)K_{mn} + K_{m(n-1)} + 2C_m - K_m - 3}{K_m - C_m},
\]

(14)

and

\[
\sum_{k=1}^{n} (-1)^{k-1}K_{mk} = \frac{(-1)^{n+1}(K_{m(n+1)} + (1 + C_m)K_{mn} - K_{m(n-1)}) + 2C_m + K_m + 3}{K_m + C_m + 2}.
\]

(15)

Before providing proofs, we present some explicit evaluations. The identities in equations (7), (8) and (9) follow from the first part of Theorem 2.2 for \( m = 1, 2 \) and 4, respectively. For \( m = 3 \) we obtain

\[
\sum_{k=1}^{n} T_{3k} = \frac{T_{3n+3} - 4T_{3n} + T_{3n-3} - 1}{2},
\]

(16)

which may be stated in the equivalent form

\[
\sum_{k=1}^{n} T_{3k} = \frac{T_{3n+2} - T_{3n} - 1}{2}.
\]

(17)

Moreover, we have

\[
\sum_{k=1}^{n} (-1)^{k-1}T_{k} = \frac{(-1)^{n+1}(T_{n+1} - T_{n-1}) + 1}{2},
\]

(18)

\[
\sum_{k=1}^{n} (-1)^{k-1}T_{2k} = \frac{(-1)^{n+1}(T_{2n} + T_{2n-1})}{2},
\]

(19)

\[
\sum_{k=1}^{n} K_{k} = \frac{K_{n+2} + K_{n} - 6}{2},
\]

(20)

\[
\sum_{k=1}^{n} (-1)^{k-1}K_{k} = \frac{(-1)^{n+1}(K_{n+1} - K_{n-1}) + 2}{2},
\]

(21)

\[
\sum_{k=1}^{n} K_{2k} = \frac{K_{2n+1} + K_{2n} - 4}{2},
\]

(22)

and

\[
\sum_{k=1}^{n} (-1)^{k-1}K_{2k} = \frac{(-1)^{n+1}(K_{2n} + K_{2n-1}) + 2}{2}.
\]

(23)
3 The Proofs

In this section we prove Theorems 2.2 and 2.3. The method of proof is completely elementary. The idea is to combine Lemma 2.1 with the following sum identities:

**Proposition 3.1** Let \( f(k) \) be a real sequence and \( m, n \) and \( j \) be positive integers. Then

\[
\sum_{k=1}^{n} (f(m(k + j)) - f(m(k - j))) = \sum_{k=n+1-j}^{n+j} f(mk) - \sum_{k=1-j}^{j} f(mk), \tag{24}
\]

and

\[
\sum_{k=1}^{n} (-1)^{k-1} (f(m(k + j)) - f(m(k - j))) = \sum_{k=n+1-j}^{n+j} (-1)^{k+j-1} f(mk) - \sum_{k=1-j}^{j} (-1)^{k+j-1} f(mk). \tag{25}
\]

**PROOF:** These sum relations may be proved straightforwardly by shifting the summation index. See [6] for details and first applications. \( \square \)

**PROOF of the main Theorems:**

To prove Theorem 2.2 we start with equation (10). Replacing \( n \) by \( m \) and \( k \) by \( mk \) results in

\[
T_{m(k+1)} - T_{m(k-1)} = K_m T_{mk} - (1 + C_m) T_{m(k-1)} + T_{m(k-2)}. \tag{26}
\]

Set \( f(k) = T_k \) and apply (24) with \( j = 1 \) to get

\[
T_{m(n+1)} + T_{mn} - T_m = K_m \sum_{k=1}^{n} T_{mk} - (1 + C_m) \sum_{k=1}^{n} T_{m(k-1)} + \sum_{k=1}^{n} T_{m(k-2)}. \tag{27}
\]

Since,

\[
\sum_{k=1}^{n} T_{m(k-1)} = \sum_{k=1}^{n} T_{mk} - T_{mn},
\]

and

\[
\sum_{k=1}^{n} T_{m(k-2)} = \sum_{k=1}^{n} T_{mk} - T_{mn} - T_{m(n-1)} + T_{-m},
\]

the first part of Theorem 2.2 follows immediately after rearrangement. To prove the second part, set \( f(k) = T_k \) and apply (25) with \( j = 1 \) to get

\[
(-1)^{n+1} T_{m(n+1)} + (-1)^n T_{mn} + T_m = \]
\[ K_m \sum_{k=1}^{n} (-1)^{k-1}T_{mk} - (1 + C_m) \sum_{k=1}^{n} (-1)^{k-1}T_{m(k-1)} + \]
\[ \sum_{k=1}^{n} (-1)^{k-1}T_{m(k-2)}. \]

(28)

Simplifying gives the stated relation.
The proof of Theorem 2.3 is very similar. Combine (11) with (24) and (25) setting \( f(k) = K_k \) and \( j = 1 \). As the proof is straightforwardly completed, we omit the details.

\[ \square \]

References


Received: December 19, 2017; Published: January 18, 2018