Traces and Fractional Sobolev Extension Domains
with Variable Exponent

Azeddine Baalal and Mohamed Berghout

Department of Mathematics
Faculty of Sciences Ain Chock
University Hassan II Casablanca
B.P. 5366 Maarif, Casablanca, Morocco

Abstract
Assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain of class $C^{0,1}$ and denoted by $W^{s,q(.),p(\cdot)}(\Omega)$ the fractional Sobolev space with variable exponent. We show that $\Omega$ is a $W^{s,q(.),p(\cdot)}$—extension domain for $s \in (0,1)$. As an application, we study complemented subspaces in $W^{s,q(.),p(\cdot)}$ via the trace operator.

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1 Introduction
This paper is devoted to the problem of extendability in the fractional Sobolev spaces with variable exponent and its relation with the trace operator. The problem of how to extend Sobolev functions was recognized early in the development of the Sobolev spaces. In this direction we mention, in particular, the works of Sobolev [20],[21], Deny and Lions [8] and Gagliardo [12]; see also [2],[18],[19],[22],[6],[15],[11],[10] and the references therein. All these previous results are held under certain crucial regularity assumptions on the domain $\Omega$. 
Extension properties play important roles in applications; for instance, they can be used to establish some embedding properties.

Let us begin by making precise the notion of an open set of class $C^{0,1}$. Given $x \in \mathbb{R}^N$, write

$$x = \left( x', x_N \right) \text{ with } x' \in \mathbb{R}^{N-1}, \quad x_1, x_2, \ldots, x_{N-1},$$

and set

$$|x'| = \left( \sum_{i=1}^{N-1} x_i^2 \right)^{\frac{1}{2}}.$$

We define

$$Q := \left\{ x = \left( x', x_N \right) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x'| < 1 \text{ and } |x_N| < 1 \right\},$$

$$Q^+ := \left\{ x = \left( x', x_N \right) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x'| < 1 \text{ and } 0 < x_N < 1 \right\},$$

and

$$Q_0 := \{ x \in Q : x_N = 0 \}.$$

As usual, we say that $\Omega$ is of class $C^{0,1}$ if there exists $M > 0$ such that for any $x \in \partial \Omega$ there exist a ball $B = B_r(x)$, $r > 0$, and an isomorphism

$$T : Q \rightarrow B \text{ such that }$$

$$T \in C^{0,1}(\overline{Q}), \quad T^{-1} \in C^{0,1}(\overline{B}), \quad T\left( Q^+ \right) = B \cap \Omega,$$

$$T\left( Q_0 \right) = B \cap \partial \Omega \text{ and } \|T\|_{C^{0,1}(\overline{Q})} + \|T^{-1}\|_{C^{0,1}(\overline{B})} \leq M.$$

The map $T$ is called a local chart.

From now on let $\Omega$ be a bounded domain in $\mathbb{R}^N$ of class $C^{0,1}$. We fix $s \in (0, 1)$ and we consider two variable exponents, that is, $q : \overline{\Omega} \rightarrow (1, \infty)$ and $p : \overline{\Omega} \times \overline{\Omega} \rightarrow (1, \infty)$ be two continuous functions, we set

$$p^- := \text{essinf}_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} p(x, y), \quad p^+ := \text{esssup}_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} p(x, y),$$

$$q^- := \text{essinf}_{x \in \overline{\Omega}} q(x) \text{ and } q^+ := \text{esssup}_{x \in \overline{\Omega}} q(x).$$

For technical reasons, we will assume that

$$1 < p^- \leq p(x, y) \leq p^+ < \infty,$$

$$1 < q^- \leq q(x) \leq q^+ < \infty,$$

$$sp(x, y) < N, q(x) > p(x, x).$$
and
\[ p^* (x) > p^+ q (x) \quad \text{for} \quad x \in \overline{\Omega}, \]
where \( p^* (x) := \frac{Np(x,x)}{N-sp(x,x)} \) the fractional critical variable Sobolev exponent.

The variable exponent Lebesgue space \( L^{q(.)} (\Omega) \) is defined by
\[ L^{q(.)} (\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable}, \int_{\Omega} |u(x)|^{q(x)} \, dx < \infty \right\}. \]

We define a norm, the so called Luxembourg norm, in this space by
\[ \|u\|_{L^{q(.)}(\Omega)} := \inf \left\{ \lambda > 0; \int_{\Omega} \frac{|u(x)|^{q(x)}}{\lambda} \, dx \leq 1 \right\}. \]

As in the classical case, the dual variable exponent function \( q' \) of \( q \) is given by
\[ \frac{1}{q(x)} + \frac{1}{q'(x)} = 1, \]
and dual space for \( L^{q(.)} (\Omega) \) is \( L^{q'(.)} (\Omega) \). If \( u \in L^{q(.)} (\Omega) \) and \( v \in L^{q'(.)} (\Omega) \) then the following Hölder’s inequality holds:
\[ \int_{\Omega} |uv| \, dx \leq 2 \|u\|_{L^{q(.)}(\Omega)} \|v\|_{L^{q'(.)}(\Omega)}. \]

We define the fractional Sobolev space with variable exponents via the Gagliardo approach as follows:
\[ W^{s,q(.)} (\Omega) := \left\{ u \in L^{q(.)} (\Omega), \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x-y|^{N+sp(x,y)}} \, dxdy < \infty, \text{ for some } \lambda > 0 \right\}. \]

Let
\[ [u]^{s,p(.,.)}(\Omega) := \inf \left\{ \lambda > 0, \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x-y|^{N+sp(x,y)}} \, dxdy < 1 \right\}, \]
be the corresponding variable exponent Gagliardo semi norm. It is easy to see that \( W^{s,q(.)} (\Omega) \) is a Banach space with the norm
\[ \|u\|_{W^{s,q(.)} (\Omega)} := \|u\|_{L^{q(.)}(\Omega)} + [u]^{s,p(.,.)}(\Omega). \]

It is clear that \( W^{s,q(.)} (\Omega) \) can be seen as a natural extension of the classical fractional Sobolev space; however, fractional Sobolev spaces with variable exponents are also used to study some nonlocal problems as described in [3], [16],[9]. In [16], the authors prove the following theorem:

**Theorem 1.1** Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain of class \( C^{0,1} \) and \( s \in (0,1) \).
Let \( q(x), p(x,y) \) be continuous variable exponents with \( sp(x,y) < N \) for \( (x,y) \in \overline{\Omega} \times \overline{\Omega} \) and \( q(x) > p(x,x) \) for \( x \in \overline{\Omega} \). Assume that \( r : \overline{\Omega} \to (1,\infty) \)
is a continuous function such that $p^*(x) > r(x) \geq r^- > 1$, for $x \in \overline{\Omega}$.
Then there exists a constant $C = C(n,s,p,q,r,\Omega)$ such that for every $f \in W^{s,q(\cdot),p(\cdot)}(\Omega)$, it is held that
\[ \|f\|_{L^r(\Omega)} \leq C \|f\|_{W^{s,q(\cdot),p(\cdot)}(\Omega)}. \]
Thus, the space $W^{s,q(\cdot),p(\cdot)}(\Omega)$ is continuously embedded in $L^r(\Omega)$ for any $r \in (1, p^*)$. Moreover, this embedding is compact.

The definition of the spaces $L^q(\mathbb{R}^N)$ and $W^{s,q(\cdot),p(\cdot)}(\mathbb{R}^N)$ is analogous to $L^q(\Omega)$ and $W^{s,q(\cdot),p(\cdot)}(\Omega)$; one just changes every occurrence of $\Omega$ by $\mathbb{R}^N$.

We say that $\Omega \subset \mathbb{R}^N$ is a $W^{s,q(\cdot),p(\cdot)} –$ extension domain if there exists a continuous linear extension operator
\[ \mathcal{E} : W^{s,q(\cdot),p(\cdot)}(\Omega) \to W^{s,q(\cdot),p(\cdot)}(\mathbb{R}^N) \]
such that $\mathcal{E} u|_{\Omega} = u$ for each $u \in W^{s,q(\cdot),p(\cdot)}(\Omega)$. Our main result is the following theorem:

**Theorem 1.2** Suppose that $\Omega$ is of class $C^{0,1}$. Then $\Omega$ is a $W^{s,q(\cdot),p(\cdot)} –$ extension domain.

For classical fractional Sobolev space, this result has been proved by [11].

Recall that if $(A, \|\cdot\|_A)$ is a Banach space of measurable functions on $\mathbb{R}^N$ and $E \subset \mathbb{R}^N$ is a measurable set of positive Lebesgue measure, then $A|_E$ is the trace space defined as
\[ A|_E := \{ f : E \to \mathbb{R} ; \text{ there exists } F \in A \text{ such that } F|_E = f \text{ a.e.} \}. \]
This space is equipped with the norm
\[ \|f\|_{A|_E} = \inf \{ \|F\|_A : F \in A , F|_E = f \text{ a.e.} \}. \]

Denoting the trace operator by $T F = F|_E$. The above construction applies, in particular, to the Sobolev space $A = W^{s,q(\cdot),p(\cdot)}(\mathbb{R}^N)$. A closed subspace $Y$ of a Banach space $X$ is complemented if there is another closed subspace $Z$ of $X$ such that $X = Y \oplus Z$. That is, $Y \cap Z = \{0\}$ and every element $x \in X$ can be written as $x = y + z$, with $y \in Y$ and $z \in Z$. Let us consider the trace operator
\[ T : W^{s,q(\cdot),p(\cdot)}(\mathbb{R}^N) \to W^{s,q(\cdot),p(\cdot)}(\Omega) \]
defined by $T u = u|_{\Omega}$. Our next result relates to the complemented subspace problem. More precisely, we have the following theorem:
Theorem 1.3 Suppose that $\Omega$ is of class $C^{0,1}$. Then
\[ W^{s,q(.),p(\cdot)}(\mathbb{R}^N) = \text{Ker} \mathcal{T} \oplus \mathcal{E} (W^{s,q(.),p(\cdot)}(\Omega)) . \]

Note that the lack of the Hilbert structure of the space $W^{s,q(.),p(\cdot)}(\mathbb{R}^N)$ makes the complemented subspace problem in this space very difficult. Notation used in the paper is standard. The symbol $C$ will be used to designate a general constant whose value may change even within a single string of estimates.

This paper is organized as follows. In Section 2 we prove the main extension result. We begin by some preliminary lemmas, in which we will construct the extension to the whole of $\mathbb{R}^N$ of a function $u$ defined on $\Omega$ in two separated cases: when the function $u$ is identically zero in a neighborhood of the boundary $\partial \Omega$ and when it coincides with the half-space $\mathbb{R}^N_+$. In Section 3 we apply our result to study complemented subspaces in $W^{s,q(.),p(\cdot)}(\mathbb{R}^N)$.

2 Sobolev extension domains

To study the properties of the fractional Sobolev spaces with variable exponent $W^{s,q(.),p(\cdot)}(\Omega)$, it is often preferable to begin with the case $\Omega = \mathbb{R}^N$. It is therefore useful to be able to extend a function $u \in W^{s,q(.),p(\cdot)}(\Omega)$ to a function $\tilde{u} \in W^{s,q(.),p(\cdot)}(\mathbb{R}^N)$. We start with some technical lemmas, in which we will construct the extension to the whole of $\mathbb{R}^N$ of a function $u$ defined on $\Omega$.

Lemma 2.1 Let $\Omega$ be an open set in $\mathbb{R}^N$, $u$ a function in $W^{s,q(.),p(\cdot)}(\Omega)$. If there exists a compact subset $K \subset \Omega$ such that $u \equiv 0$ in $\Omega \setminus K$, then the extension function $\tilde{u}$ defined as
\[ \tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega, \end{cases} \]
belongs to $W^{s,q(.),p(\cdot)}(\mathbb{R}^N)$.

Proof. Clearly $\tilde{u} \in L^{q(\cdot)}(\Omega)$. Hence, we show that
\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}(x) - \tilde{u}(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{N + sp(x,y)}} dx dy < \infty, \text{ for some } \lambda > 0. \]

Let $\lambda > 0$. Set $\alpha = \min(\lambda^{p-}, \lambda^{p+})$ and $d(y) = \text{dist}(y, \partial K)$. Using the symmetry of the integral in the Gagliardo semi norm with respect to $x$ and $y$ and the fact that $\tilde{u} \equiv 0$ in $\mathbb{R}^N \setminus \Omega$, we can split as follows:
\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\tilde{u}(x) - \tilde{u}(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{N + sp(x,y)}} dx dy = \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{N + sp(x,y)}} dx dy + 2 \int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega} \frac{|u(x)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{N + sp(x,y)}} dy dx. \]
Since \( u \in W^{s,q(\cdot),p(\cdot)}(\Omega) \), then
\[
\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}} \, dx \, dy < \infty.
\]

Furthermore, for any \( y \in \mathbb{R}^{N} \setminus K \),
\[
\frac{|u(x)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}} = \frac{\chi_{K}(x)}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}} \left( |u(x)|^{p(x,y)} + |u(x)|^{-p(x,y)} \right) \left( \frac{1}{(d(y))^{N+sp(x,y)}} \right).
\]

Using the Fubini’s theorem and Hölder inequality, we get
\[
\int_{\Omega} \int_{\mathbb{R}^{N} \setminus \Omega} \frac{|u(x)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}} \, dy \, dx = \int_{\Omega} \left( \int_{\mathbb{R}^{N} \setminus \Omega} \frac{|u(x)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}} \, dy \right) \, dx
\leq 2 \left\| \chi_{K} \right\|_{L^{q'(\cdot)}(\Omega)} \left( \left\| u \right\|_{L^{p^{+}(\cdot)}(\Omega)}^{p^{+}} + \left\| u \right\|_{L^{p^{-}(\cdot)}(\Omega)}^{p^{-}} \right) \left( \frac{1}{\alpha} \right) \left( \int_{\mathbb{R}^{N} \setminus \Omega} \frac{dy}{\min((d(y))^{N+sp^{-}}, (d(y))^{N+sp^{+}})} \right).
\]

Note that the integral \( \int_{\mathbb{R}^{N} \setminus \Omega} \frac{dy}{\min((d(y))^{N+sp^{-}}, (d(y))^{N+sp^{+}})} \) is finite since \( d(y) > 0 \), \( N + sp^{-} > N \) and \( N + sp^{+} > N \). On the other hand, by Theorem 1.1 we know that there exists a positive constant \( C \) such that
\[
\left\| u \right\|_{L^{p^{+}(\cdot)}(\Omega)}^{p^{+}} \leq C \left\| u \right\|_{W^{s,q(\cdot),p(\cdot)}(\Omega)}^{p^{+}} \quad \text{and} \quad \left\| u \right\|_{L^{p^{-}(\cdot)}(\Omega)}^{p^{-}} \leq C \left\| u \right\|_{W^{s,q(\cdot),p(\cdot)}(\Omega)}^{p^{-}}.
\]

Hence
\[
\int_{\Omega} \int_{\mathbb{R}^{N} \setminus \Omega} \frac{|u(x)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}} \, dy \, dx < \infty.
\]
Lemma 2.2 Let $\Omega$ be an open set in $\mathbb{R}^N$, symmetric with respect to the coordinate $x_N$, and consider the sets $\Omega_+ = \{ x \in \Omega : x_N > 0 \}$ and $\Omega_- = \{ x \in \Omega : x_N \leq 0 \}$. Let $u \in W^{s,q,\cdot,p(\cdot)}(\Omega_+)$, one defines the function $\tilde{u}$ on $\Omega$ to be the extension by reflection, that is,

$$
\tilde{u} = \begin{cases} 
u(x',x_N) & \text{if } x_N \geq 0, \\
u(x',-x_N) & \text{if } x_N < 0.
\end{cases}
$$

Then $\tilde{u} \in W^{s,q,\cdot,p(\cdot)}(\Omega)$.

Proof. By splitting the integrals, we get

$$
\int_{\Omega} |\tilde{u}|^{q(x)} dx = \int_{\Omega_+} |\tilde{u}|^{q(x)} dx + \int_{\Omega_-} |\tilde{u}|^{q(x)} dx

= 2 \int_{\Omega_+} |u|^{q(x)} dx.
$$

Since $u \in W^{s,q,\cdot,p(\cdot)}(\Omega_+)$ then $\tilde{u} \in L^{q(\cdot)}(\Omega)$.

Also, we have

$$
\int_{\Omega} \int_{\Omega} \frac{|\tilde{u}(x) - \tilde{u}(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x-y|^{N+sp(x,y)}} dxdy = \int_{\Omega_+} \int_{\Omega_+} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x-y|^{N+sp(x,y)}} dxdy +

2 \int_{\Omega_+} \int_{\Omega_-} \frac{|u(x) - u(y',-y_N)|^{p(x,y)}}{\lambda^{p(x,y)} |x-y|^{N+sp(x,y)}} dxdy +

\int_{\Omega_-} \int_{\Omega_-} \frac{|u(x',-x_N) - u(y',-y_N)|^{p(x,y)}}{\lambda^{p(x,y)} |x-y|^{N+sp(x,y)}} dxdy

< \infty.
$$

This concludes the proof. \qed

Now, a truncation lemma near the boundary $\partial \Omega$.

Lemma 2.3 Let $\Omega$ be an open set in $\mathbb{R}^N$. Let us consider $u \in W^{s,q,\cdot,p(\cdot)}(\Omega)$ and $\psi \in C^{0,1}(\Omega)$, $0 \leq \psi \leq 1$. Then $\psi u \in W^{s,q,\cdot,p(\cdot)}(\Omega)$.
Proof. It is clear that $\psi u \in L^{q(\cdot)}(\Omega)$ since $|\psi| \leq 1$. Furthermore,

$$
\int_{\Omega} \int_{\Omega} \frac{|\psi(x) u(x) - \psi(y) u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x-y|^{N+sp(x,y)}} \, dx \, dy
\leq 2^{p+1} \int_{\Omega} \int_{\Omega} \frac{|\psi(x) (u(x) - u(y))|^{p(x,y)}}{\lambda^{p(x,y)} |x-y|^{N+sp(x,y)}} \, dx \, dy +
2^{p+1} \int_{\Omega} \int_{\Omega} \frac{|u(y) (\psi(x) - \psi(y))|^{p(x,y)}}{\lambda^{p(x,y)} |x-y|^{N+sp(x,y)}} \, dx \, dy
$$

where the integral $\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x-y|^{N+sp(x,y)}} \, dx \, dy$ is finite since $u \in W^{s,q(\cdot),p(\cdot)}(\Omega)$.

Since $\psi \in C^{0,1}(\Omega)$, then

$$
\int_{\Omega} \int_{\Omega} \frac{|u(y)|^{p(x,y)} |(\psi(x) - \psi(y))|^{p(x,y)}}{\lambda^{p(x,y)} |x-y|^{N+sp(x,y)}} \, dx \, dy
\leq \frac{\max(k^{ps}, k_{ps})}{\alpha} \int_{\Omega} \int_{\{\Omega \cap |x-y| \leq 1\}} \frac{|u(y)|^{p+} + |u(y)|^{p-}}{|x-y|^{N+(s-1)p(x,y)}} \, dx \, dy +
2^{p+1} \alpha \int_{\Omega} \int_{\{\Omega \cap |x-y| \geq 1\}} \frac{|u(y)|^{p+} + |u(y)|^{p-}}{|x-y|^{N+sp(x,y)}} \, dx \, dy
\leq \frac{\max(k^{ps}, k_{ps})}{\alpha} \int_{\Omega} \int_{\{\Omega \cap |x-y| \leq 1\}} \frac{|u(y)|^{p+} + |u(y)|^{p-}}{|x-y|^{N+(s-1)p}} \, dx \, dy +
2^{p+1} \frac{\alpha}{\alpha} \int_{\Omega} \int_{\{\Omega \cap |x-y| \geq 1\}} \frac{|u(y)|^{p+} + |u(y)|^{p-}}{|x-y|^{N+sp}} \, dx \, dy.
$$

Where $k$ denotes the Lipschitz constant of $\psi$ and $\alpha = \min(\lambda^{p+}, \lambda^{p-})$. Note that the kernel $|x-y|^{N-(s-1)p}$ is summable with respect to $x$ if $|x-y| \leq 1$ since $N + (s-1)p < N$ and, on the other hand, the kernel $|x-y|^{N-sp}$ is summable when $|x-y| \geq 1$ since $N + sp > N$. Finally, using the Fubini’s theorem in combination with Hölder inequality and Theorem 1.1, we get

$$
\int_{\Omega} \int_{\Omega} \frac{|u(y) (\psi(x) - \psi(y))|^{p(x,y)}}{\lambda^{p(x,y)} |x-y|^{N+sp(x,y)}} \, dx \, dy < \infty.
$$

This concludes the proof. □
Now, we are ready to state and prove the extension theorem for any domain \( \Omega \) satisfying certain regularity assumptions.

**Theorem 2.1** Suppose that \( \Omega \) is of class \( C^{0,1} \). Then \( \Omega \) is a \( W^{s,q-1,p(-)} \) extension domain.

**Proof.** We rectify \( \partial \Omega \) by local charts and use a partition of unity. More precisely, since \( \partial \Omega \) is compact and of class \( C^{0,1} \), we can find a finite number of balls \( B_j \) such that \( \partial \Omega \subset \bigcup_{j=1}^{k} B_j \) and there exist \( k \) smooth functions \( \psi_0, \psi_1, \ldots, \psi_k \) such that \( \text{spt} \psi_0 \subset \mathbb{R}^N \setminus \partial \Omega \), \( \text{spt} \psi_j \subset B_j \) for any \( j \in \{1, \ldots, k\} \), \( 0 \leq \psi_j \leq 1 \) for any \( j \in \{0, \ldots, k\} \) and \( \sum_{j=0}^{k} \psi_j = 1 \). Given \( u \in W^{s,q-1,p(-)}(\Omega) \), write

\[
uu = \sum_{j=0}^{k} \psi_j u.
\]

Now we extend each of the functions \( \psi_j u \) to \( \mathbb{R}^N \), distinguishing \( \psi_0 u \) and \( \psi_j u \) for \( j \in \{1, \ldots, k\} \).

By Lemma 2.3, we know that \( \psi_0 u \in W^{s,q-1,p(-)}(\Omega) \). Furthermore by Lemma 2.1, we can extend it to the whole of \( \mathbb{R}^N \), by setting

\[
\tilde{\psi_0} u (x) = \begin{cases} 
\psi_0 u (x) & \text{if } x \in \Omega, \\
0 & \text{if } x \in \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

and \( \tilde{\psi_0} u \in W^{s,q-1,p(-)}(\mathbb{R}^N) \).

For any \( j \in \{1, \ldots, k\} \), let us consider the restriction of \( u \) to \( B_j \cap \Omega \) and \( T_j : Q \rightarrow B_j \) is the isomorphism of class \( C^{0,1} \) defined in Section 1. We transfer this function to \( Q^+ \) with the help of \( T_j \). More precisely, set

\[
v_j (y) := u (T_j (y))
\]

for any \( y \in Q^+ \).

Now, we state that \( v_j \in W^{s,q-1,p(-)}(Q^+) \). Using the standard changing variable formula by setting \( x = T_j (\zeta) \) and \( y = T_j (\zeta) \) we have

\[
\int_{Q^+} |v_j (\zeta)|^{q(\zeta)} d\zeta \leq \int_{Q^+} |v_j (\zeta)|^{q^+} d\zeta + \int_{Q^+} |v_j (\zeta)|^{q^-} d\zeta \\
\leq \int_{Q^+} |u (T_j (\zeta))|^{q^+} d\zeta + \int_{Q^+} |u (T_j (\zeta))|^{q^-} d\zeta \\
\leq \int_{B_j \cap \Omega} |u (x)|^{q^+} \det J_{T_j^{-1}} dx + \int_{B_j \cap \Omega} |u (x)|^{q^-} \det J_{T_j^{-1}} dx \\
\leq C \int_{B_j \cap \Omega} |u (x)|^{q^+} dx + C \int_{B_j \cap \Omega} |u (x)|^{q^-} dx.
\]
Using the Hölder’s inequality and Theorem 1.1, we get that \( v_j \in L^{q_j}(Q^+) \). Moreover,

\[
\int_{Q^+} \int_{Q^+} \frac{|v_j(\zeta) - v_j(\xi)|^{p/(\xi,\zeta)}}{|\zeta - \xi|^{N + sp(\xi,\zeta)}} d\zeta d\xi \leq \int_{Q^+} \int_{Q^+} \frac{|v_j(\zeta) - v_j(\xi)|^{p_+}}{|\zeta - \xi|^{N + sp(\xi,\zeta)}} d\zeta d\xi + \int_{Q^+} \int_{Q^+} \frac{|v_j(\zeta) - v_j(\xi)|^{p_-}}{|\zeta - \xi|^{N + sp(\xi,\zeta)}} d\zeta d\xi.
\]

Putting

\[
I = \int_{Q^+} \int_{Q^+} \frac{|v_j(\zeta) - v_j(\xi)|^{p_+}}{|\zeta - \xi|^{N + sp(\xi,\zeta)}} d\zeta d\xi
\]

and

\[
J = \int_{Q^+} \int_{Q^+} \frac{|v_j(\zeta) - v_j(\xi)|^{p_-}}{|\zeta - \xi|^{N + sp(\xi,\zeta)}} d\zeta d\xi.
\]

Now, we show that \( I \) and \( J \) are finite. Indeed, we have

\[
I = \int_{Q^+} \int_{Q^+} \frac{|v_j(\zeta) - v_j(\xi)|^{p_+}}{|\zeta - \xi|^{N + sp(\xi,\zeta)}} |\zeta - \xi|^{N + sp} d\zeta d\xi
\]

\[
= \int_{Q^+} \int_{Q^+} \frac{|v_j(\zeta) - v_j(\xi)|^{p_+}}{|\zeta - \xi|^{N + sp} - |\zeta - \xi|^{s(p_+ - p(\xi,\zeta))}} d\zeta d\xi
\]

\[
\leq \max \left(1, (diam (Q^+))^{s(p_+ - p_-)}\right) \int_{Q^+} \int_{Q^+} \frac{|v_j(\zeta) - v_j(\xi)|^{p_+}}{|\zeta - \xi|^{N + sp} - |\zeta - \xi|^{s(p_+ - p(\xi,\zeta))}} d\zeta d\xi
\]

\[
\leq C \int_{Q^+} \int_{Q^+} \frac{|v_j(\zeta) - v_j(\xi)|^{p_+}}{|\zeta - \xi|^{N + sp}} d\zeta d\xi
\]

\[
\leq C \int_{B_j \cap \Omega} \int_{B_j \cap \Omega} \frac{|u(x) - u(y)|^{p_+}}{|T_j^{-1}(x) - T_j^{-1}(y)|^{N + sp}} |\det J_{T_j^{-1}}| dxdy.
\]

Since \( T_j \) is bi-Lipschitz, then

\[
I \leq C \int_{B_j \cap \Omega} \int_{B_j \cap \Omega} \frac{|u(x) - u(y)|^{p_+}}{|x - y|^{N + sp}} dxdy,
\]

where the constant \( C \) is different step by step.

Now, we only need to show that

\[
F_{p_+}(x, y) = \frac{|u(x) - u(y)|^{p_+}}{|x - y|^{rac{N}{p_+} + s}} \in L^{p_+}(B_j \cap \Omega \times B_j \cap \Omega).
\]
Observe that
\[
F_{p^+}(x, y) = \frac{|u(x) - u(y)| |x - y|^{\frac{N}{p(x,y)} + s}}{|x - y|^{\frac{N}{p(x,y)} + s}} = \frac{|u(x) - u(y)| |x - y|^{\frac{N}{p(x,y)} + s}}{|x - y|^{\frac{N}{p(x,y)} + s}}.
\]

Since \( u \in W^{s,q(.),p(\cdot)}(B_j \cap \Omega) \), we have that \( \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p(x,y)} + s}} \in L^{p(\cdot)}(B_j \cap \Omega \times B_j \cap \Omega) \), hence by Hölder’s inequality we only need to show that \( |x - y|^{\frac{N}{p(x,y)} + s} \in L^{q(\cdot)}(B_j \cap \Omega \times B_j \cap \Omega) \), where \( q(x,y) = \frac{p^+ - p(x,y)}{p^+ - p(x,y)} \). That is, it is enough to show that
\[
\int_{B_j \cap \Omega} \int_{B_j \cap \Omega} |x - y|^{\frac{N}{p(x,y)} + s} dxdy < \infty.
\]

We have
\[
\int_{B_j \cap \Omega} \int_{B_j \cap \Omega} |x - y|^{\frac{N}{p(x,y)} + s} dxdy = \int_{B_j \cap \Omega} \int_{B_j \cap \Omega} |x - y|^N dxdy 
\leq \int_{B_j \cap \Omega} \int_{B_j \cap \Omega} (diam (B_j \cap \Omega))^N dxdy 
\leq (diam (B_j \cap \Omega))^N \int_{B_j \cap \Omega} \int_{B_j \cap \Omega} dxdy.
\]

Since \( \Omega \) is bounded, we have that \( I \) is finite. By the same way we show that \( J \) is finite. Finally \( v_j \in W^{s,q(.),p(\cdot)}(Q^+) \).

Moreover, using Lemma 2.2, we can extend \( v_j \) to all \( Q \) so that extension \( \overline{v_j} \) belongs to \( W^{s,q(.),p(\cdot)}(Q) \). Retransfer \( \overline{v_j} \) to \( B_j \) using \( T_j^{-1} \) and call it \( w_j : \)
\[
w_j(x) := \overline{v_j}(T_j^{-1}(x))
\]
for any \( x \in B_j \). Since is bi-Lipschitz, by arguing as above it follows that \( w_j \in W^{s,q(.),p(\cdot)}(B_j) \). On the other hand \( w_j \equiv u \) on \( B_j \cap \Omega \), consequently \( \psi_j w_j \equiv \psi_j u \) on \( B_j \cap \Omega \). By definition \( \psi_j w_j \) has a compact support in \( B_j \) and therefore, as done for \( \psi_0 u \), we can consider the extension \( \widetilde{\psi_j w_j} \) to all \( \mathbb{R}^N \) in such a way that \( \widetilde{\psi_j w_j} \in W^{s,q(.),p(\cdot)}(\mathbb{R}^N) \). Finally, define a linear operator
\[ \mathcal{E} : W^{s,q}(-,p(-)) (\Omega) \to W^{s,q}(-,p(-)) (\mathbb{R}^N) \] by \( \mathcal{E} u = \tilde{\psi}_0 u + \sum_{j=1}^k \tilde{\psi}_j u \). By construction, it is clear that \( \mathcal{E}_{u|\Omega} = u \) for all \( u \in W^{s,q}(-,p(-)) (\Omega) \). Moreover, it is easy to see that the graph of \( \mathcal{E} \) is closed in \( W^{s,q}(-,p(-)) (\Omega) \times W^{s,q}(-,p(-)) (\mathbb{R}^N) \). By the closed graph theorem, the linear operator \( \mathcal{E} \) is continuous. \( \square \)

3 Complemented subspaces in \( W^{s,q}(-,p(-)) (\mathbb{R}^N) \)

In this section we look for the complemented subspaces problem in \( W^{s,q}(-,p(-)) (\mathbb{R}^N) \). From Theorem 2.1, we know that if \( \Omega \) is a bounded domain of class \( C^{0,1} \), then \( \Omega \) is a \( W^{s,q}(-,p(-)) \) extension domain. Actually, every function \( u \in W^{s,q}(-,p(-)) (\Omega) \) admits an extension to \( W^{s,q}(-,p(-)) (\mathbb{R}^N) \). Therefore the trace operator (1) is surjective.

**Theorem 3.1** Suppose that \( \Omega \) is of class \( C^{0,1} \). Then

\[ W^{s,q}(-,p(-)) (\mathbb{R}^N) = \text{Ker} \mathcal{T} \oplus \mathcal{E} (W^{s,q}(-,p(-)) (\Omega)). \]

**Proof.** From Theorem 2.1 we know that there exists a linear operator

\[ \mathcal{E} : W^{s,q}(-,p(-)) (\Omega) \to W^{s,q}(-,p(-)) (\mathbb{R}^N) \]

such that for all \( u \in W^{s,q}(-,p(-)) (\Omega) \),

(i) \( \mathcal{E}_{u|\Omega} = u \),

(ii) \( \| \mathcal{E} u \|_{W^{s,q}(-,p(-)) (\mathbb{R}^N)} \leq C \| u \|_{W^{s,q}(-,p(-)) (\Omega)} \),

where \( C \) is a positive constant.

Note that \( \mathcal{E} (W^{s,q}(-,p(-)) (\Omega)) \subset W^{s,q}(-,p(-)) (\mathbb{R}^N) \) is a closed subspace. Every element \( u \in W^{s,q}(-,p(-)) (\mathbb{R}^N) \) can be written as \( u = u - \mathcal{E} (\mathcal{T} (u)) + \mathcal{E} (\mathcal{T} (u)) \). Since \( u - \mathcal{E} (\mathcal{T} (u)) \in \text{Ker} \mathcal{T} \), \( \mathcal{E} (\mathcal{T} (u)) \in \mathcal{E} (W^{s,q}(-,p(-)) (\Omega)) \), and \( \text{Ker} \mathcal{T} \cap \mathcal{E} (W^{s,q}(-,p(-)) (\Omega)) = \{ 0 \} \), we conclude that

\[ W^{s,q}(-,p(-)) (\mathbb{R}^N) = \text{Ker} \mathcal{T} \oplus \mathcal{E} (W^{s,q}(-,p(-)) (\Omega)). \]

The proof is complete. \( \square \)

**References**


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