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A Neumann Boundary Value Problem of Sand Transport: Existence and Homogenization of a Short Term Case[1]

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Abstract

In this paper we consider a model built for short-term dynamics of dunes in tidal area in a non periodic case. It is a degenerated parabolic equation which moreover, is singularly perturbed. Considering the Neumann boundary value problem, we, then give an existence and uniqueness result for the model. And finally with the

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two-scale convergence tool, an homogenized model is derived with the corrector term.

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1 Introduction and modeling

Dunes and megaripples generation and dynamics, on the seabed over a continental shelf, are the results of interaction between the seabed and water currents. The dunes formation is due to transport of sediments and in particular sand transportation. Sand transportation problem near the coast in areas subjected to tide has always interested the humanity and in particular the scientific community.
Many works related to the transport of sand exist in the literature, among which one can quote those of DeVriend [8], Engelund and Hansen [9], Kennedy [17], Blondeau [5], Dawson, Johns and Soulsby [7], Johns, Soulsby and Chesher [16], Idier [14] and Idier, Astruc and Hulsher [15] etc. In these works, the authors study the coupling of an equation of fluid dynamics (water) modeled by the shallow water equations and a transport equation modeling the sand near the coasts. In general, models are based on in situ observations and the results obtained after modeling prove ineffective. Then the problem of mathematical modeling for studying the sand transport is necessary. Mathematical modeling of the evolution of submarine dunes and movement of the sea can allow the study of the phenomena of erosion particularly near the coast. Coastal erosion is a serious problem, the man whose environmental dangers are recognized. It takes place when the sea is gaining ground on the earth, resulting in a breakdown of natural protection of the coasts, and the progressive loss of land. It is in this sense that Faye et al [12] proposed mathematical models, valid for short, mean and long term of sand dynamics near the coast in tidal areas. These models have been studied by using asymptotic methods in a domain without boundary , the torus of dimension 2 denoted by $\mathbb{T}^2$. In this paper, we shall consider a case with boundary conditions. The objective of this work is to study the models proposed by Faye et al [12] posed in a domain Ω of class $\mathcal{C}^1$ with boundary Γ. More precisely, we focus on linear models for seabed evolution and on methods which allow the removal of the explicit presence of the tide oscillations from them in a domain Ω of class $\mathcal{C}^1$ with boundary Γ.

It is therefore possible to define an adapted boundary condition to the model by considering the transport fluctuation q. One can define a Dirichlet or Neumann condition in the boundary of Ω.
The equation modeling sand transport is given by the Exner equation [14, 12]

$$\begin{cases} \dfrac{\partial z}{\partial t} - \nabla \cdot q = 0, \text{ in } [0,T) \times \Omega \\ \qquad q = q_f - \lambda |q_f| \nabla z. \end{cases} \tag{1.1}$$

where q_f stands for the water velocity induced sand flow on a flat seabed and where $|q_f|$ stands for its norm. The constant λ is the inverse value of the maximum slope of the sediment surface

when the water velocity is 0.
From this equation and considering the transport flow due to Van Rijn see [14], Faye et al [12] showed that

$$\begin{cases} \frac{\partial z^\epsilon}{\partial t} - \frac{1}{\epsilon}\nabla\cdot\Big(\mathcal{A}^\epsilon\nabla z^\epsilon\Big) = \frac{1}{\epsilon}\nabla\cdot\mathcal{C}^\epsilon,\ [0,T)\times\mathbb{T}^2 \\ z^\epsilon(0,x) = z_0(x),\ \mathbb{T}^2, \end{cases} \tag{1.2}$$

is a relevant model for short term dynamics of dunes near the sea bed, where $z_0 \in H^1(\mathbb{T}^2)$ is a given function and $\mathbb{T}^2$ is the two dimensional torus. The coefficients $\mathcal{A}^\epsilon$ and $\mathcal{C}^\epsilon$ are given by

$$\mathcal{A}^\epsilon(t,x) = a(1-b\epsilon\mathbf{m})g_a(|\mathbf{u}|),\ \text{and}\ \mathcal{C}^\epsilon(t,x) = c(1-b\epsilon\mathbf{m})g_c(|\mathbf{u}|)\frac{\mathbf{u}}{|\mathbf{u}|} \tag{1.3}$$

where $a>0,\ b,\ c$ are reals numbers. The fields $\mathbf{u}:[0,T)\times\mathbb{T}^2\to\mathbb{R}^2$ and $\mathbf{m}:[0,T)\times\mathbb{T}^2\to\mathbb{R}$ are respectively the water velocity and the height variation due to the tide. We suppose that they are given by

$$\mathbf{u}(t,x) = \mathcal{U}(t,\frac{t}{\epsilon},x)\ \text{and}\ \mathbf{m}(t,x) = \mathcal{M}(t,\frac{t}{\epsilon},x), \tag{1.4}$$

where

$$\begin{cases} \mathcal{U} = \mathcal{U}(t,\theta,x)\ \text{and}\ \mathcal{M} = \mathcal{M}(t,\theta,x)\ \text{are regular functions on}\ \mathbb{R}\times\mathbb{R}\times\mathbb{T}^2, \\ \theta\longmapsto(\mathcal{U},\mathcal{M})\ \text{is periodic of period 1}, \\ |\mathcal{U}|,\ |\frac{\partial\mathcal{U}}{\partial t}|,\ |\frac{\partial\mathcal{U}}{\partial\theta}|,\ |\nabla\mathcal{U}|,\ |\mathcal{M}|,\ |\frac{\partial\mathcal{M}}{\partial t}|,\ |\frac{\partial\mathcal{M}}{\partial\theta}|,\ |\nabla\mathcal{M}|\ \text{are bounded by}\ d, \\ \forall(t,\theta,x)\in\mathbb{R}^+\times\mathbb{R}\times\mathbb{T}^2,\ |\mathcal{U}(t,\theta,x)|\le U_{thr}\Longrightarrow \\ \frac{\partial\mathcal{U}}{\partial t} = 0,\ \frac{\partial\mathcal{M}}{\partial t} = 0,\ \nabla\mathcal{M}(t,\theta,x) = 0\ \text{and}\ \nabla\mathcal{U}(t,\theta,x) = 0, \\ \exists\theta_\alpha<\theta_\omega\in[0,1]\ \text{such that}\ \forall\ \theta\in[\theta_\alpha,\theta_\omega]\Longrightarrow|\mathcal{U}(t,\theta,x)|\ge U_{thr}. \end{cases} \tag{1.5}$$

g_a and g_c are positive functions satisfying the following hypotheses

$$\begin{cases} g_a\ge g_c\ge 0,\ g_c(0) = g_c'(0) = 0, \\ \exists\, d\ge 0,\ \ \sup_{u\in\mathbb{R}^+}|g_a(u)| + \sup_{u\in\mathbb{R}^+}|g_a'(u)|\le d,\ \ \sup_{u\in\mathbb{R}^+}|g_c(u)| + \sup_{u\in\mathbb{R}^+}|g_c'(u)|\le d, \\ \exists\, U_{thr}\ge 0,\ \exists\, G_{thr}>0,\ \text{such that}\ u\ge U_{thr}\Longrightarrow g_a(u)\ge G_{thr}. \end{cases} \tag{1.6}$$

Let Ω be an open set of $\mathbb{R}^2$ with boundary $\partial\Omega$ of class at least piecewise $\mathcal{C}^1$ and containing the torus $\mathbb{T}^2$. We assume that the sand does not go out of Ω. Thus the flux q is zero on the boundary of $\partial\Omega$,what is translated by $q\cdot n = 0$ on $\partial\Omega$, where n is the normal external vector and

$$q = q_f - |q_f|\lambda\nabla z. \tag{1.7}$$

Using (1.7) we get

$$q\cdot n = q_f\cdot n - |q_f|\lambda\nabla z\cdot n = 0\ \text{on}\ \partial\Omega, \tag{1.8}$$

then, assuming that $q_f\neq 0$ on $\partial\Omega$, we get:

$$\nabla z\cdot n = \frac{\partial z}{\partial n} = \frac{q_f\cdot n}{|q_f|\lambda} = g\ \text{on}\ \partial\Omega. \tag{1.9}$$

That equation defines the boundary condition we shall consider in the sequel with an additional consideration on the dependence on time of g.
Thus the model that we shall consider in this work is therefore the following

$$\left\{\begin{array}{c} \frac{\partial z^\epsilon}{\partial t} - \frac{1}{\epsilon}\nabla\cdot\Big(\mathcal{A}^\epsilon\nabla z^\epsilon\Big) = \frac{1}{\epsilon}\nabla\cdot\mathcal{C}^\epsilon \text{ in } (0,T)\times\Omega \\ z^\epsilon(0,x) = z_0(x) \text{ in } \in\Omega \\ \frac{\partial z^\epsilon}{\partial n} = g \text{ on } (0,T)\times\partial\Omega, \end{array}\right. \tag{1.10}$$

where g is a given function in $L^2([0,T),H^1(\mathbb{R}^2))$.
In the following section, we shall look at the well-posed of the problem (1.10).This type of problem is more general than the one studied in Faye et al [12]. But the main difficulty lies in fact that, there are integrals terms on the boundary of Ω to be controlled.

2 Existence and estimates

In [12], the authors considered models for short, mean and long terms for dune morphodynamics in tide influenced environments for seabed evolution. Existence and uniqueness results of these models have been already studied in [12], for the short and mean term and in [13] for the long term model. But it is important to quote that there studies are done in the torus that is a compact manifold without boundary. This simplified the estimations in our previous works. The objective of this paper is to study the short term model by considering domain Ω with boundary. In our context, we shall consider a Neuman boundary value problem. In the same mind of hypothesis (1.4), let

$$\mathcal{A}^\epsilon(t,x) = \widetilde{\mathcal{A}}_\epsilon(t,\frac{t}{\epsilon},x),\ \mathcal{C}^\epsilon(t,x) = \widetilde{\mathcal{C}}_\epsilon(t,\frac{t}{\epsilon},x) \tag{2.1}$$

where

$$\widetilde{\mathcal{A}}_\epsilon(t,\theta,x) = a(1-b\epsilon\mathcal{M}(t;\theta,x))g_a(|\mathcal{U}(t,\theta,x)|) \tag{2.2}$$

$$\widetilde{\mathcal{C}}_\epsilon(t,\theta,x) = c(1-b\epsilon\mathcal{M}(t,\theta,x))g_c(\mathcal{U}(t,\theta,x))\frac{\mathcal{U}(t,\theta,x)}{|\mathcal{U}(t,\theta,x)|} \tag{2.3}$$

where $\mathcal{U}$ and $\mathcal{M}$ are given in (1.4). A relevant model for short term dynamics of dunes is the following:

$$\left\{\begin{array}{c} \frac{\partial z^\epsilon}{\partial t} - \frac{1}{\epsilon}\nabla\cdot\Big(\mathcal{A}^\epsilon\nabla z^\epsilon\Big) = \frac{1}{\epsilon}\nabla\cdot\mathcal{C}^\epsilon \text{ in } (0,T)\times\Omega \\ z^\epsilon(0,x) = z_0(x),\ \text{in } \Omega \\ \frac{\partial z^\epsilon}{\partial n} = g \text{ in } (0,T)\times\partial\Omega \end{array}\right. \tag{2.4}$$

where $z_0\in H^1(\Omega),\ g\in L^2([0,T),H^1(\mathbb{R}^2)$ and Ω is an open set of $\mathbb{R}^2$.

Theorem 2.1 *Let Ω be a measurable set of classe $\mathcal{C}^1$. Under assumptions (1.4), (1.5) and (1.6), for any $T>0,\ a>0$ and $b,c\in\mathbb{R}$ and if $z_0\in H^1(\Omega)$ and $g\in L^2([0,T),H^1(\mathbb{R}^2))$, there exists a unique function $z^\epsilon\in L^\infty([0,T),H^1(\Omega))$ solution to (2.4). This solution satisfies*

$$\left\|z^\epsilon\right\|_{L^\infty([0,T),H^1(\Omega))} \leq \tilde{\gamma} \tag{2.5}$$

where $\tilde{\gamma}$ is a constant depending only on $z_0,\ G_{thr}$ and g.

Before giving the proof of theorem 2.1, we show that the coefficients of equation (2.4), as well as their derivatives, are bounded independently of ϵ. Existence for a given ϵ follows from results of Lions [20], Ladyzenkaja et al [19]. But, since our aim is to study the asymptotic behavior of z^ϵ as ϵ goes to 0, we need estimates which do not depend on ϵ. For this, based on the work of Faye et al [12], we are going to show the existence of the following Neumann boundary value problems:
$\forall \mu > 0,\ \nu > 0$ find $\mathcal{S}^\nu = \mathcal{S}^\nu(t,\theta,x)$ and $\mathcal{S}^\nu_\mu = \mathcal{S}^\nu_\mu(t,\theta,x)$ periodic of period 1 in θ and solution to

$$\begin{cases} \frac{\partial \mathcal{S}^\nu}{\partial \theta} - \nabla\cdot\Big(\big(\widetilde{\mathcal{A}}_\epsilon(t,\theta,\cdot)+\nu\big)\nabla \mathcal{S}^\nu\Big) = \nabla\cdot\widetilde{\mathcal{C}}_\epsilon(t,\theta,\cdot) \text{ in } (0,T)\times\mathbb{R}\times\Omega \\ \mathcal{S}^\nu(0,0,x) = z_0(x) \text{ in } \Omega \\ \frac{\partial \mathcal{S}^\nu}{\partial n} = g \text{ on } (0,T)\times\mathbb{R}\times\partial\Omega \end{cases} \tag{2.6}$$

and

$$\begin{cases} \mu\mathcal{S}^\nu_\mu + \frac{\partial \mathcal{S}^\nu_\mu}{\partial \theta} - \nabla\cdot\Big(\big(\widetilde{\mathcal{A}}_\epsilon(t,\theta,\cdot)+\nu\big)\nabla \mathcal{S}^\nu_\mu\Big) = \nabla\cdot\widetilde{\mathcal{C}}_\epsilon(t,\theta,\cdot) \text{ in } (0,T)\times\mathbb{R}\times\Omega \\ \mathcal{S}^\nu_\mu(0,0,x) = z_0(x) \text{ in } \Omega \\ \frac{\partial \mathcal{S}^\nu_\mu}{\partial n} = g \text{ on } (0,T)\times\mathbb{R}\times\partial\Omega. \end{cases} \tag{2.7}$$

In equations (2.6) and (2.7), t is only a parameter. We assume that the following hypothesis holds

$$\forall \epsilon > 0,\ \nu > 0,\ \mu > 0,\ (\widetilde{\mathcal{A}}_\epsilon + \nu)g + \widetilde{\mathcal{C}}_\epsilon\cdot n = 0 \text{ on } \partial\Omega, \tag{2.8}$$

where n is the outward normal on $\partial\Omega$.
This hypothesis translates a free mean divergence:

$$\int_\Omega \nabla\cdot\Big(\big(\widetilde{\mathcal{A}}_\epsilon+\nu\big)\nabla\mathcal{S}^\nu_\mu + \widetilde{\mathcal{C}}_\epsilon\Big)dx = 0.$$

It can be considered in order to simplify the expressions of the boundary value problem (2.7). In fact in the case where the mean divergence of $-(\widetilde{\mathcal{A}}_\epsilon(t,\theta,x)+\nu)\nabla\mathcal{S}^\nu_\mu - \widetilde{\mathcal{C}}_\epsilon(t,\theta,x)$ over Ω is not zero, we can add the expression $\frac{1}{|\Omega|}\int_{\partial\Omega}\Big(\big(\widetilde{\mathcal{A}}_\epsilon+\nu\big)g + \widetilde{\mathcal{C}}_\epsilon\cdot n\Big)$ in the first equation of (2.7). This hypothesis plays an important role in our method. Actually, we shall construct processes $\big(\mathcal{S}^\nu_\mu\big)_{\mu,\nu}$ that give the desired results in terms of asymptotic analysis and two scale convergence. Let us focus on existence and uniqueness of $\mathcal{S}^\nu$ and $\mathcal{S}^\nu_\mu$ solutions to (2.6) and (2.7). But before proceeding further on, we point out the following remark.

Remark 2.1 We have to notice that, under assumptions (1.4), (1.5) and (1.6), the coefficients $\widetilde{\mathcal{A}}_\epsilon$, $\widetilde{\mathcal{C}}_\epsilon$ and its derivatives are bounded on $\mathbb{R}^+\times\mathbb{R}\times\Omega$ by a constant γ not depending on ϵ. Moreover, for all $0\le\epsilon<1$, $\theta\longmapsto(\widetilde{\mathcal{A}}_\epsilon,\ \widetilde{\mathcal{C}}_\epsilon)$ is periodic of period 1 and there exists a constant $\widetilde{G}_{thr}$ and $\theta_\alpha,\ \theta_\omega\in[0,1],\ \theta_\alpha<\theta_\omega$ such that

$$\widetilde{\mathcal{A}}_\epsilon(t,\theta,x) \ge \widetilde{G}_{thr}, \forall (t,x)\in\mathbb{R}\times\Omega \tag{2.9}$$

and such that $\forall\,(t,\theta,x)\in\mathbb{R}^+\times\mathbb{R}\times\Omega$

$$\widetilde{\mathcal{A}}_\epsilon(t,\theta,x)\le\widetilde{G}_{thr}\Longrightarrow\begin{cases}\dfrac{\partial\widetilde{\mathcal{A}}_\epsilon}{\partial t}(t,\theta,x)=0,\ \nabla\widetilde{\mathcal{A}}_\epsilon(t,,\theta,x)=0,\\ \dfrac{\partial\widetilde{\mathcal{C}}_\epsilon}{\partial t}(t,,\theta,x)=\ \nabla\cdot\widetilde{\mathcal{C}}_\epsilon(t,\theta,x)=0.\end{cases}\tag{2.10}$$

We have the following theorem:

Theorem 2.2 *Under the same assumptions as in theorem 2.1 and under assumptions (2.8), (2.9) and (2.10), $\forall\epsilon>0,\ \nu>0,\ \mu>0$, there exists a unique $\mathcal{S}_\mu^\nu=\mathcal{S}_\mu^\nu(t,\theta,x)$ 1-periodic in θ, solution to (2.7). Moreover there exists constants $\gamma_2,\ \gamma_3,\ \gamma_6$ which depends only on $\Omega,\ \gamma,\ \nu,\ g$ such that*

$$\sup_{\theta\in\mathbb{R}}\Big|\int_\Omega\mathcal{S}_\mu^\nu(\theta,x)dx\Big|=0,\tag{2.11}$$

$$\Big\|\mathcal{S}_\mu^\nu\Big\|_{L^2_\#(\mathbb{R},H^1(\Omega))}\le\sqrt{\frac{\gamma^2|\Omega|^2}{\nu^2}+2\frac{\gamma^2|\Omega|^2}{\nu}+\Big\|z_0\Big\|_2^2},\tag{2.12}$$

$$\Big\|\nabla\mathcal{S}_\mu^\nu\Big\|_{L^2_\#(\mathbb{R},L^2(\Omega))}\le\frac{\gamma|\Omega|}{\nu},\tag{2.13}$$

$$\Big\|\frac{\partial\mathcal{S}_\mu^\nu}{\partial\theta}\Big\|_{L^2_\#(\mathbb{R},L^2(\Omega))}\le\gamma\Big(\frac{1}{2}\frac{\gamma^2|\Omega|^2}{\nu^2}+\frac{\gamma|\Omega|}{\nu}\Big),\tag{2.14}$$

$$\Big\|\Delta\mathcal{S}_\mu^\nu\Big\|^2_{L^2_\#(\mathbb{R},L^2(\Omega))}\le\gamma_2,\tag{2.15}$$

$$\Big\|\nabla\mathcal{S}_\mu^\nu\Big\|_{L^\infty_\#(\mathbb{R},L^2(\Omega))}\le\gamma_3,\tag{2.16}$$

$$\Big\|\mathcal{S}_\mu^\nu\Big\|_{L^\infty_\#(\mathbb{R},L^2(\Omega))}\le\gamma_3,\tag{2.17}$$

$$\Big\|\frac{\partial\mathcal{S}_\mu^\nu}{\partial t}\Big\|^2_{L^\infty_\#(\mathbb{R},H^1(\Omega))}\le\gamma_6.\tag{2.18}$$

Proof Integrating (2.7) over Ω, we get

$$\mu\int_\Omega\mathcal{S}_\mu^\nu dx+\int_\Omega\frac{\partial\mathcal{S}_\mu^\nu}{\partial\theta}dx-\int_\Omega\nabla\cdot\Big((\widetilde{\mathcal{A}}_\epsilon+\nu)\nabla\mathcal{S}_\mu^\nu\Big)dx=\int_\Omega\nabla\cdot\widetilde{\mathcal{C}}_\epsilon dx\tag{2.19}$$

Using Green formula in the third term of (2.19) and in the right hand side of the above equality, we have:

$$\mu\int_\Omega\mathcal{S}_\mu^\nu dx+\frac{d}{d\theta}\int_\Omega\mathcal{S}_\mu^\nu dx-\int_{\partial\Omega}(\widetilde{\mathcal{A}}^\epsilon+\nu)\frac{\partial S_\mu^\nu}{\partial n}d\sigma=\int_{\partial\Omega}\widetilde{\mathcal{C}}_\epsilon\cdot nd\sigma.$$

Then,

$$\mu\int_\Omega\mathcal{S}_\mu^\nu dx+\frac{d}{d\theta}\int_\Omega\mathcal{S}_\mu^\nu dx=\int_{\partial\Omega}\Big((\widetilde{\mathcal{A}}_\epsilon+\nu)g+\widetilde{\mathcal{C}}_\epsilon\cdot n\Big)d\sigma.\tag{2.20}$$

Because of (2.8), we have

$$\mu \int_\Omega \mathcal{S}_\mu^\nu dx + \frac{d}{d\theta} \int_\Omega \mathcal{S}_\mu^\nu dx = 0. \tag{2.21}$$

This equation is an ordinary differential equation whose unknown is $\int_\Omega \mathcal{S}_\mu^\nu(\theta,\cdot)dx$. For an initial condition given in $\theta_0 \in [0,1]$ the solution of (2.21) can be written as follows

$$\int_\Omega \mathcal{S}_\mu^\nu(t,\theta,\cdot)dx = e^{-\mu(\theta_0-\theta)} \int_\Omega \mathcal{S}_\mu^\nu(t,\theta_0,\cdot)dx. \tag{2.22}$$

For $\theta = \theta_0 + 1$ we have

$$\int_\Omega \mathcal{S}_\mu^\nu(t,\theta_0+1,\cdot)dx = e^{-\mu} \int_\Omega \mathcal{S}_\mu^\nu(t,\theta_0,\cdot)dx. \tag{2.23}$$

But thanks to the periodicity of $\mathcal{S}_\mu^\nu$ in θ, $\mathcal{S}_\mu^\nu(t,\theta_0+1,x) = \mathcal{S}_\mu^\nu(t,\theta_0,x)$ for and t and x in their respective space.
Because of the periodicity of $\mathcal{S}_\mu^\nu$, the only possibility of satisfying the last equality is:

$$\int_\Omega \mathcal{S}_\mu^\nu(t,\theta,x)dx = 0 \quad \text{for any} \quad \theta \in [0;1].$$

Therefore, the relation (2.11) is true.
Multiplying (2.7) by $\mathcal{S}_\mu^\nu$ and integrating over Ω, we get

$$\mu \int_\Omega \left|\mathcal{S}_\mu^\nu\right|^2 dx + \frac{1}{2}\frac{d}{d\theta} \int_\Omega \left|\mathcal{S}_\mu^\nu\right|^2 dx + \int_\Omega (\widetilde{\mathcal{A}}_\epsilon + \nu)\left|\nabla \mathcal{S}_\mu^\nu\right|^2 dx - \int_{\partial\Omega} (\widetilde{\mathcal{A}}_\epsilon + \nu)\frac{\partial S_\mu^\nu}{\partial n}\mathcal{S}_\mu^\nu d\sigma = \int_\Omega \nabla\cdot\widetilde{\mathcal{C}}_\epsilon\, \mathcal{S}_\mu^\nu dx. \tag{2.24}$$

Using Green formula's in the left hand side, we get

$$\mu \int_\Omega \left|\mathcal{S}_\mu^\nu\right|^2 dx + \frac{1}{2}\frac{d}{d\theta} \int_\Omega \left|\mathcal{S}_\mu^\nu\right|^2 dx + \int_\Omega (\widetilde{\mathcal{A}}_\epsilon + \nu)\left|\nabla \mathcal{S}_\mu^\nu\right|^2 dx = \int_{\partial\Omega} \Big((\widetilde{\mathcal{A}}_\epsilon+\nu)g + \widetilde{\mathcal{C}}_\epsilon\cdot n\Big)\mathcal{S}_\mu^\nu d\sigma - \int_\Omega \widetilde{\mathcal{C}}_\epsilon\cdot\nabla\mathcal{S}_\mu^\nu dx \tag{2.25}$$

Because of (2.8) we have

$$\mu\left\|\mathcal{S}_\mu^\nu\right\|_2^2 + \frac{1}{2}\frac{d}{d\theta}\Big(\left\|\mathcal{S}_\mu^\nu\right\|_2^2\Big) + \int_\Omega (\widetilde{\mathcal{A}}_\epsilon + \nu)\left|\nabla\mathcal{S}_\mu^\nu\right|^2 dx \le \gamma|\Omega|\left\|\nabla\mathcal{S}_\mu^\nu\right\|_{L^2(\Omega)}. \tag{2.26}$$

Integrating (2.26) over $\theta \in [0,1]$, we get

$$\mu\left\|\mathcal{S}_\mu^\nu\right\|^2_{L^2_\#(\mathbb{R},L^2(\Omega))} + \int_0^1 \int_\Omega (\widetilde{\mathcal{A}}_\epsilon + \nu)\left|\nabla\mathcal{S}_\mu^\nu\right|^2 dx \le \gamma|\Omega|\left\|\nabla\mathcal{S}_\mu^\nu\right\|_{L^2_\#(\mathbb{R},L^2(\Omega))}.$$

From this last inequality, and thanks to $\widetilde{\mathcal{A}}_\epsilon + \nu \ge \nu$ and because of the positivity of the first term, we get

$$\nu \int_0^1 \int_\Omega \left|\nabla\mathcal{S}_\mu^\nu\right|^2 dx \le \gamma|\Omega|\left\|\nabla\mathcal{S}_\mu^\nu\right\|_{L^2_\#(\mathbb{R},L^2(\Omega))} \tag{2.27}$$

and finally we have:

$$\left\|\nabla\mathcal{S}_\mu^\nu\right\|_{L^2_\#(\mathbb{R},L^2(\Omega))} \le \frac{\gamma|\Omega|}{\nu}. \tag{2.28}$$

Integrating (2.26) from 0 to another $\theta \in [0,1]$ we obtain

$$\frac{1}{2}\int_0^\theta \frac{d}{d\theta}\Big(\Big\|\mathcal{S}_\mu^\nu\Big\|_2^2\Big)d\theta \le \gamma|\Omega|\int_0^\theta \Big\|\nabla \mathcal{S}_\mu^\nu\Big\|_{L^2(\Omega)}d\theta, \tag{2.29}$$

from which we get

$$\Big\|\mathcal{S}_\mu^\nu(\theta,\cdot)\Big\|_2^2 \le 2\gamma|\Omega|\|\nabla \mathcal{S}_\mu^\nu\Big\|_{L^2_\#(\mathbb{R},H^1(\Omega))} + \Big\|\mathcal{S}_\mu^\nu(0,\cdot)\Big\|_2^2 \le 2\frac{\gamma^2|\Omega|^2}{\nu} + \Big\|z_0\Big\|_2^2. \tag{2.30}$$

Integrating this last equality from 0 to 1 with respect to the variable θ, we get

$$\Big\|\mathcal{S}_\mu^\nu\Big\|_{L^2_\#(\mathbb{R},L^2(\Omega))} \le \sqrt{2\frac{\gamma^2|\Omega|^2}{\nu} + \Big\|z_0\Big\|_2^2}. \tag{2.31}$$

From (2.28) and (2.31), we get

$$\Big\|\mathcal{S}_\mu^\nu\Big\|_{L^2_\#(\mathbb{R},H^1(\Omega))} \le \sqrt{\frac{\gamma^2|\Omega|^2}{\nu^2} + 2\frac{\gamma^2|\Omega|^2}{\nu} + \Big\|z_0\Big\|_2^2}. \tag{2.32}$$

Multiplying (2.7) by $\frac{\partial \mathcal{S}_\mu^\nu}{\partial\theta}$ and integrating over Ω, we get:

$$\frac{1}{2}\mu\frac{d}{d\theta}(\int_\Omega \Big|\mathcal{S}_\mu^\nu\Big|^2 dx) + \int_\Omega \Big|\frac{\partial \mathcal{S}_\mu^\nu}{\partial\theta}\Big|^2 dx + \int_\Omega (\widetilde{\mathcal{A}}_\epsilon+\nu)\nabla \mathcal{S}_\mu^\nu \cdot \nabla(\frac{\partial \mathcal{S}_\mu^\nu}{\partial\theta})dx - \int_{\partial\Omega}(\widetilde{\mathcal{A}}_\epsilon+\nu)g\frac{\partial \mathcal{S}_\mu^\nu}{\partial\theta}d\sigma = \int_\Omega \nabla\cdot\widetilde{\mathcal{C}}_\epsilon\frac{\partial \mathcal{S}_\mu^\nu}{\partial\theta}dx,$$

Using the fact that

$$\int_\Omega (\widetilde{\mathcal{A}}_\epsilon+\nu)\nabla \mathcal{S}_\mu^\nu \cdot \nabla \frac{\partial \mathcal{S}_\mu^\nu}{\partial\theta}dx = \frac{1}{2}\frac{d\Big(\int_\Omega(\widetilde{\mathcal{A}}_\epsilon+\nu)|\nabla \mathcal{S}_\mu^\nu|^2 dx\Big)}{d\theta} - \frac{1}{2}\int_\Omega \frac{\partial \widetilde{\mathcal{A}}_\epsilon}{\partial\theta}|\nabla \mathcal{S}_\mu^\nu|^2 dx, \tag{2.33}$$

we get

$$\frac{1}{2}\mu\frac{d}{d\theta}(\Big\|\mathcal{S}_\mu^\nu\Big\|_2^2) + \Big\|\frac{\partial \mathcal{S}_\mu^\nu}{\partial\theta}\Big\|_2^2 + \frac{1}{2}\frac{d\Big(\int_\Omega(\widetilde{\mathcal{A}}_\epsilon+\nu)|\nabla \mathcal{S}_\mu^\nu|^2 dx\Big)}{d\theta} =$$

$$\frac{1}{2}\int_\Omega \frac{\partial \widetilde{\mathcal{A}}_\epsilon}{\partial\theta}|\nabla \mathcal{S}_\mu^\nu|^2 dx + \int_{\partial\Omega}(\widetilde{\mathcal{A}}_\epsilon+\nu)g\frac{\partial \mathcal{S}_\mu^\nu}{\partial\theta}d\sigma + \int_{\partial\Omega}\widetilde{\mathcal{C}}_\epsilon\cdot n\frac{\partial \mathcal{S}_\mu^\nu}{\partial\theta}d\sigma - \int_\Omega \widetilde{\mathcal{C}}_\epsilon\cdot\nabla\Big(\frac{\partial \mathcal{S}_\mu^\nu}{\partial\theta}\Big)dx \tag{2.34}$$

Because of hypothesis (2.8)

$$\frac{1}{2}\mu\frac{d}{d\theta}(\Big\|\mathcal{S}_\mu^\nu\Big\|_2^2) + \Big\|\frac{\partial \mathcal{S}_\mu^\nu}{\partial\theta}\Big\|_2^2 + \frac{1}{2}\frac{d\Big(\int_\Omega(\widetilde{\mathcal{A}}_\epsilon+\nu)|\nabla \mathcal{S}_\mu^\nu|^2 dx\Big)}{d\theta} = \frac{1}{2}\int_\Omega \frac{\partial \widetilde{\mathcal{A}}_\epsilon}{\partial\theta}|\nabla \mathcal{S}_\mu^\nu|^2 dx - \int_\Omega \widetilde{\mathcal{C}}_\epsilon\cdot\nabla\Big(\frac{\partial \mathcal{S}_\mu^\nu}{\partial\theta}\Big)dx \tag{2.35}$$

Integrating (2.35) over $\theta \in [0,1]$ and using 1-periodicity of $\Big\|\mathcal{S}_\mu^\nu\Big\|^2$ and $\int_\Omega(\widetilde{\mathcal{A}}_\epsilon+\nu)|\nabla \mathcal{S}_\mu^\nu|^2 dx$ with respect to θ we have:

$$\Big\|\frac{\partial \mathcal{S}_\mu^\nu}{\partial\theta}\Big\|^2_{L^2_\#(\mathbb{R},L^2(\Omega))} \le \frac{1}{2}\int_0^1\int_\Omega \frac{\partial \widetilde{\mathcal{A}}_\epsilon}{\partial\theta}|\nabla \mathcal{S}_\mu^\nu|^2 dx d\theta + \int_0^1\int_\Omega \frac{\partial \widetilde{\mathcal{C}}_\epsilon}{\partial\theta}\cdot\nabla \mathcal{S}_\mu^\nu dx d\theta.$$

$$\leq \gamma\Big(\frac{1}{2}\Big\|\nabla\mathcal{S}_\mu^\nu\Big\|^2_{L^2_\#(\mathbb{R},L^2(\Omega))} + \Big\|\nabla\mathcal{S}_\mu^\nu\Big\|_{L^2_\#(\mathbb{R},L^2(\Omega))}\Big)$$

Using (2.28), we get finally

$$\Big\|\frac{\partial\mathcal{S}_\mu^\nu}{\partial\theta}\Big\|^2_{L^2_\#(\mathbb{R},L^2(\Omega))} \leq \gamma\Big(\frac{1}{2}\frac{\gamma^2|\Omega|^2}{\nu^2} + \frac{\gamma|\Omega|}{\nu}\Big).$$

Multiplying the equation (2.7) by $-\Delta\mathcal{S}_\mu^\nu$ and integrating over Ω, we get:

$$\mu\int_\Omega\Big|\nabla\mathcal{S}_\mu^\nu\Big|^2dx - \mu\int_{\partial\Omega}\frac{\partial\mathcal{S}_\mu^\nu}{\partial n}\mathcal{S}_\mu^\nu d\sigma + \int_\Omega\nabla\mathcal{S}_\mu^\nu\cdot\nabla(\frac{\partial\mathcal{S}_\mu^\nu}{\partial\theta})dx - \int_{\partial\Omega}\frac{\partial\mathcal{S}_\mu^\nu}{\partial\theta}\frac{\partial\mathcal{S}_\mu^\nu}{\partial n}d\sigma$$
$$+\int_\Omega\nabla\widetilde{\mathcal{A}}_\epsilon\cdot\nabla\mathcal{S}_\mu^\nu\Delta\mathcal{S}_\mu^\nu dx + \int_\Omega(\widetilde{\mathcal{A}}_\epsilon+\nu)\Big|\Delta\mathcal{S}_\mu^\nu\Big|^2dx = -\int_\Omega\nabla\cdot\widetilde{\mathcal{C}}_\epsilon\Delta\mathcal{S}_\mu^\nu dx.$$

Or

$$\mu\int_\Omega\Big|\nabla\mathcal{S}_\mu^\nu\Big|^2dx + \int_\Omega\nabla\mathcal{S}_\mu^\nu\cdot\nabla(\frac{\partial\mathcal{S}_\mu^\nu}{\partial\theta})dx + \int_\Omega\nabla\widetilde{\mathcal{A}}_\epsilon\cdot\nabla\mathcal{S}_\mu^\nu\Delta\mathcal{S}_\mu^\nu dx + \int_\Omega(\widetilde{\mathcal{A}}_\epsilon+\nu)\Big|\Delta\mathcal{S}_\mu^\nu\Big|^2dx =$$
$$-\int_\Omega\nabla\cdot\widetilde{\mathcal{C}}_\epsilon\Delta\mathcal{S}_\mu^\nu dx + \mu\int_{\partial\Omega}g\mathcal{S}_\mu^\nu d\sigma + \int_{\partial\Omega}\frac{\partial\mathcal{S}_\mu^\nu}{\partial\theta}g d\sigma.$$

And we deduce that

$$\mu\Big\|\nabla\mathcal{S}_\mu^\nu\Big\|_2^2 + \frac{1}{2}\frac{d}{d\theta}(\Big\|\nabla\mathcal{S}_\mu^\nu\Big\|_2^2) + \int_\Omega(\widetilde{\mathcal{A}}_\epsilon+\nu)\Big|\Delta(\mathcal{S}_\mu^\nu\Big|^2dx \leq -\int_\Omega\nabla\cdot\widetilde{\mathcal{C}}_\epsilon\Delta\mathcal{S}_\mu^\nu dx - \int_\Omega\nabla\widetilde{\mathcal{A}}_\epsilon\cdot\nabla\mathcal{S}_\mu^\nu\Delta\mathcal{S}_\mu^\nu dx$$
$$\mu\Big\|g\Big\|_{L^2(\partial\Omega)}\Big\|\mathcal{S}_\mu^\nu\Big\|_{L^2(\partial\Omega)} + \frac{d}{d\theta}\Big(\int_{\partial\Omega}g\mathcal{S}_\mu^\nu d\sigma\Big)$$

Using the following relation :

$$\Big|UV\Big| \leq \frac{\widetilde{\mathcal{A}}_\epsilon+\nu}{4}U^2 + \frac{1}{\widetilde{\mathcal{A}}_\epsilon+\nu}V^2, \tag{2.36}$$

for $U=\Delta\mathcal{S}_\mu^\nu$, $V=\nabla\cdot\widetilde{\mathcal{C}}_\epsilon$ for the first term of the right hand side and $V=\nabla\widetilde{\mathcal{A}}_\epsilon\cdot\nabla\mathcal{S}_\mu^\nu$, $U=\Delta\mathcal{S}_\mu^\nu$ for the second term of the right hand side we get:

$$\mu\|\nabla\mathcal{S}_\mu^\nu\|_2^2 + \frac{1}{2}\frac{d}{d\theta}(\|\nabla\mathcal{S}_\mu^\nu\|_2^2) + \int_\Omega(\widetilde{\mathcal{A}}_\epsilon+\nu)\Big|\Delta\mathcal{S}_\mu^\nu\Big|^2dx \leq \int_\Omega\frac{(\widetilde{\mathcal{A}}_\epsilon+\nu)}{4}\Big|\Delta\mathcal{S}_\mu^\nu\Big|^2 + \int_\Omega\frac{\Big|\nabla\cdot\widetilde{\mathcal{C}}_\epsilon\Big|^2}{\widetilde{\mathcal{A}}_\epsilon+\nu} + \tag{2.37}$$

$$\int_\Omega\frac{(\widetilde{\mathcal{A}}_\epsilon+\nu)}{4}\Big|\Delta\mathcal{S}_\mu^\nu\Big|^2dx + \int_\Omega\frac{\Big|\nabla\widetilde{\mathcal{A}}_\epsilon\Big|^2}{\widetilde{\mathcal{A}}_\epsilon+\nu}\Big|\nabla\mathcal{S}_\mu^\nu\Big|^2 + \mu C(\Omega)\Big\|g\Big\|_{L^2(\partial\Omega)}\Big\|\mathcal{S}_\mu^\nu\Big\|_{H^1(\Omega)} + \frac{d}{d\theta}\Big(\int_{\partial\Omega}g\mathcal{S}_\mu^\nu d\sigma\Big) \tag{2.38}$$

From (2.37)-(2.38) we get

$$\mu\Big\|\nabla\mathcal{S}_\mu^\nu\Big\|_2^2 + \frac{1}{2}\frac{d}{d\theta}(\Big\|\nabla\mathcal{S}_\mu^\nu\Big\|_2^2) + \frac{1}{2}\int_\Omega(\widetilde{\mathcal{A}}_\epsilon+\nu)\Big|\Delta\mathcal{S}_\mu^\nu\Big|^2dx \leq \int_\Omega\frac{\Big|\nabla\widetilde{\mathcal{A}}_\epsilon\Big|^2}{\widetilde{\mathcal{A}}_\epsilon+\nu}\Big|\nabla\mathcal{S}_\mu^\nu\Big|^2 + \int_\Omega\frac{\Big|\nabla\cdot\widetilde{\mathcal{C}}_\epsilon\Big|^2}{\widetilde{\mathcal{A}}_\epsilon+\nu} \tag{2.39}$$
$$+\mu C(\Omega)\Big\|g\Big\|_{L^2(\partial\Omega)}\Big\|\mathcal{S}_\mu^\nu\Big\|_{H^1(\Omega)} + \frac{d}{d\theta}\Big(\int_{\partial\Omega}g\mathcal{S}_\mu^\nu d\sigma\Big) \tag{2.40}$$

We have to notice that, since $\theta \to \mathcal{S}_\mu^\nu$ is periodic of period 1, $\theta \to |\nabla \mathcal{S}_\mu^\nu|^2$ and $\int_{\partial\Omega} \mathcal{S}_\mu^\nu d\sigma$ are also periodic of period 1, then, integrating (2.39) with respect to $\theta \in [0,1]$and using the fact that the first term of (2.39) in the left hand side is positive, we get:

$$\frac{1}{2}\nu\Big\|\Delta \mathcal{S}_\mu^\nu\Big\|^2_{L^2_\#(\mathbb{R},L^2(\Omega))} \leq \frac{\gamma^2}{\nu}(\Big\|\nabla \mathcal{S}_\mu^\nu\Big\|^2_{L^2_\#(\mathbb{R},L^2(\Omega))} + |\Omega|) + \mu C(\Omega)\|g\|_{L^2(\partial\Omega)}\Big\|\mathcal{S}_\mu^\nu\Big\|_{L^2_\#(\mathbb{R},H^1(\Omega))}. \quad (2.41)$$

Thus,

$$\Big\|\Delta \mathcal{S}_\mu^\nu\Big\|^2_{L^2_\#(\mathbb{R},L^2(\Omega))} \leq \frac{2\gamma^2}{\nu^2}(\frac{\gamma^2|\Omega|^2}{\nu^2} + |\Omega|) + 2\mu C(\Omega)\|g\|_{L^2(\partial\Omega)}\sqrt{\frac{\gamma^2\Omega^2}{\nu^2} + 2\frac{\gamma^2|\Omega|^2}{\nu} + \Big\|z_0\Big\|_2^2}. \quad (2.42)$$

and then

$$\Big\|\Delta \mathcal{S}_\mu^\nu\Big\|^2_{L^2_\#(\mathbb{R},L^2(\Omega))} \leq \gamma_2. \quad (2.43)$$

From (2.28), we deduce that there exists a $\theta_0 \in [0,1]$ such that

$$\|\nabla \mathcal{S}_\mu^\nu(\theta_0,\cdot)\|_2 \leq \frac{\gamma|\Omega|}{\nu}. \quad (2.44)$$

From (2.39) we can also deduce

$$\frac{1}{2}\frac{d}{d\theta}(\Big\|\nabla \mathcal{S}_\mu^\nu\Big\|_2^2) \leq \int_\Omega \frac{\Big|\nabla \widetilde{\mathcal{A}}_\epsilon\Big|^2}{\widetilde{\mathcal{A}}_\epsilon + \nu}\Big|\nabla \mathcal{S}_\mu^\nu\Big|^2 + \int_\Omega \frac{\Big|\nabla \cdot \widetilde{\mathcal{C}}_\epsilon\Big|^2}{\widetilde{\mathcal{A}}_\epsilon + \nu} + \mu C(\Omega)\Big\|g\Big\|_{L^2(\partial\Omega)}\Big\|\mathcal{S}_\mu^\nu\Big\|_{H^1(\Omega)} + \frac{d}{d\theta}\Big(\int_{\partial\Omega} g\mathcal{S}_\mu^\nu d\sigma\Big) \quad (2.45)$$

Integrating (2.45) from θ_0 given in (2.44) to $\theta_1 \in [0,1]$, we get

$$\frac{1}{2}\int_{\theta_0}^{\theta_1}\frac{d}{d\theta}(\Big\|\nabla \mathcal{S}_\mu^\nu\Big\|_2^2)d\theta \leq \int_{\theta_0}^{\theta_1}\int_\Omega \frac{\Big|\nabla \widetilde{\mathcal{A}}_\epsilon\Big|^2}{\widetilde{\mathcal{A}}_\epsilon + \nu}\Big|\nabla \mathcal{S}_\mu^\nu\Big|^2 + \int_{\theta_0}^{\theta_1}\int_\Omega \frac{\Big|\nabla \cdot \widetilde{\mathcal{C}}_\epsilon\Big|^2}{\widetilde{\mathcal{A}}_\epsilon + \nu}$$
$$+\mu c(\Omega)\Big\|g\Big\|_{L^2(\partial\Omega)}\int_{\theta_0}^{\theta_1}\Big\|\mathcal{S}_\mu^\nu\Big\|_{H^1(\Omega)} + \int_{\theta_0}^{\theta_1}\frac{d}{d\theta}\Big(\int_{\partial\Omega} g\mathcal{S}_\mu^\nu d\sigma\Big)d\theta \quad (2.46)$$

From this last inequality, we have,

$$\Big\|\nabla \mathcal{S}_\mu^\nu(\theta_1,\cdot)\Big\|_2^2 - \Big\|\nabla \mathcal{S}_\mu^\nu(\theta_0,\cdot)\Big\|_2^2 \leq \frac{2\gamma^2}{\nu}\int_{\theta_0}^{\theta_1}\Big(\int_\Omega\Big|\nabla \mathcal{S}_\mu^\nu\Big|^2 + |\Omega|\Big) +$$
$$\mu c(\Omega)\Big\|g\Big\|_{L^2(\partial\Omega)}\int_{\theta_0}^{\theta_1}\Big\|\mathcal{S}_\mu^\nu\Big\|_{H^1(\Omega)} + \int_{\theta_0}^{\theta_1}\frac{d}{d\theta}\Big(\int_{\partial\Omega} g\mathcal{S}_\mu^\nu d\sigma\Big)d\theta$$
$$\leq \frac{2\gamma^2}{\nu}\Big(\Big\|\nabla \mathcal{S}_\mu^\nu\Big\|^2_{L^2_\#(\mathbb{R},L^2(\Omega))} + |\Omega|\Big) + \mu C(\Omega)\Big\|g\Big\|_{L^2(\partial\Omega)}\Big\|\mathcal{S}_\mu^\nu\Big\|_{L^2_\#(\mathbb{R},L^2(\Omega))} \quad (2.47)$$

Using (2.28) and (2.32), we get finally

$$\Big\|\nabla \mathcal{S}_\mu^\nu(\theta_1,\cdot)\Big\|_2^2 \leq \frac{2\gamma^2}{\nu}\Big(\frac{\gamma^2|\Omega|^2}{\nu^2} + |\Omega|\Big) + 2\mu C(\Omega)\|g\|_{L^2(\partial\Omega)}\sqrt{\frac{\gamma^2|\Omega|^2}{\nu^2} + 2\frac{\gamma^2|\Omega|^2}{\nu} + \Big\|z_0\Big\|_2^2} + \frac{\gamma^2|\Omega|^2}{\nu^2}.$$

So, taking the supremum for all $\theta_1 \in [0,1]$, we get the following inequality

$$\left\|\nabla \mathcal{S}_\mu^\nu\right\|_{L^\infty_\#(\mathbb{R},L^2(\Omega))} \leq \gamma_3,$$

giving (2.16).
At any $\theta \in \mathbb{R}$, from the Fourier expansion of $\mathcal{S}_\mu^\nu(t,\theta,\cdot)$:

$$\mathcal{S}_\mu^\nu(t,\theta,\cdot) = \sum_{k\in\mathbb{N}^2} \mathcal{S}_k(t,\theta) e^{ik\cdot x}, \tag{2.48}$$

we get that

$$\|\mathcal{S}_\mu^\nu(t,\theta,\cdot)\|_2^2 \leq \|\nabla \mathcal{S}_\mu^\nu(t,\theta,\cdot)\|_2^2 \tag{2.49}$$

for any $\theta \in \mathbb{R}$. Finally we get

$$\|\mathcal{S}_\mu^\nu\|_{L^\infty_\#(\mathbb{R},L^2(\Omega))} \leq \gamma_3. \tag{2.50}$$

Following the same idea as in [12], $\frac{\partial \mathcal{S}_\mu^\nu}{\partial t}$ is solution to

$$\mu \frac{\partial \mathcal{S}_\mu^\nu}{\partial t} + \frac{\partial}{\partial \theta}(\frac{\partial \mathcal{S}_\mu^\nu}{\partial t}) - \nabla \cdot ((\widetilde{\mathcal{A}}_\epsilon + \nu)\nabla(\frac{\partial \mathcal{S}_\mu^\nu}{\partial t})) = \nabla \cdot \tilde{\mathcal{C}}^\epsilon \tag{2.51}$$

where

$$\tilde{\mathcal{C}}^\epsilon = \frac{\partial \widetilde{\mathcal{C}}_\epsilon}{\partial t} + \frac{\partial \widetilde{\mathcal{A}}_\epsilon}{\partial t} \nabla \mathcal{S}_\mu^\nu \tag{2.52}$$

and

$$\nabla \cdot \tilde{\mathcal{C}}^\epsilon = \frac{\partial(\nabla \cdot \widetilde{\mathcal{C}}_\epsilon)}{\partial t} + \frac{\partial(\nabla \widetilde{\mathcal{A}}_\epsilon)}{\partial t} \cdot \nabla S_\mu^\nu + \frac{\partial \widetilde{\mathcal{A}}_\epsilon}{\partial t} \Delta S_\mu^\nu \tag{2.53}$$

From relations (2.52) and (2.53) and the above estimates, we deduce that their exits a constant γ_4 depending only on ν, γ, Ω and γ_2 such that

$$\begin{cases} \left\|\tilde{\mathcal{C}}^\epsilon\right\|_{L^2_\#(\mathbb{R},L^2(\Omega))} \leq \gamma_4 \\ \left\|\nabla \cdot \tilde{\mathcal{C}}^\epsilon\right\|_{L^2_\#(\mathbb{R},L^2(\Omega))} \leq \gamma_4 \end{cases} \tag{2.54}$$

with

$$\gamma_4 = \gamma\Big(1 + \frac{\gamma|\Omega|}{\nu} + \sqrt{\gamma_2}\Big)$$

Multiplying (2.51) by $\frac{\partial \mathcal{S}_\mu^\nu}{\partial t}$ and integrating over Ω, we have

$$\mu \int_\Omega \Big|\frac{\partial \mathcal{S}_\mu^\nu}{\partial t}\Big|^2 + \frac{1}{2}\frac{d}{d\theta}(\int_\Omega \Big|\frac{\partial \mathcal{S}_\mu^\nu}{\partial t}\Big|^2 dx) + \int_\Omega (\widetilde{\mathcal{A}}_\epsilon + \nu)\Big|\nabla(\frac{\partial \mathcal{S}_\mu^\nu}{\partial t})\Big|^2 dx - \int_{\partial\Omega} (\widetilde{\mathcal{A}}_\epsilon + \nu)\frac{\partial(\frac{\partial \mathcal{S}_\mu^\nu}{\partial t})}{\partial n}\frac{\partial \mathcal{S}_\mu^\nu}{\partial t} d\sigma = \int_\Omega \nabla \cdot \tilde{\mathcal{C}}^\epsilon \frac{\partial \mathcal{S}_\mu^\nu}{\partial t} dx.$$

This is equivalent to

$$\mu \Big\|\frac{\partial \mathcal{S}_\mu^\nu}{\partial t}\Big\|_2^2 + \frac{1}{2}\frac{d}{d\theta}(\Big\|\frac{\partial \mathcal{S}_\mu^\nu}{\partial t}\Big\|_2^2) + \int_\Omega (\widetilde{\mathcal{A}}_\epsilon + \nu)\Big|\nabla(\frac{\partial \mathcal{S}_\mu^\nu}{\partial t})\Big|^2 dx = \int_{\partial\Omega} \Big((\widetilde{\mathcal{A}}_\epsilon + \nu)\frac{\partial g}{\partial t} + \tilde{\mathcal{C}}^\epsilon \cdot n\Big)\frac{\partial \mathcal{S}_\mu^\nu}{\partial t} d\sigma - \int_\Omega \tilde{\mathcal{C}}^\epsilon \nabla \frac{\partial \mathcal{S}_\mu^\nu}{\partial t} dx \tag{2.55}$$

Using (2.52) and (2.8), we have

$$\int_{\partial\Omega}\Big((\widetilde{\mathcal{A}}_\epsilon+\nu)\frac{\partial g}{\partial t}+\widetilde{\mathcal{C}}^\epsilon\cdot n\Big)\frac{\partial\mathcal{S}_\mu^\nu}{\partial t}d\sigma=\int_{\partial\Omega}\Big((\widetilde{\mathcal{A}}_\epsilon+\nu)\frac{\partial g}{\partial t}+\frac{\partial\widetilde{\mathcal{C}}_\epsilon}{\partial t}\cdot n+\frac{\partial\widetilde{\mathcal{A}}_\epsilon}{\partial t}\underbrace{\nabla\mathcal{S}_\mu^\nu\cdot n}_{=g}\Big)\frac{\partial\mathcal{S}_\mu^\nu}{\partial t}d\sigma \tag{2.56}$$

$$=\int_{\partial\Omega}\frac{\partial}{\partial t}\Big((\widetilde{\mathcal{A}}_\epsilon+\nu)\ g+\widetilde{\mathcal{C}}^\epsilon\cdot n\Big)\frac{\partial\mathcal{S}_\mu^\nu}{\partial t}d\sigma=0, \tag{2.57}$$

then integrating (2.55) from 0 to 1 with respect to θ, we have

$$\mu\Big\|\frac{\partial\mathcal{S}_\mu^\nu}{\partial t}\Big\|^2_{L^2_\#(\mathbb{R},L^2(\Omega))}+\int_0^1\int_\Omega\Big(\widetilde{\mathcal{A}}_\epsilon+\nu\Big)\Big|\nabla\frac{\partial\mathcal{S}_\mu^\nu}{\partial t}\Big|^2dxd\theta\le\gamma_4\int_0^1\Big\|\frac{\partial\mathcal{S}_\mu^\nu}{\partial t}\Big\|_{L^2(\Omega)}d\theta \tag{2.58}$$

From (2.58), we have,

$$\Big\|\nabla\frac{\partial\mathcal{S}_\mu^\nu}{\partial t}\Big\|^2_{L^2_\#(\mathbb{R},L^2(\Omega))}\le\frac{\gamma_4}{\nu}. \tag{2.59}$$

From (2.59), there exists $\theta_0\in[0,1]$ such that

$$\Big\|\nabla\frac{\partial\mathcal{S}_\mu^\nu(\theta_0,\cdot)}{\partial t}\Big\|_{H^1(\Omega)}\le\frac{\gamma_4}{\nu}. \tag{2.60}$$

Since $\Big\|\frac{\partial\mathcal{S}_\mu^\nu}{\partial t}\Big\|_2^2$ is positive, we get from (2.55)

$$\frac{1}{2}\frac{d}{d\theta}(\Big\|\frac{\partial\mathcal{S}_\mu^\nu}{\partial t}\Big\|_2^2)+\int_\Omega(\widetilde{\mathcal{A}}_\epsilon+\nu)\Big|\nabla(\frac{\partial\mathcal{S}_\mu^\nu}{\partial t})\Big|^2dx\le\gamma_4\Big\|\nabla\frac{\partial\mathcal{S}_\mu^\nu}{\partial t}\Big\|_2. \tag{2.61}$$

Integrating (2.61) from θ_0 given in (2.60) to $\theta_1\in[0,1]$ and because of the positivity of the second (2.61), we obtain

$$\frac{1}{2}\int_{\theta_0}^{\theta_1}\frac{d}{d\theta}(\Big\|\frac{\partial\mathcal{S}_\mu^\nu}{\partial t}\Big\|_2^2)d\theta\le\gamma_4\int_{\theta_0}^{\theta_1}\Big\|\nabla\frac{\partial\mathcal{S}_\mu^\nu}{\partial t}\Big\|_{H^1(\Omega)}d\theta \tag{2.62}$$

which gives, using (2.59) and (2.60)

$$\Big\|\frac{\partial\mathcal{S}_\mu^\nu(\theta_1,\cdot)}{\partial t}\Big\|_2^2\le2\gamma_4\sqrt{\frac{\gamma_4}{\nu}}+\frac{\gamma_4}{\nu}.$$

Then we have,

$$\Big\|\frac{\partial\mathcal{S}_\mu^\nu}{\partial t}\Big\|^2_{L^\infty_\#(\mathbb{R},L^2(\Omega))}\le2\gamma_4\sqrt{\frac{\gamma_4}{\nu}}+\frac{\gamma_4}{\nu}$$

and then

$$\Big\|\frac{\partial\mathcal{S}_\mu^\nu}{\partial t}\Big\|^2_{L^\infty_\#(\mathbb{R},L^2(\Omega))}\le\gamma_6.$$

■

Because of theorem 2.2, the sequence of solutions $(\mathcal{S}_\mu^\nu)_\mu$ to (2.7) is bounded in $L^2_\#(\mathbb{R},H^1(\Omega))$, which is a reflexive space, then there exists a sub-sequence still denoted by $(\mathcal{S}_\mu^\nu)_\mu$ and a profile $(\mathcal{S}^\nu)$ such that $(\mathcal{S}_\mu^\nu)_\mu\rightharpoonup\mathcal{S}^\nu$ in $L^2_\#(\mathbb{R},L^2(\Omega))$ and $(\nabla\mathcal{S}_\mu^\nu)_\mu\rightharpoonup\nabla\mathcal{S}^\nu$ in $L^2_\#(\mathbb{R},L^2(\Omega))$. Because of

the θ dependance of $\mathcal{S}_\mu^\nu$, we have also $\frac{\partial \mathcal{S}_\mu^\nu}{\partial\theta} \rightharpoonup \frac{\partial \mathcal{S}^\nu}{\partial\theta}$ in $L^2_\#(\mathbb{R}, L^2(\Omega))$.
Then, multiplying (2.7) by a test function $v \in H^1(\Omega)$ and integrating over $\mathbb{R}\times\Omega$, we get

$$\mu\int_0^1\int_\Omega \mathcal{S}_\mu^\nu v dx d\theta + \int_0^1\int_\Omega \frac{\partial \mathcal{S}_\mu^\nu}{\partial\theta} v dx d\theta + \int_0^1\int_\Omega \big(\widetilde{\mathcal{A}}_\epsilon + \nu\big)\nabla \mathcal{S}_\mu^\nu \nabla v dx d\theta = \int_0^1\int_\Omega \nabla\cdot\widetilde{\mathcal{C}}_\epsilon v\, dx d\theta$$
$$+\int_0^1\int_{\partial\Omega}\big(\widetilde{\mathcal{A}}_\epsilon + \nu\big) g v d\sigma d\theta. \tag{2.63}$$

Passing to the limit as $\mu\to 0$ we get

$$\int_0^1\int_\Omega \frac{\partial \mathcal{S}^\nu}{\partial\theta} v dx d\theta + \int_0^1\int_\Omega \big(\widetilde{\mathcal{A}}_\epsilon + \nu\big)\nabla \mathcal{S}^\nu \nabla v dx d\theta = \int_0^1\int_\Omega \nabla\cdot\widetilde{\mathcal{C}}_\epsilon v\, dx d\theta \tag{2.64}$$
$$+\int_0^1\int_{\partial\Omega}\big(\widetilde{\mathcal{A}}_\epsilon + \nu\big) g v d\sigma d\theta,\ \forall v\in H^1(\Omega). \tag{2.65}$$

Taking $v\in H^1(\Omega)$ such that $v=0$ on $\partial\Omega$, we get

$$\int_0^1\int_\Omega \frac{\partial \mathcal{S}^\nu}{\partial\theta} v dx d\theta + \int_0^1\int_\Omega \big(\widetilde{\mathcal{A}}_\epsilon + \nu\big)\nabla \mathcal{S}^\nu \nabla v dx d\theta = \int_0^1\int_\Omega \nabla\cdot\widetilde{\mathcal{C}}_\epsilon v\, dx d\theta. \tag{2.66}$$

Using Green's formula, in the second term with respect to the variable x, we get, in the distributions sense,

$$\frac{\partial \mathcal{S}^\nu}{\partial\theta} - \nabla\cdot\Big(\big(\widetilde{\mathcal{A}}_\epsilon + \nu\big)\nabla\mathcal{S}^\nu\Big) = \nabla\cdot\widetilde{\mathcal{C}}_\epsilon \tag{2.67}$$

Using again Green's formula in the second term of (2.64) and using (2.66), we get

$$\forall\ v\in H^1(\Omega),\ \int_0^1\int_{\partial\Omega}\big(\widetilde{\mathcal{A}}_\epsilon + \nu\big)\frac{\partial\mathcal{S}^\nu}{\partial n} v d\sigma\, d\theta = \int_0^1\int_{\partial\Omega}\big(\widetilde{\mathcal{A}}_\epsilon + \nu\big) g v\, d\sigma d\theta \tag{2.68}$$

giving

$$\frac{\partial\mathcal{S}^\nu}{\partial n} = g \text{ on } \partial\Omega. \tag{2.69}$$

THEOREM 2.3 *Under the same assumptions as in theorem 2.2,* $\forall\epsilon>0,\ \nu>0,$ *there exists a unique* $\mathcal{S}^\nu = \mathcal{S}^\nu(t,\theta,x)$ *periodic of period 1 solution to (2.6). Moreover the following inequalities holds*

$$\sup_{\theta\in\mathbb{R}}\Big|\int_\Omega \mathcal{S}^\nu(t,\theta,x)dx\Big| = 0, \tag{2.70}$$

$$\Big\|\sqrt{\widetilde{\mathcal{A}}_\epsilon}\nabla\mathcal{S}^\nu\Big\|_{L^2_\#(\mathbb{R},L^2(\Omega))} \le \gamma|\Omega|, \tag{2.71}$$

$$\Big\|\frac{\partial\mathcal{S}^\nu}{\partial\theta}\Big\|_{L^2_\#(\mathbb{R},L^2(\Omega))} \le \gamma|\Omega|, \tag{2.72}$$

$$\Big\|\sqrt{\widetilde{\mathcal{A}}_\epsilon}\Delta\mathcal{S}^\nu\Big\|^2_{L^2_\#(\mathbb{R},L^2(\Omega))} \le \gamma|\Omega|(\frac{\gamma_1^2}{\widetilde{G}_{thr}} + 1), \tag{2.73}$$

$$\left\|\nabla\mathcal{S}^{\nu}\right\|^{2}_{L^{\infty}_{\#}(\mathbb{R},L^{2}(\Omega))}\leq\gamma|\Omega|(\gamma^{2}|\Omega|+1)+\frac{\gamma^{2}|\Omega|^{2}}{\tilde{G}_{thr}},\tag{2.74}$$

$$\left\|\mathcal{S}^{\nu}\right\|^{2}_{L^{\infty}_{\#}(\mathbb{R},L^{2}(\Omega))}\leq 2\gamma|\Omega|+\|z_{0}\|_{2}^{2},\tag{2.75}$$

$$\left\|\frac{\partial\mathcal{S}^{\nu}(\theta,\cdot)}{\partial t}\right\|^{2}_{L^{\infty}_{\#}(\mathbb{R},H^{1}(\Omega))}\leq\frac{\tilde{\gamma}_{4}}{\tilde{G}_{thr}}.\tag{2.76}$$

$\tilde{\gamma}_4$ depends on Ω, z_0 and g.

Before proposing a proof let us denote by

$$\left\|\mathcal{S}^{\nu}\right\|^{2}_{L^{2}_{\#}(\mathbb{R},H^{1}(\Omega))}=\left\|\mathcal{S}^{\nu}\right\|^{2}_{L^{2}_{\#}(\mathbb{R},L^{2}(\Omega))}+\left\|\nabla\mathcal{S}^{\nu}\right\|^{2}_{L^{2}_{\#}(\mathbb{R},L^{2}(\Omega))}$$

and

$$\left\|\mathcal{S}^{\nu}\right\|^{2}_{L^{\infty}_{\#}(\mathbb{R},H^{1}(\Omega))}=\sup_{\theta\in[0,1]}\Big(\left\|\mathcal{S}^{\nu}(\theta,\cdot)\right\|^{2}_{2}+\left\|\nabla\mathcal{S}^{\nu}(\theta,\cdot)\right\|^{2}_{2}\Big).$$

Proof All the estimates of $\mathcal{S}^{\nu}_{\mu}$ solution to (2.7) and its derivatives are bounded by constants not depending on μ. Then, existence of $\mathcal{S}^{\nu}$ ensure by letting μ tends to zero.
Following the same idea as in the proof of (2.11), we get

$$\int_{\Omega}\mathcal{S}^{\nu}(t,\theta,\cdot)dx=e^{-\mu(\theta_{0}-\theta)}\int_{\Omega}\mathcal{S}^{\nu}(t,\theta_{0},\cdot)dx.$$

Since $\theta\mapsto\mathcal{S}^{\nu}$ is periodic of period 1, then $\theta\mapsto\int_{\Omega}\mathcal{S}^{\nu}(\cdot,\theta,\cdot)dx$ is also periodic of periodic of period 1, then the only possibility of satisfying the last equality is:

$$\int_{\Omega}\mathcal{S}^{\nu}(\theta,\cdot)dx=0\quad\text{for all}\quad\theta\in[0,1].$$

Finally

$$\sup_{\theta\in\mathbb{R}}\left|\int_{\Omega}\mathcal{S}^{\nu}(\theta,x)dx\right|=0\tag{2.77}$$

To show that solution (2.6) is unique, let $\mathcal{S}^{\nu}$ and $\bar{\mathcal{S}^{\nu}}$ be two solutions of (2.6), then $\mathcal{S}^{\nu}-\bar{\mathcal{S}^{\nu}}$ is also solution to

$$\left\{\begin{array}{c}\frac{\partial(\mathcal{S}^{\nu}-\bar{\mathcal{S}^{\nu}})}{\partial\theta}-\nabla\cdot\Big((\widetilde{\mathcal{A}}_{\epsilon}+\nu)\nabla(\mathcal{S}^{\nu}-\bar{\mathcal{S}^{\nu}})\Big)=0\text{ in }]0,T]\times\mathbb{R}\times\Omega\\ \mathcal{S}^{\nu}(0,0,x)-\bar{\mathcal{S}^{\nu}}(0,0,x)=0\text{ in }\Omega\\ \frac{\partial(\mathcal{S}^{\nu}-\bar{\mathcal{S}^{\nu}})}{\partial n}=0\text{ on }]0,T]\times\mathbb{R}\times\partial\Omega.\end{array}\right.\tag{2.78}$$

Multiplying (2.78) by $\mathcal{S}^{\nu}-\bar{\mathcal{S}^{\nu}}$ and integrating over Ω, we get:

$$\frac{1}{2}\frac{d}{d\theta}\int_{\Omega}\left|(\mathcal{S}^{\nu}-\bar{\mathcal{S}^{\nu}})\right|^{2}dx+\int_{\Omega}(\widetilde{\mathcal{A}}_{\epsilon}+\nu)\left|\nabla(\mathcal{S}^{\nu}-\bar{\mathcal{S}^{\nu}})\right|^{2}dx-\int_{\Omega}(\widetilde{\mathcal{A}}_{\epsilon}+\nu)\frac{\partial(\mathcal{S}^{\nu}-\bar{\mathcal{S}^{\nu}})}{\partial n}(\mathcal{S}^{\nu}-\bar{\mathcal{S}^{\nu}})d\sigma=0.$$

Since $\frac{\partial(\mathcal{S}^{\nu}-\bar{\mathcal{S}^{\nu}})}{\partial n}=0$ on $]0,T]\times\mathbb{R}\times\partial\Omega$ there is no boundary term, hence we get

$$\frac{1}{2}\frac{d}{d\theta}\Big(\left\|(\mathcal{S}^{\nu}-\bar{\mathcal{S}^{\nu}})\right\|^{2}_{2}\Big)+\nu\left\|\nabla(\mathcal{S}^{\nu}-\bar{\mathcal{S}^{\nu}})\right\|^{2}_{2}=0\tag{2.79}$$

which gives, as the second term of (2.79) is positive

$$\frac{1}{2}\frac{d}{d\theta}\Big(\Big\|(\mathcal{S}^\nu - \bar{\mathcal{S}}^\nu)\Big\|_2^2\Big) \leq 0 \tag{2.80}$$

Integrating (2.80) from 0 to 1 and using periodicity of $\mathcal{S}^\nu$ we get from (2.79) the following inequality

$$\nu\int_0^1 \Big\|\nabla(\mathcal{S}^\nu - \bar{\mathcal{S}}^\nu)\Big\|_2^2 d\theta \leq 0.$$

Thus,

$$\nu\Big\|\nabla(\mathcal{S}^\nu - \bar{\mathcal{S}}^\nu)\Big\|^2_{L^2_\#(\mathbb{R},L^2(\Omega))} = 0.$$

$\forall t \in [0,T),\ \forall\theta \in \mathbb{R},$ then $\mathcal{S}^\nu(\theta) - \bar{\mathcal{S}}^\nu(\theta) \in H^1_0(\Omega)$ then using Poincare's inequality, there exists a constant $c(\Omega)$ such that

$$\mathcal{S}^\nu - \bar{\mathcal{S}}^\nu \in H^1_0(\Omega), \quad \Big\|\mathcal{S}^\nu - \bar{\mathcal{S}}^\nu\Big\|^2_{L^2_\#(\mathbb{R},L^2(\Omega))} \leq c(\Omega)\Big\|\nabla(\mathcal{S}^\nu - \bar{\mathcal{S}}^\nu)\Big\|^2_{L^2_\#(\mathbb{R},L^2(\Omega))}$$

Thus, we have

$$\Big\|\mathcal{S}^\nu - \bar{\mathcal{S}}^\nu\Big\|^2_{L^2_\#(\mathbb{R},L^2(\Omega))} = 0,$$

giving

$$S^\nu = \bar{\mathcal{S}}^\nu.$$

Multiplying the equation (2.6) by $\mathcal{S}^\nu$ and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{d\theta}\int_\Omega \Big|\mathcal{S}^\nu\Big|^2 dx + \int_\Omega (\widetilde{\mathcal{A}}_\epsilon + \nu)\Big|\nabla\mathcal{S}^\nu\Big|^2 dx - \int_{\partial\Omega}(\widetilde{\mathcal{A}}_\epsilon + \nu)\frac{\partial\mathcal{S}^\nu}{\partial n}\mathcal{S}^\nu d\sigma = \int_\Omega \nabla\cdot\widetilde{\mathcal{C}}_\epsilon\,\mathcal{S}^\nu dx. \tag{2.81}$$

Using the Green formula in the right hand side of the following equality we get

$$\frac{1}{2}\frac{d}{d\theta}\int_\Omega \Big|\mathcal{S}^\nu\Big|^2 dx + \int_\Omega (\widetilde{\mathcal{A}}_\epsilon + \nu)\Big|\nabla\mathcal{S}^\nu\Big|^2 dx - \int_{\partial\Omega}(\widetilde{\mathcal{A}}_\epsilon + \nu)\frac{\partial\mathcal{S}^\nu}{\partial n}\mathcal{S}^\nu d\sigma = -\int_\Omega \widetilde{\mathcal{C}}_\epsilon\cdot\nabla\mathcal{S}^\nu dx + \int_{\partial\Omega}\widetilde{\mathcal{C}}_\epsilon\cdot n\,\mathcal{S}^\nu\,d\sigma. \tag{2.82}$$

Using the boundary condition and hypothesis (2.8) we have

$$\frac{1}{2}\frac{d}{d\theta}\int_\Omega \Big|\mathcal{S}^\nu\Big|^2 dx + \int_\Omega (\widetilde{\mathcal{A}}_\epsilon + \nu)\Big|\nabla\mathcal{S}^\nu\Big|^2 dx = \int_{\partial\Omega}\Big((\widetilde{\mathcal{A}}_\epsilon + \nu)g + \widetilde{\mathcal{C}}_\epsilon\cdot n\Big)\mathcal{S}^\nu\,d\sigma - \int_\Omega \widetilde{\mathcal{C}}_\epsilon\cdot\nabla\mathcal{S}^\nu dx \tag{2.83}$$

From (2.8) and because of the fact that the coefficient $\big|\widetilde{\mathcal{C}}_\epsilon\big|^2$ is bounded by $\gamma\widetilde{\mathcal{A}}_\epsilon$, we get finally

$$\frac{1}{2}\frac{d}{d\theta}\int_\Omega \Big|\mathcal{S}^\nu\Big|^2 dx + \int_\Omega (\widetilde{\mathcal{A}}_\epsilon + \nu)\Big|\nabla\mathcal{S}^\nu\Big|^2 dx \leq \gamma|\Omega|\Big(\int_\Omega \widetilde{\mathcal{A}}_\epsilon\big|\nabla\mathcal{S}^\nu\big|^2 dx\Big)^{\frac{1}{2}} \tag{2.84}$$

Then, integrating (2.84) from 0 to 1 and using periodicity of $\mathcal{S}^\nu$ we get

$$\int_0^1\int_\Omega (\widetilde{\mathcal{A}}_\epsilon + \nu)\Big|\nabla\mathcal{S}^\nu\Big|^2 dx d\theta \leq \gamma|\Omega|\int_0^1 \Big\|\sqrt{\widetilde{\mathcal{A}}_\epsilon}\nabla\mathcal{S}^\nu\Big\|_{L^2(\Omega)} d\theta. \tag{2.85}$$

From (2.85), since $\widetilde{\mathcal{A}}_\epsilon + \nu \geq \widetilde{\mathcal{A}}_\epsilon$ we deduce

$$\Big\|\sqrt{\widetilde{\mathcal{A}}_\epsilon}\nabla\mathcal{S}^\nu\Big\|_{L^2_\#(\mathbb{R},L^2(\Omega))} \leq \gamma|\Omega| \tag{2.86}$$

Using (2.9) and integrating (2.82) with respect to $\theta \in [\theta_\alpha, \theta_\omega]$,

$$\sqrt{\tilde{G}}_{thr}\Big(\int_{\theta_\alpha}^{\theta_\omega}\int_\Omega |\nabla\mathcal{S}^\nu|^2 dx\Big)^2 \leq \gamma|\Omega|\int_0^1 \Big\|\sqrt{\widetilde{\mathcal{A}}_\epsilon}\nabla\mathcal{S}^\nu\Big\|_{L^2(\Omega)} d\theta. \tag{2.87}$$

Then

$$\Big(\int_{\theta_\alpha}^{\theta_\omega}\int_\Omega |\nabla\mathcal{S}^\nu|^2 dxd\theta\Big)^{\frac{1}{2}} \leq \frac{\gamma|\Omega|}{\sqrt{\tilde{G}_{thr}}}. \tag{2.88}$$

From this last inequality, there exists a $\theta_0 \in (\theta_\alpha, \theta_\omega)$ such that

$$\Big\|\nabla\mathcal{S}^\nu(\theta_0,\cdot)\Big\|_{L^2(\Omega)} \leq \frac{\gamma|\Omega|}{\tilde{G}_{thr}}. \tag{2.89}$$

Integrating (2.84) from 0 to θ and using the fact that the second term is positive, we have

$$\frac{1}{2}\Big(\Big\|\mathcal{S}^\nu(\theta,\cdot)\Big\|_2^2\Big) - \frac{1}{2}\Big(\Big\|\mathcal{S}^\nu(0,\cdot)\Big\|_2^2\Big) \leq \gamma|\Omega|\int_0^1 \Big\|\sqrt{\widetilde{\mathcal{A}}_\epsilon}\nabla S^\nu\Big\|_{L^2(\Omega)}. \tag{2.90}$$

From (2.86) we have

$$\Big\|\mathcal{S}^\nu(\theta,\cdot)\Big\|_2^2 \leq 2\gamma|\Omega| + \Big\|\mathcal{S}^\nu(0,\cdot)\Big\|_2^2 \tag{2.91}$$

Taking the supremum for all $\theta \in (0,1)$ we get

$$\Big\|\mathcal{S}^\nu\Big\|^2_{L^\infty_\#(\mathbb{R};L^2(\Omega))} \leq 2\gamma|\Omega| + \|z_0\|_2^2. \tag{2.92}$$

Multiplying (2.6) by $-\Delta\mathcal{S}^\nu$ and integrating over Ω we get,

$$\frac{1}{2}\frac{d}{d\theta}\int_\Omega \Big|\nabla\mathcal{S}^\nu\Big|^2 dx - \int_{\partial\Omega}\frac{\partial\mathcal{S}^\nu}{\partial n}\frac{\partial\mathcal{S}^\nu}{\partial\theta}d\sigma + \int_\Omega(\widetilde{\mathcal{A}}_\epsilon+\nu)\Big|\Delta\mathcal{S}^\nu\Big|^2 dx = -\int_\Omega \nabla\cdot\widetilde{\mathcal{C}}_\epsilon\,\Delta\mathcal{S}^\nu dx - \int_\Omega \nabla\widetilde{\mathcal{A}}_\epsilon\cdot\nabla\mathcal{S}^\nu\Delta\mathcal{S}^\nu dx, \tag{2.93}$$

Integrating from 0 to 1 with respect to θ, and using (2.36) for $U = \Delta\mathcal{S}^\nu$, $V = \nabla\cdot\widetilde{\mathcal{C}}_\epsilon$ for the first term of the right hand side and $V = \nabla\widetilde{\mathcal{A}}_\epsilon\cdot\nabla\mathcal{S}^\nu$, $U = \Delta S^\nu$ for the second term of the right hand side, and using periodicity of $\theta \to \int_\Omega \Big|\nabla\mathcal{S}^\nu(\theta,\cdot)\Big|^2$, we get:

$$\frac{1}{2}\int_0^1\int_\Omega(\widetilde{\mathcal{A}}_\epsilon+\nu)\Big|\Delta\mathcal{S}^\nu\Big|^2 dxd\theta \leq \int_0^1\int_\Omega \frac{|\nabla\widetilde{\mathcal{A}}_\epsilon|^2}{\widetilde{\mathcal{A}}_\epsilon+\nu}|\nabla\mathcal{S}^\nu|^2 dxd\theta + \int_0^1\int_\Omega \frac{|\nabla\cdot\widetilde{\mathcal{C}}_\epsilon|^2}{\widetilde{\mathcal{A}}_\epsilon+\nu}dxd\theta, \tag{2.94}$$

We have to notice that $|\nabla\widetilde{\mathcal{A}}_\epsilon|$ and $|\nabla\cdot\widetilde{\mathcal{C}}_\epsilon|$ are bounded by $\gamma|\widetilde{\mathcal{A}}_\epsilon|$, then using (2.86), we get

$$\Big\|\sqrt{\widetilde{\mathcal{A}}_\epsilon}\Delta\mathcal{S}^\nu\Big\|^2_{L^2_\#(\mathbb{R},L^2(\Omega))} \leq \gamma(\Big\|\sqrt{\widetilde{\mathcal{A}}_\epsilon}\nabla\mathcal{S}^\nu\Big\|^2_{L^2_\#(\mathbb{R},L^2(\Omega))} + |\Omega|), \tag{2.95}$$

which gives

$$\leq \gamma|\Omega|(\gamma^2|\Omega|+1) \tag{2.96}$$

Integrating (2.93) from θ_0 given in (2.89) to $\theta \in [0,1]$ and because of (2.96) we have

$$\begin{aligned}\Big\|\nabla\mathcal{S}^\nu(\theta,\cdot)\Big\|_2^2 &\leq \int_0^1\int_\Omega \frac{|\nabla\widetilde{\mathcal{A}}_\epsilon|^2}{\widetilde{\mathcal{A}}_\epsilon+\nu}|\nabla\mathcal{S}^\nu|^2 dx d\theta + \int_0^1\int_\Omega \frac{|\nabla\widetilde{\mathcal{A}}_\epsilon|^2}{\widetilde{\mathcal{A}}_\epsilon+\nu} dx d\theta + \Big\|\nabla\mathcal{S}^\nu(\theta_0,\cdot)\Big\|_2^2 \\ &\leq \gamma|\Omega|(\gamma^2|\Omega|+1) + \frac{\gamma^2|\Omega|^2}{\tilde{G}_{thr}},\end{aligned} \tag{2.97}$$

giving (2.74).
Multiplying (2.6) by $\frac{\partial\mathcal{S}^\nu}{\partial\theta}$ and integrating with respect to the variable x, we get

$$\int_\Omega \Big|\frac{\partial\mathcal{S}^\nu}{\partial\theta}\Big|^2 dx + \int_\Omega (\widetilde{\mathcal{A}}_\epsilon+\nu)\nabla\mathcal{S}^\nu\cdot\nabla(\frac{\partial\mathcal{S}^\nu}{\partial\theta})dx - \int_{\partial\Omega}(\widetilde{\mathcal{A}}_\epsilon+\nu)g\frac{\partial\mathcal{S}^\nu}{\partial\theta}d\sigma = \int_\Omega \nabla\cdot\widetilde{\mathcal{C}}_\epsilon\frac{\partial\mathcal{S}^\nu}{\partial\theta}dx,$$

Using the fact that

$$\int_\Omega \big(\widetilde{\mathcal{A}}_\epsilon+\nu\big)\nabla\mathcal{S}^\nu\cdot\nabla\frac{\partial\mathcal{S}^\nu}{\partial\theta}dx = \frac{1}{2}\frac{d\Big(\int_\Omega(\widetilde{\mathcal{A}}_\epsilon+\nu)|\nabla\mathcal{S}^\nu|^2dx\Big)}{d\theta} - \frac{1}{2}\int_\Omega\frac{\partial\widetilde{\mathcal{A}}_\epsilon}{\partial\theta}|\nabla\mathcal{S}^\nu|^2dx, \tag{2.98}$$

we get

$$\begin{aligned}&\Big\|\frac{\partial\mathcal{S}^\nu}{\partial\theta}\Big\|_2^2 + \frac{1}{2}\frac{d\Big(\int_\Omega(\widetilde{\mathcal{A}}_\epsilon+\nu)|\nabla\mathcal{S}^\nu|^2dx\Big)}{d\theta} = \\ &\frac{1}{2}\int_\Omega\frac{\partial\widetilde{\mathcal{A}}_\epsilon}{\partial\theta}|\nabla\mathcal{S}^\nu|^2dx + \int_{\partial\Omega}(\widetilde{\mathcal{A}}_\epsilon+\nu)g\frac{\partial\mathcal{S}^\nu}{\partial\theta}d\sigma + \int_{\partial\Omega}\widetilde{\mathcal{C}}_\epsilon\cdot n\frac{\partial\mathcal{S}^\nu}{\partial\theta}d\sigma - \int_\Omega\widetilde{\mathcal{C}}_\epsilon\cdot\nabla\Big(\frac{\partial\mathcal{S}^\nu}{\partial\theta}\Big)dx\end{aligned} \tag{2.99}$$

Since $\widetilde{\mathcal{A}}^\epsilon$, $\nabla\cdot\widetilde{\mathcal{C}}^\epsilon$ and $\frac{\partial\widetilde{\mathcal{A}}_\epsilon}{\partial\theta}$ are bounded by γ, we get

$$\Big\|\frac{\partial\mathcal{S}^\nu}{\partial\theta}\Big\|_2^2 + \frac{1}{2}\frac{d\Big(\int_\Omega(\widetilde{\mathcal{A}}_\epsilon+\nu)|\nabla\mathcal{S}^\nu|^2dx\Big)}{d\theta} \leq \tag{2.100}$$

$$\frac{\gamma}{2}\int_\Omega\widetilde{\mathcal{A}}_\epsilon|\nabla\mathcal{S}^\nu|^2dx + \int_{\partial\Omega}\Big((\widetilde{\mathcal{A}}_\epsilon+\nu)g+\widetilde{\mathcal{C}}_\epsilon\cdot n\Big)\frac{\partial\mathcal{S}^\nu}{\partial\theta}d\sigma + \gamma\int_\Omega|\nabla(\frac{\partial\mathcal{S}^\nu}{\partial\theta})|dx \tag{2.101}$$

Integrating (2.100)-(2.101) over $\theta\in[0,1]$ and using 1-periodicity of $\mathcal{S}^\nu$ and $\nabla\mathcal{S}^\nu$ with respect to θ we have:

$$\Big\|\frac{\partial\mathcal{S}^\nu}{\partial\theta}\Big\|^2_{L^2_\#(\mathbb{R},L^2(\Omega))} \leq \frac{\gamma}{2}\Big\|\sqrt{\widetilde{\mathcal{A}}_\epsilon}\nabla\mathcal{S}^\nu\Big\|^2_{L^2_\#(\mathbb{R},L^2(\Omega))} \leq \frac{\gamma^3}{2}|\Omega|^2.$$

The solution $\mathcal{S}^\nu$ of the equation (2.6) is differentiable over the time. $\frac{\partial\mathcal{S}^\nu}{\partial t}$ is then solution to

$$\frac{\partial\Big(\frac{\partial\mathcal{S}^\nu}{\partial t}\Big)}{\partial\theta} - \nabla\cdot\big((\widetilde{\mathcal{A}}_\epsilon+\nu)\nabla(\frac{\partial\mathcal{S}^\nu}{\partial t})\big) = \nabla\cdot\widetilde{\mathcal{C}}^\epsilon \tag{2.102}$$

where

$$\widetilde{\mathcal{C}}^\epsilon = \frac{\partial\widetilde{\mathcal{C}}_\epsilon}{\partial t} + \frac{\partial\widetilde{\mathcal{A}}_\epsilon}{\partial t}\nabla\mathcal{S}^\nu \tag{2.103}$$

and

$$\nabla\cdot\tilde{\mathcal{C}}^{\epsilon}=\frac{\partial(\nabla\cdot\widetilde{\mathcal{C}}_{\epsilon})}{\partial t}+\frac{\partial(\nabla\widetilde{\mathcal{A}}_{\epsilon})}{\partial t}\nabla S^{\nu}+\frac{\partial\widetilde{\mathcal{A}}_{\epsilon}}{\partial t}\Delta S^{\nu} \tag{2.104}$$

Since the coefficients $\widetilde{\mathcal{A}}_{\epsilon}$ and $\widetilde{\mathcal{C}}_{\epsilon}$ and its derivative are bounded by γ, then coefficients $\widetilde{\mathcal{C}}^{\epsilon}$ defined by (2.103) and $\nabla\cdot\widetilde{\mathcal{C}}^{\epsilon}$ defined by (2.104) are also bounded by a constant γ_8 depending only on γ and Ω

$$\begin{cases}\left\|\tilde{\mathcal{C}}^{\epsilon}\right\|_{L^2_{\#}(\mathbb{R},L^2(\Omega))}\leq\gamma_8\\ \left\|\nabla\cdot\tilde{\mathcal{C}}^{\epsilon}\right\|_{L^2_{\#}(\mathbb{R},L^2(\Omega))}\leq\gamma_8.\end{cases} \tag{2.105}$$

Multiplying (2.102) by $\frac{\partial\mathcal{S}^{\nu}}{\partial t}$ and integrating over Ω, we get

$$\frac{1}{2}\frac{d}{d\theta}\Big(\int_{\Omega}\Big|\frac{\partial\mathcal{S}^{\nu}}{\partial t}\Big|^2dx\Big)+\int_{\Omega}(\widetilde{\mathcal{A}}_{\epsilon}+\nu)\Big|\nabla\Big(\frac{\partial\mathcal{S}^{\nu}}{\partial t}\Big)\Big|^2dx-\int_{\partial\Omega}(\widetilde{\mathcal{A}}_{\epsilon}+\nu)\frac{\partial(\frac{\partial\mathcal{S}^{\nu}}{\partial t})}{\partial n}\frac{\partial\mathcal{S}^{\nu}}{\partial t}d\sigma=\int_{\Omega}\nabla\cdot\tilde{\mathcal{C}}^{\epsilon}\frac{\partial\mathcal{S}^{\nu}}{\partial t}dx. \tag{2.106}$$

From this last equality we have

$$\frac{1}{2}\frac{d}{d\theta}\Big(\Big\|\frac{\partial\mathcal{S}^{\nu}}{\partial t}\Big\|_2^2\Big)+\int_{\Omega}(\widetilde{\mathcal{A}}_{\epsilon}+\nu)\Big|\nabla\Big(\frac{\partial\mathcal{S}^{\nu}}{\partial t}\Big)\Big|^2dx=\int_{\partial\Omega}(\widetilde{\mathcal{A}}_{\epsilon}+\nu)\frac{\partial g}{\partial t}\frac{\partial\mathcal{S}^{\nu}}{\partial t}d\sigma-\int_{\Omega}\tilde{\mathcal{C}}^{\epsilon}\cdot\nabla\frac{\partial\mathcal{S}^{\nu}}{\partial t}dx+\int_{\partial\Omega}\tilde{\mathcal{C}}^{\epsilon}\cdot n\frac{\partial\mathcal{S}^{\nu}}{\partial t}d\sigma. \tag{2.107}$$

We can remark that, proceeding in the same way in the proof (2.56)-(2.57)

$$\int_{\partial\Omega}(\widetilde{\mathcal{A}}_{\epsilon}+\nu)\frac{\partial g}{\partial t}\frac{\partial\mathcal{S}^{\nu}}{\partial t}d\sigma+\int_{\partial\Omega}\widetilde{\mathcal{C}}^{\epsilon}\cdot n\frac{\partial\mathcal{S}^{\nu}}{\partial t}d\sigma=\int_{\partial\Omega}\Big((\widetilde{\mathcal{A}}_{\epsilon}+\nu)\frac{\partial g}{\partial t}+\widetilde{\mathcal{C}}^{\epsilon}\cdot n\Big)\frac{\partial\mathcal{S}^{\nu}}{\partial t}d\sigma$$

$$=\int_{\partial\Omega}\Big((\widetilde{\mathcal{A}}_{\epsilon}+\nu)\frac{\partial g}{\partial t}+\frac{\partial\widetilde{\mathcal{C}}_{\epsilon}}{\partial t}\cdot n+\frac{\partial\widetilde{\mathcal{A}}_{\epsilon}}{\partial t}\underbrace{\nabla\mathcal{S}^{\nu}\cdot n}_{=g}\Big)\frac{\partial\mathcal{S}^{\nu}}{\partial t}d\sigma=\int_{\partial\Omega}\frac{\partial}{\partial t}\underbrace{\Big((\widetilde{\mathcal{A}}_{\epsilon}+\nu)g+\widetilde{\mathcal{C}}_{\epsilon}\cdot n\Big)}_{=0}\frac{\partial\mathcal{S}^{\nu}}{\partial t}d\sigma=0. \tag{2.108}$$

Then, integrating (2.107) from 0 to 1 with respect to θ and using 1-periodicity of $\theta\mapsto\Big\|\frac{\partial\mathcal{S}^{\nu}}{\partial t}(\theta,\cdot)\Big\|_2^2$, we have

$$\Big\|\sqrt{\widetilde{\mathcal{A}}_{\epsilon}}\nabla\frac{\partial\mathcal{S}^{\nu}}{\partial t}\Big\|^2_{L^2_{\#}(\mathbb{R},L^2(\Omega))}\leq\tilde{\gamma}_8\Big\|\sqrt{\widetilde{\mathcal{A}}_{\epsilon}}\frac{\partial\nabla\mathcal{S}^{\nu}}{\partial t}\Big\|_{L^2_{\#}(\mathbb{R},L^2(\Omega))},$$

which gives

$$\Big\|\sqrt{\widetilde{\mathcal{A}}_{\epsilon}}\nabla\frac{\partial\mathcal{S}^{\nu}}{\partial t}\Big\|_{L^2_{\#}(\mathbb{R},L^2(\Omega))}\leq\tilde{\gamma}_8, \tag{2.109}$$

where $\tilde{\gamma}_8$ depends only on $\gamma 8$.
From (2.109), we deduce that, there exists $\theta\in[0,1]$ such that

$$\Big\|\nabla\frac{\partial\mathcal{S}^{\nu}(\theta,\cdot)}{\partial t}\Big\|_2\leq\frac{\tilde{\gamma}_8}{\widetilde{G}_{thr}}. \tag{2.110}$$

Using Fourier expansion of $\frac{\partial\mathcal{S}^{\nu}}{\partial t}$, we get

$$\Big\|\frac{\partial\mathcal{S}^{\nu}(\theta,\cdot)}{\partial t}\Big\|_2\leq\Big\|\nabla\frac{\partial\mathcal{S}^{\nu}(\theta,\cdot)}{\partial t}\Big\|_2\leq\frac{\tilde{\gamma}_8}{\widetilde{G}_{thr}}. \tag{2.111}$$

From (2.110) and (2.111), we have

$$\Big\|\frac{\partial \mathcal{S}^\nu}{\partial t}\Big\|_{L^\infty_\#(\mathbb{R},H^1(\Omega))} \leq \frac{\tilde{\gamma}_8}{\widetilde{G}_{thr}}$$

which gives (2.76). ■

Passing to the limit as ν tends to zero we obtain the following theorem.

THEOREM 2.4 *Under the same assumptions as in theorem 2.2, $\forall \epsilon > 0$ there exists a unique $\mathcal{S} = \mathcal{S}(t, \theta, x)$ periodic of period 1 solution to*

$$\begin{cases} \frac{\partial \mathcal{S}}{\partial \theta} - \nabla \cdot \left(\widetilde{\mathcal{A}}_\epsilon \nabla \mathcal{S} \right) = \nabla \cdot \widetilde{\mathcal{C}}_\epsilon \ in\ (0,T) \times \mathbb{R} \times \Omega \\ \mathcal{S}(0,0,x) = z_0(x) \ in\ \Omega \\ \frac{\partial \mathcal{S}}{\partial n} = g \ on \quad (0,T) \times \mathbb{R} \times \partial\Omega. \end{cases} \tag{2.112}$$

Moreover there exists constants $\tilde{\gamma}_1$ depending on γ and Ω such that the following estimates hold.

$$\sup_{\theta \in \mathbb{R}} \Big| \int_\Omega \mathcal{S}(\theta, x) dx \Big| = 0 \tag{2.113}$$

$$\Big\| \mathcal{S} \Big\|_{L^\infty_\#(\mathbb{R},L^2(\Omega))} \leq \tilde{\gamma}_1 \tag{2.114}$$

$$\Big\| \frac{\partial \mathcal{S}}{\partial t} \Big\|^2_{L^\infty_\#(\mathbb{R},L^2(\Omega))} \leq \frac{\tilde{\gamma}_8}{\tilde{G}_{thr}}. \tag{2.115}$$

Proof To show that solution (2.112) is unique, let $\mathcal{S}$ and $\bar{\mathcal{S}}$ two solutions of (2.112). Then, $\mathcal{S} - \bar{\mathcal{S}}$ is solution of

$$\begin{cases} \frac{\partial (\mathcal{S} - \bar{\mathcal{S}})}{\partial \theta} - \nabla \cdot \left(\widetilde{\mathcal{A}}_\epsilon \nabla (\mathcal{S} - \bar{\mathcal{S}}) \right) = 0 \text{ in } (0,T) \times \mathbb{R} \times \Omega \\ \mathcal{S}(0,0,x) - \bar{\mathcal{S}}(0,0,x) = 0 \text{ in } \Omega \\ \frac{\partial (\mathcal{S} - \bar{\mathcal{S}})}{\partial n} = 0 \text{ on } (0,T) \times \mathbb{R} \times \partial\Omega. \end{cases} \tag{2.116}$$

Multiplying (2.112) by $\mathcal{S} - \bar{\mathcal{S}}$ and integrating over Ω, we get:

$$\frac{1}{2} \frac{d}{d\theta} \int_\Omega \Big| (\mathcal{S} - \bar{\mathcal{S}}) \Big|^2 dx + \int_\Omega \widetilde{\mathcal{A}}_\epsilon \Big| \nabla (\mathcal{S} - \bar{\mathcal{S}}) \Big|^2 dx - \int_{\partial\Omega} \widetilde{\mathcal{A}}_\epsilon \frac{\partial (\mathcal{S} - \bar{\mathcal{S}})}{\partial n} (\mathcal{S} - \bar{\mathcal{S}}) d\sigma = 0.$$

Taking account the boundary condition , we have

$$\frac{1}{2} \frac{d}{d\theta} \int_\Omega \Big| (\mathcal{S} - \bar{\mathcal{S}}) \Big|^2 dx + \int_\Omega \widetilde{\mathcal{A}}_\epsilon \Big| \nabla (\mathcal{S} - \bar{\mathcal{S}}) \Big|^2 = 0.$$

Then

$$\frac{d}{d\theta} \Big\| (\mathcal{S} - \bar{\mathcal{S}}) \Big\|_2^2 \leq 0$$

Since the solutions $\mathcal{S}$ et $\bar{\mathcal{S}}$ are periodic of period 1, we get

$$\Big\| (\mathcal{S}(\theta) - \bar{\mathcal{S}}(\theta)) \Big\|_2^2 = 0$$

Finally

$$\mathcal{S} = \bar{\mathcal{S}}$$

The inequalities (2.113) , (2.114) and (2.115) are obtained by letting ν tend towards 0 in the inequalities (2.70), (2.75) and (2.76). ■

Proof of theorem 2.1 Let z_1^ϵ and z_2^ϵ be two solutions of (2.4), then, $z_1^\epsilon - z_2^\epsilon$ is solution to

$$\left\{ \begin{array}{c} \frac{\partial(z_1^\epsilon - z_2^\epsilon)}{\partial t} - \frac{1}{\epsilon}\nabla\cdot\Big(\mathcal{A}^\epsilon\nabla(z_1^\epsilon - z_2^\epsilon)\Big) = 0 \text{ in } (0,T)\times\Omega \\ z_1^\epsilon(0,x) - z_2^\epsilon(0,x) = 0 \text{ in } \Omega \\ \frac{\partial(z_1^\epsilon - z_2^\epsilon)}{\partial n} = 0 \text{ on } (0,T)\times\partial\Omega. \end{array} \right. \tag{2.117}$$

Multiplying (2.117) by $z_1^\epsilon - z_2^\epsilon$ and integrating over Ω, we get:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \Big|(z_1^\epsilon - z_2^\epsilon)\Big|^2 dx + \int_\Omega \mathcal{A}^\epsilon \Big|\nabla(z_1^\epsilon - z_2^\epsilon)\Big|^2 dx - \int_\Gamma \mathcal{A}^\epsilon \frac{\partial(z_1^\epsilon - z_2^\epsilon)}{\partial n}(z_1^\epsilon - z_2^\epsilon)d\sigma = 0.$$

Taking into account the boundary condition, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \Big|(z_1^\epsilon - z_2^\epsilon)\Big|^2 dx + \int_\Omega \mathcal{A}^\epsilon \Big|\nabla(z_1^\epsilon - z_2^\epsilon)\Big|^2 \leq 0.$$

which gives as the second term is positive,

$$\frac{d(\Big\|z_1^\epsilon(t,\cdot) - z_2^\epsilon(t,\cdot)\Big\|^2)}{dt} \leq 0 \tag{2.118}$$

Integrating (2.118) over $t \in [0;T)$, we get

$$\Big\|z_1^\epsilon(t,\cdot) - z_2^\epsilon(t,\cdot)\Big\|^2 \leq 0 \tag{2.119}$$

then

$$\Big\|z_1^\epsilon(t,\cdot) - z_2^\epsilon(t,\cdot)\Big\|^2 = 0 \tag{2.120}$$

Finally

$$z_1^\epsilon = z_2^\epsilon$$

Now, we consider the function $Z^\epsilon = Z^\epsilon(t,x) = \mathcal{S}(t, \frac{t}{\epsilon}, x)$ where S is given by the theorem 2.4. Since

$$\frac{\partial Z^\epsilon}{\partial t} = \frac{\partial \mathcal{S}}{\partial t}(t, \frac{t}{x}, x) + \frac{1}{\epsilon}\frac{\partial \mathcal{S}}{\partial \theta}(t, \frac{t}{\epsilon}, x), \tag{2.121}$$

We deduce from (2.112) that Z^ϵ is solution to

$$\frac{\partial Z^\epsilon}{\partial t} - \frac{1}{\epsilon}\nabla\cdot(\mathcal{A}^\epsilon\nabla Z^\epsilon) = \frac{1}{\epsilon}\nabla\cdot\mathcal{C}^\epsilon + \frac{\partial \mathcal{S}}{\partial t}(t, \frac{t}{\epsilon}, x). \tag{2.122}$$

From relations (2.4) and (2.122), we deduce that $z^\epsilon - Z^\epsilon$ is solution of

$$\left\{ \begin{array}{lcl} \frac{\partial(z^\epsilon - Z^\epsilon)}{\partial t} - \frac{1}{\epsilon}\nabla\cdot(\mathcal{A}^\epsilon\nabla(z^\epsilon - Z^\epsilon) & = & \frac{\partial \mathcal{S}}{\partial t}(t, \frac{t}{\epsilon}, x) \text{ in } (0,T)\times\Omega \\ (z^\epsilon - Z^\epsilon)_{|t=0} & = & z_0 - \mathcal{S}(0,0,\cdot) \text{ in } \Omega \\ \frac{\partial(z^\epsilon - Z^\epsilon)}{\partial n} = 0 \text{ on } (0,T)\times\partial\Omega & & \end{array} \right. \tag{2.123}$$

Multiplying (2.123) by $z^\epsilon - Z^\epsilon$ and integrating over Ω, we get with the previous estimates the following inequality:

$$\frac{d(\left\|z^\epsilon - Z^\epsilon\right\|_2^2)}{dt} \leq \sqrt{\frac{\tilde{\gamma}_8}{\tilde{G}_{thr}}}\left\|z^\epsilon - Z^\epsilon\right\|_2, \tag{2.124}$$

Then

$$\left\|z^\epsilon(t,\cdot) - Z^\epsilon(t,\cdot)\right\|_2 \leq \left\|z_0 - S(0,0,\cdot)\right\|_2\sqrt{\frac{\tilde{\gamma}_8}{\tilde{G}_{thr}}t}. \tag{2.125}$$

We have

$$\left\|z^\epsilon(t,\cdot)\right\|_2 = \left\|z^\epsilon(t,\cdot) - Z^\epsilon(t,\cdot) + Z^\epsilon(t,\cdot)\right\|_2 \leq \left\|z^\epsilon(t,\cdot) - Z^\epsilon(t,\cdot)\right\|_2 + \left\|Z^\epsilon(t,\cdot)\right\|_2 \tag{2.126}$$

Integrating from 0 to T with respect to the variable t, and using (2.125) we get

$$\left\|z^\epsilon\right\|_{L^2([0,T),L^2(\Omega))} \leq \left\|z_0 - S(0,0,\cdot)\right\|_2\sqrt{\frac{\tilde{\gamma}_8}{\tilde{G}_{thr}}T} + \left\|z_0\right\|_2^2 + 2\gamma|\Omega|. \tag{2.127}$$

Finally

$$\tilde{\gamma} = \left\|z_0 - S(0,0,\cdot)\right\|_2\sqrt{\frac{\tilde{\gamma}_8}{\tilde{G}_{thr}}T} + \left\|z_0\right\|_2^2 + 2\gamma|\Omega|.$$

■

Remark 2.2 We can also replace the boundary condition in (2.4) by a Dirichlet condition $z^\epsilon(t,x) = \tilde{g}(t,x)\ \partial\Omega$. In this case we have

$$\begin{cases} \frac{\partial z^\epsilon}{\partial t} - \frac{1}{\epsilon}\nabla\cdot\left(\mathcal{A}^\epsilon\nabla z^\epsilon\right) = \frac{1}{\epsilon}\nabla\cdot\mathcal{C}^\epsilon \text{ in },(0,T)\times\Omega \\ z^\epsilon(0,x) = z_0(x) \text{ in } \Omega \\ z^\epsilon = \tilde{g} \text{ on } (0,T)\times\partial\Omega \end{cases} \tag{2.128}$$

where $\tilde{g} \in L^2([0,T),H^1(\Omega))$.
We have to remark that, under the same condition as in theorem 2.1, if we consider system (1.10) with homogenous boundary condition i.e. $\dfrac{\partial z^\epsilon}{\partial n} = 0$ on $[0,T)\times\partial\Omega$ or equation (2.128) with $\tilde{g} = 0$ on $[0,T)\times\partial\Omega$, the solution z^ϵ to (1.10) or (2.128) belong bounded in $L^\infty([0,T),H^1(\mathbb{T}^2))$. In this two cases, we no longer have an integral on the bounder to be managed. So the method developed in Faye et al [12], can be applied without issues.

3 Homogenization Result

3.1 On twoscales convergence

The Two-scales is a concept due to Nguetseng [21] and is developed by Allaire [1]. It is a branch of the theory of homogenization. The aim of homogenization and thus the two-scales convergence is to study the behaviour of u^ϵ when $\epsilon \longrightarrow 0$, where u^ϵ is solution to a partial differential equation like

$$\left\{\begin{array}{lcll} L^\epsilon u^\epsilon & = & f^\epsilon & \text{sur} \quad \Omega \subset \mathbb{R}^N \\ B^\epsilon u^\epsilon & = & g^\epsilon & \text{sur} \quad \partial\Omega \end{array}\right. \tag{3.1}$$

where Ω is open set of $\mathbb{R}^N$ of boundary $\partial\Omega$. Therefore, it is to identify an equation of the form

$$\left\{\begin{array}{lcll} Lu & = & f & \text{sur} \quad \Omega \subset \mathbb{R}^N \\ Bu & = & g & \text{sur} \quad \partial\Omega \end{array}\right. \tag{3.2}$$

Definition 3.1 A sequence $(u^\epsilon)_{\epsilon>0} \subset L^p(\Omega)$ is said to two scale converge to a function $U \in L^p(\Omega, L^p_\#(Y))$ if, for every function $\psi \in \mathcal{C}^\infty(\Omega, L^\infty_\#(Y))$ we have:

$$\lim_{\epsilon \longrightarrow 0} \int_\Omega u_\epsilon(x)\psi(x, \frac{x}{\epsilon})dx = \int_\Omega \int_Y U(x,y)\psi(x,y)dxdy. \tag{3.3}$$

U is the limit to two scales of u^ϵ on $L^p(\Omega, L^p_\#(Y))$ where $Y = [0,1]^N$.

Let us recall the basic but fundamental theorem cf [21] , [1] for more details.

Theorem 3.1 *Let $(u_\epsilon)_\epsilon$ be a bounded sequence in $L^p(\Omega)$. Then there exists a subsequence still denoted $(u_\epsilon)_{\epsilon>0}$ and a function $U \in L^\infty_\#(\Omega, L^\infty(\Omega))$ such that*

$$u^\epsilon \longrightarrow U \ \textit{Two-Scale} \tag{3.4}$$

and in addition,

$$u^\epsilon \longrightarrow u \quad \textit{in } L^p \textit{ weakly-*}. \tag{3.5}$$

where the function u is defined by $x \mapsto u(x) = \int_Y U(x,y)dy$.

In the case of parabolic equations we have the following definition of two scale convergence.

Definition 3.2 A sequence of functions (z^ϵ) in $L^\infty([0,T), L^2(\Omega))$ is said to two scale converge to $U \in L^\infty([0,T), L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2)))$ if for every $\psi \in \mathcal{C}([0,T), \mathcal{C}_\#(\mathbb{R}, \mathcal{C}(\Omega)))$ we have

$$\lim_{\epsilon \to 0} \int_\Omega \int_0^T z^\epsilon(t,x)\psi(t, \frac{t}{\epsilon}, x)dt\,dx = \int_\Omega \int_0^T \int_0^1 U(t,\theta,x)\psi(t,\theta,x)d\theta\,dt\,dx. \tag{3.6}$$

In [1] and [21], it is stated the following fundamental theorem that we shall use.

Theorem 3.2 *If a sequence (z^ϵ) is bounded in $L^\infty([0,T), L^2(\Omega))$, there exists a subsequence still denoted (z^ϵ) and a function $U \in L^\infty([0,T), L^\infty_\#(\mathbb{R}, L^2(\Omega)))$ such that*

$$u^\epsilon \longrightarrow U \ \textit{two scale}. \tag{3.7}$$

3.2 Homogenization of the problem of sand transport

Since z^ϵ is bounded in $L^\infty([0,T],L^2(\Omega))$, we deduce that it two-scale converges to a profile $U \in L^\infty([0,T), L^\infty_\#(\mathbb{R},L^2(\Omega)))$. Our aim here is to characterize the equation satified by the two-scale limit U.
It is obvious that

$$\mathcal{A}^\epsilon(t,x) \text{ two scale converges to } \widetilde{\mathcal{A}}(t,\theta,x) \in L^\infty([0,T], L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2)))$$
$$\text{and } \mathcal{C}^\epsilon(t,x) \text{ two scale converges to } \widetilde{\mathcal{C}}(t,\theta,x), \tag{3.8}$$

with

$$\widetilde{\mathcal{A}}(t,\theta,x) = a\, g_a(|\mathcal{U}(t,\theta,x)|) \text{ and } \widetilde{\mathcal{C}}(t,\theta,x) = c\, g_c(|\mathcal{U}(t,\theta,x)|)\, \frac{\mathcal{U}(t,\theta,x)}{|\mathcal{U}(t,\theta,x)|}, \tag{3.9}$$

and we have the following theorem.

Theorem 3.3 *Under assumptions (2.9) and (2.10), for any T not depending on ϵ, the sequence of solutions z^ϵ to (2.4) two-scale converges to the profile $U \in L^\infty([0,T], L^\infty_\#(\mathbb{R}, L^2(\Omega)))$ solution to*

$$\begin{cases} \frac{\partial Z}{\partial \theta} - \nabla\cdot\left(\widetilde{\mathcal{A}}\nabla Z\right) = \nabla\cdot\widetilde{\mathcal{C}} & (0,T)\times\mathbb{R}\times\Omega \\ \frac{\partial Z}{\partial n} = g & (0,T)\times\mathbb{R}\times\partial\Omega, \end{cases} \tag{3.10}$$

where $\widetilde{\mathcal{A}}(t,\theta,x)$ and $\widetilde{\mathcal{C}}(t,\theta,x)$ are given (3.9).

Proof Let $\psi^\epsilon(t,x) = \psi(t,\frac{t}{\epsilon},x)$ be a regular function with compact support on $|0,T)\times\Omega$ and periodic of period 1. Multiplying the first equation by (2.4) by ψ^ϵ and integrating over $[0,T)\times\Omega$ we get :

$$\int_\Omega\int_0^T \frac{\partial z^\epsilon}{\partial t}\psi^\epsilon dtdx - \frac{1}{\epsilon}\int_\Omega\int_0^T \nabla\cdot(\mathcal{A}^\epsilon\nabla z^\epsilon)\psi^\epsilon dtdx = \frac{1}{\epsilon}\int_\Omega\int_0^T \nabla\cdot\mathcal{C}^\epsilon\psi^\epsilon dtdx. \tag{3.11}$$

Using integration by parts over $[0,T)$ in the first term and Green's formula over Ω in the second integral, we get

$$-\int_\Omega z_0(x)\psi(0,0,x)dx - \int_\Omega\int_0^T \frac{\partial\psi^\epsilon}{\partial t}z^\epsilon dtdx + \frac{1}{\epsilon}\int_\Omega\int_0^T \mathcal{A}^\epsilon\nabla z^\epsilon\nabla\psi^\epsilon dtdx - \frac{1}{\epsilon}\int_0^T\int_{\partial\Omega}\mathcal{A}^\epsilon\frac{\partial z^\epsilon}{\partial n}\psi^\epsilon d\sigma =$$
$$-\frac{1}{\epsilon}\int_\Omega\int_0^T \mathcal{C}^\epsilon\cdot\nabla\psi^\epsilon dtdx + \frac{1}{\epsilon}\int_0^T\int_{\partial\Omega}\mathcal{C}^\epsilon\psi^\epsilon.nd\sigma. \tag{3.12}$$

Computing $\frac{\partial\psi^\epsilon}{\partial t}$ we have

$$\frac{\partial\psi^\epsilon}{\partial t} = \Big(\frac{\partial\psi}{\partial t}\Big)^\epsilon + \frac{1}{\epsilon}\Big(\frac{\partial\psi}{\partial\theta}\Big)^\epsilon, \tag{3.13}$$

where

$$\Big(\frac{\partial\psi}{\partial t}\Big)^\epsilon(t,x) = \frac{\partial\psi}{\partial t}(t,\frac{t}{\epsilon},x) \text{ and } \Big(\frac{\partial\psi}{\partial\theta}\Big)^\epsilon(t,x) = \frac{\partial\psi}{\partial\theta}(t,\frac{t}{\epsilon},x), \tag{3.14}$$

And then, the above integrals becomes

$$\int_\Omega\int_0^T z^\epsilon\Big(\Big(\frac{\partial\psi}{\partial t}\Big)^\epsilon + \frac{1}{\epsilon}\Big(\frac{\partial\psi}{\partial\theta}\Big)^\epsilon + \frac{1}{\epsilon}\nabla\cdot(\mathcal{A}^\epsilon\nabla\psi^\epsilon)\Big)dtdx - \frac{1}{\epsilon}\int_0^T\int_{\partial\Omega}\mathcal{A}^\epsilon g\psi^\epsilon d\sigma$$

$$= -\frac{1}{\epsilon}\int_\Omega\int_0^T \mathcal{C}^\epsilon\cdot\nabla\psi^\epsilon dtdx - \int_\Omega z_0(x)\psi(0,0,x)dx. \tag{3.15}$$

Multiplying by ϵ

$$\epsilon\int_\Omega\int_0^T z^\epsilon\Big(\frac{\partial\psi}{\partial t}\Big)^\epsilon dtdx + \int_\Omega\int_0^T\Big(\frac{\partial\psi}{\partial\theta}\Big)^\epsilon z^\epsilon dtdx + \int_\Omega\int_0^T \nabla\cdot\big(\mathcal{A}^\epsilon\nabla\psi^\epsilon\big)z^\epsilon dtdx - \int_0^T\int_{\partial\Omega}\mathcal{A}^\epsilon g\psi^\epsilon d\sigma$$

$$= \int_\Omega\int_0^T \nabla\cdot\mathcal{C}^\epsilon\psi^\epsilon dtdx - \epsilon\int_\Omega z_0(x)\psi(0,0,x)dx. \tag{3.16}$$

Since ψ^ϵ is regular enough with compact support on $[0,T)\times\Omega$, and $\mathcal{A}^\epsilon$ is a regular function, the functions $\big(\frac{\partial\psi}{\partial t}\big)^\epsilon$, $\big(\frac{\partial\psi}{\partial\theta}\big)^\epsilon$, $\nabla\cdot(\mathcal{A}^\epsilon\nabla\psi^\epsilon)\big)$ and $\nabla\psi^\epsilon$ can be considered as test functions. Then using two-scale convergence we get when ϵ goes to 0,

$$\int_0^1\int_\Omega\int_0^T\frac{\partial\psi}{\partial\theta}Zdtd\theta dx + \int_0^1\int_\Omega\int_0^T\nabla\cdot\big(\widetilde{\mathcal{A}}\nabla\psi)\big)Zdtd\theta dx - \int_0^1\int_0^T\int_{\partial\Omega}\widetilde{\mathcal{A}}g\,\psi\,d\sigma dtd\theta$$

$$= -\int_0^1\int_\Omega\int_0^T\widetilde{\mathcal{C}}\cdot\nabla\psi dtd\theta dx. \tag{3.17}$$

Using test function ψ such that $\psi = 0$ on $\partial\Omega$ and using Green Formula, we get

$$\int_\Omega\int_0^1\int_0^T\Big(\frac{\partial Z}{\partial\theta} - \nabla\cdot(\widetilde{A}\nabla Z)\Big)\psi dtd\theta dx = \int_\Omega\int_0^1\int_0^T\nabla\cdot\mathcal{C}\psi dtd\theta dx \tag{3.18}$$

which is the weak formulation of

$$\frac{\partial Z}{\partial\theta} - \nabla\cdot(\widetilde{A}\nabla Z = \nabla\cdot\mathcal{C}. \tag{3.19}$$

Using again Green formula in the second term of (3.17) and using (3.18) we get finally

$$\frac{\partial Z}{\partial\theta} = g. \tag{3.20}$$

Let us characterize the homogenized equation for $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{C}}$. Multiplying (2.2) by ψ^ϵ and integrating over Ω we get

$$\int_\Omega\int_0^T\widetilde{\mathcal{A}}_\epsilon\psi^\epsilon dtdx = \int_\Omega\int_0^T a(1-b\epsilon\mathcal{M}(t,\theta,x)g_a(|U(t,\theta,x)|)\psi^\epsilon dtdx.$$

Then, we have

$$\int_\Omega\int_0^T\int_0^1 ag_a(|U(t,\theta,x)|)\psi dtdx = \int_\Omega\int_0^T\int_0^1\mathcal{A}\psi d\theta dtdx.$$

Multiplying (2.3) by ψ^ϵ and integrating over Ω we get

$$\int_\Omega\int_0^T\widetilde{\mathcal{C}}_\epsilon\psi^\epsilon dtdx = \int_\Omega\int_0^T c(1-b\epsilon\mathcal{M}(t,\theta,x))g_c(|U(t,\theta,x)|)\frac{\mathcal{U}(t,\theta,x)}{|\mathcal{U}(t,\theta,x)|}\psi^\epsilon dtdx$$

we have

$$\int_\Omega \int_0^T \int_0^1 cg_c(|U(t,\theta,x)|)\frac{\mathcal{U}(t,\theta,x)}{|\mathcal{U}(t,\theta,x)|}\psi dtdx = \int_\Omega \int_0^T \int_0^1 \mathcal{C}\psi d\theta dtdx$$

then

$$\mathcal{A} = ag_a(|\mathcal{U}(t,\theta,x)|) \text{ and } \mathcal{C} = cg_c(|\mathcal{U}(t,\theta,x)|)\frac{\mathcal{U}(t,\theta,x)}{|\mathcal{U}(t,\theta,x)|}.$$

■

3.3 Corrector result

Since the coefficients $\mathcal{A}^\epsilon(t,x)$ and $\mathcal{C}^\epsilon(t,x)$ of (2.4) two scale converge to $\widetilde{\mathcal{A}}(t,\theta,x)$ and $\widetilde{\mathcal{C}}(t,\theta,x)$, then theycan be set in the following expressions:

$$\mathcal{A}^\epsilon(t,x) = \widetilde{\mathcal{A}}^\epsilon(t,x) + \epsilon\widetilde{\mathcal{A}}_1^\epsilon(t,x) \text{ and } \mathcal{C}^\epsilon(t,x) = \widetilde{\mathcal{C}}^\epsilon(t,x) + \epsilon\widetilde{\mathcal{C}}_1^\epsilon(t,x) \tag{3.21}$$

where

$$\mathcal{A}^\epsilon(t,x) = \widetilde{\mathcal{A}}(t,\frac{t}{\epsilon},x),\ \mathcal{C}^\epsilon(t,x) = \widetilde{\mathcal{C}}(t,\frac{t}{\epsilon},x) \tag{3.22}$$

and

$$\widetilde{\mathcal{A}}_1^\epsilon(t,x) = \widetilde{\mathcal{A}}_1(t,\frac{t}{\epsilon},x),\ \widetilde{\mathcal{C}}_1^\epsilon(t,x) = \widetilde{\mathcal{C}}_1(t,\frac{t}{\epsilon},x) \tag{3.23}$$

We have also to notice that, under the same assumptions as in theorem 2.1, the coefficients

$$\widetilde{\mathcal{A}},\ \widetilde{\mathcal{C}},\ \widetilde{\mathcal{A}}_1, \widetilde{\mathcal{C}}_1,\ \widetilde{\mathcal{A}}^\epsilon,\ \widetilde{\mathcal{C}}^\epsilon,\ \widetilde{\mathcal{A}}_1^\epsilon, \text{ and } \widetilde{\mathcal{C}}_1^\epsilon \text{ are regular and bounded.} \tag{3.24}$$

THEOREM 3.4 *Under assumptions (2.9), (2.10) and (3.24), if z^ϵ is solution to (2.4) and $Z^\epsilon = Z(t,\frac{t}{\epsilon},x)$ where Z is solution to (3.10).*
Then for any T not depending on ϵ, the following estimates holds

$$\left\|\frac{z^\epsilon - Z^\epsilon}{\epsilon}\right\|_{L^\infty([0,T),L^2(\mathbb{T}^2))} \leq \alpha, \tag{3.25}$$

where α is a constant not depending on ϵ.

Furthermore, sequence $\left(\frac{z^\epsilon - Z^\epsilon}{\epsilon}\right)$ two-scale converges to a profile $U^1 \in L^\infty([0,T], L^\infty_\#(\mathbb{R}, L^2(\mathbb{T}^2)))$ which is the unique solution to

$$\begin{cases} \frac{\partial Z^1}{\partial\theta} - \nabla\cdot\left(\widetilde{\mathcal{A}}\nabla Z^1\right) = \nabla\cdot\widetilde{\mathcal{C}}_1 + \frac{\partial Z}{\partial t} + \nabla\cdot(\widetilde{\mathcal{A}}_1\nabla Z) & (0,T)\times\mathbb{R}\times\Omega \\ \frac{\partial Z^1}{\partial n} = 0 \quad (0,T)\times\mathbb{R}\times\partial\Omega. \end{cases} \tag{3.26}$$

Proof Because of (3.21), equation (2.4) becomes

$$\begin{cases} \frac{\partial z^\epsilon}{\partial t} - \frac{1}{\epsilon}\nabla\cdot\left(\widetilde{\mathcal{A}}^\epsilon\nabla z^\epsilon\right) = \frac{1}{\epsilon}\nabla\cdot\widetilde{\mathcal{C}}^\epsilon + \nabla\cdot\left(\widetilde{\mathcal{A}}_1^\epsilon\nabla z^\epsilon\right) + \nabla\cdot\left(\widetilde{\mathcal{C}}^\epsilon\nabla z^\epsilon\right) \\ \frac{\partial z^\epsilon}{\partial n} = g \end{cases} \tag{3.27}$$

From (3.10) and using the fact that

$$\frac{\partial Z^\epsilon}{\partial t} = \left(\frac{\partial Z}{\partial t}\right)^\epsilon + \frac{1}{\epsilon}\left(\frac{\partial Z}{\partial\theta}\right)^\epsilon, \tag{3.28}$$

where

$$\Big(\frac{\partial Z}{\partial t}\Big)^\epsilon(t,x) = \frac{\partial Z}{\partial t}(t,\frac{t}{\epsilon},x) \text{ and } \Big(\frac{\partial Z}{\partial \theta}\Big)^\epsilon(t,x) = \frac{\partial Z}{\partial \theta}(t,\frac{t}{\epsilon},x)$$

Z^ϵ is solution to

$$\begin{cases} \dfrac{\partial Z^\epsilon}{\partial t} - \dfrac{1}{\epsilon}\nabla\cdot\big(\widetilde{\mathcal{A}}^\epsilon\nabla Z^\epsilon\big) = \dfrac{1}{\epsilon}\nabla\cdot\widetilde{\mathcal{C}}^\epsilon + \Big(\dfrac{\partial Z}{\partial t}\Big)^\epsilon \\ \dfrac{\partial Z^\epsilon}{\partial n} = g \end{cases} \tag{3.29}$$

From formulas (3.27) and (3.29) we deduce that $\frac{z^\epsilon - Z^\epsilon}{\epsilon}$ is solution to

$$\begin{cases} \dfrac{\partial\Big(\frac{z^\epsilon-Z^\epsilon}{\epsilon}\Big)}{\partial t} - \dfrac{1}{\epsilon}\nabla\cdot\Big((\widetilde{\mathcal{A}}^\epsilon + \epsilon\widetilde{\mathcal{A}}_1^\epsilon)\nabla\Big(\dfrac{z^\epsilon - Z^\epsilon}{\epsilon}\Big)\Big) = \dfrac{1}{\epsilon}\Big(\nabla\cdot\widetilde{\mathcal{C}}_1^\epsilon + \Big(\dfrac{\partial Z}{\partial t}\Big)^\epsilon + \nabla\cdot\big(\widetilde{\mathcal{A}}_1^\epsilon\nabla Z^\epsilon\big) \quad (0,T)\times\Omega \\ \dfrac{\partial\Big(\frac{z^\epsilon-Z^\epsilon}{\epsilon}\Big)}{\partial n} = 0 \quad (0,T)\times\partial\Omega \end{cases} \tag{3.30}$$

All the coefficients of (3.30) are regular and bounded, then existence of $\Big(\frac{z^\epsilon-Z^\epsilon}{\epsilon}\Big)$ is a consequence result of Ladyzenskaja, Sollonnikov and Ural' Ceva [19]. We have to notice that, as the boundary condition of (3.30) is homogeneous see remark 2.2, there is no the boundary term to be considered. Then using the same argument as in the proof of theorem 2.1, we get that $\Big(\frac{z^\epsilon-Z^\epsilon}{\epsilon}\Big)$ solution to (3.30) is bounded in $L^2([0,T),L^2(\Omega))$. This solution two scale converges to a profile $Z^1 \in L^\infty([0,T),L^\infty_\#(\mathbb{R},L^2(\Omega))$ solution to (3.26). ■

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