Block Nyström Type Method and Its Block Extension for Fourth Order Initial and Boundary Value Problems

E.O. Adeyefa

Department of Mathematics
Federal University Oye-Ekiti, Nigeria

A.O. Akintunde

Department of Mathematics
Federal University Oye-Ekiti, Nigeria

R.I. Ndu

Department of Mathematics
Rivers State University, Rivers State, Nigeria

J.A. Oladunjoye

Department of Computer Science
Federal University University Wukari, Nigeria

A. A. Ibrahim

Department of Mathematics
Oduduwa University Ipetumodu, Nigeria

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Abstract

The derivation of Block Nyström type Method (BNTM) which is not normally used as numerical integrator of boundary value problems
(BVPs) is considered and directly applied to solve both initial value problems (IVPs) and BVPs in ordinary differential equations (ODEs). Collocation technique is adopted in the derivation of the BNTM which is applied as simultaneous integrator to fourth order ODEs. The BNTM possesses the desirable feature of being self-starting as the implementation is in block form. The paper concludes by solving Numerical examples which establish the effectiveness and accuracy of the method. The superiority of BNTM is established by the numerical values presented.

Keywords: block Nyström type method; fourth order initial and boundary value problems

1 Introduction

Nyström type methods are widely used for the numerical integration of initial value problems (IVPs) in ordinary differential equations (ODEs). Specifically, they are extensively used for directly solving second order IVPs. Nevertheless, they are not normally used for the numerical integration of boundary value problems (BVPs). This paper focuses on the formulation of block Nyström type method (BNTM) for the numerical solution of fourth order IVPs and BVPs in ODEs. The demand for the solution of higher order differential equations which has connections with several fields such as engineering, science, management and has its application in fluid dynamics (see Alomari et al. [2]), beam theory (see Jator [20]), electric circuits (see Boutayeb and Chetouani [6]), ship dynamics (see Wu et al. [33], Twizell [30], Cortell [9]), and neural networks (see Malek and Beidokhti [26]), the reaction and diffusion of chemicals, the dynamics of populations in biology, the development and treatment of diseases in medicine, molecular dynamics, the motion of rocket and several other areas is on the increase. The quest for numerical methods has increasingly been of much interest to researchers owing to the fact that most of these DEs are difficult to solve or their analytical solutions do not exist. The focus here is to develop a continuous one-step hybrid method for the solution of initial value problem (IVP)

\[ y^{(iv)} = f(x, y, y', y'', y'''), \quad x \in [x_0, x_N], \]  

subject to initial conditions

\[ y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad y''(x_0) = y''_0, \quad y'''(x_0) = y'''_0, \]

which are not a restriction on the proposed method, since the method can also handle ODEs with Dirichlet, Neumann or Robin boundary conditions with
only minor modifications in the boundary conditions. IVP of the form (1) is solved by reduction to an equivalent system of first order ordinary differential equation and appropriate numerical method could be employed to solve the resultant system (see Adesanya et al. [1], Butcher [8], Lambert [24], Henrici [17], Hairer et al. [16], Dormand [12]). This approach had been reported to increase the number of equations four times and thereby more function evaluations need to be evaluated as this results to a longer execution time and more computational effort (see Jator [20], Awoyemi ([3],[5]), Waeleh et al. [32], Mehrkanoon [27]). Moreover, Bun and Vasil’yer [7] reported that the system of equations to be solved when the method of reduction is applied cannot be solved explicitly with respect to the derivatives of the highest order.

A successful application of numerical algorithm to directly solve a general fourth order initial value problems of the form (1) has been demonstrated in the literature (see Awoyemi ([4],[5]), Kayode ([22],[23])). However, all these methods were implemented in predictor-corrector mode and hence, according to Jator, (Jator [20]) the implementation of such schemes is more costly since the subroutines for incorporating the starting values lead to lengthy computational time. Besides, they advance the numerical integration of the ordinary differential equations in one-step at a time, which leads to overlapping of the piecewise polynomials solution model (Yusuph, [35]). To address the setback of the predictor-corrector method, Vigo-Aguiar and Ramos [31], Jator [19], Yap and Ismail [34], Hussain et al. [18], among others independently proposed block method for solving higher order ordinary differential equation which does not require the development of separate predictors but simultaneously generate approximation at different grid points within the interval of integration without overlapping as experienced in the predictor-corrector method. The aim of developing new methods has always been to improve on the efficiency and convergence of existing methods with the ultimate aim of reducing the error of approximation. Thus, in what immediately follows, we shall formulate the proposed method to directly integrate fourth order initial value problem and subsequently, analysis of the method shall be discussed and, in the following section, numerical examples are given to show the efficiency of the methods. Finally, the conclusion of the paper is discussed.

2 Continuous approximation for the BNTM

We seek the polynomial

\[ Y(x) = \sum_{j=0}^{m+w-1} a_j x^j \]
on the interval $[x_n, x_{n+1}]$ where $a_j$ are unknown coefficients, $m$ is the number of interpolation points and $w$ is the number of collocation points. The polynomial $Y(x)$ passes through the points $(x_n, y_n), (x_{n+r}, y_{n+r}), (x_{n+s}, y_{n+s}), (x_{n+v}, y_{n+v}), (x_{n+v1}, y_{n+v1}), (x_{n+1}, y_{n+1})$ and satisfies $m + w$ equations. We then construct a collocation method by imposing that

$$y(x_{n+j}) = y_{n+j}, j = 0, 1, 2, ..., m - 1$$  \hspace{0.5cm} (2)

$$y^{(w)}(x_{n+j}) = f_{n+j}, j = 0, 1, 2, ..., w - 1$$  \hspace{0.5cm} (3)

Equations (2) and (3) lead to a system of $(m+w)$-equations, which is solved to obtain the $a_j's$. To determine the continuous method, theorem 2.1 hereunder which greatly enhances its derivation is considered.

**Theorem 2.1:** Let $Y(x)$ satisfies the system of equations obtained in (2) and (3), then the continuous approximation

$$Y(x) = \sum_{j=0}^{m-1} \alpha_j(x)y_{n+j} + h^4 \sum_{j=0}^{w-1} \beta_j(x)f_{n+j}, j = \{0, r, s, v, v1, 1\}$$  \hspace{0.5cm} (5)

and $Y'(x)$, $Y''(x)$, $Y'''(x)$ used to produce (2) and (3) are given as

$$\begin{cases}
    Y(x) = V^T(M^{-1})^T U(x) \\
    Y'(x) = \frac{d}{dx}(V^T(M^{-1})^T U(x)) \\
    Y''(x) = \frac{d^2}{dx^2}(V^T(M^{-1})^T U(x)) \\
    Y'''(x) = \frac{d^3}{dx^3}(V^T(M^{-1})^T U(x))
\end{cases}$$  \hspace{0.5cm} (4)

where $V$ and $U(x)$ are vectors defined as

$$V = [y_n, y'_n, y''_n, y'''_n, f_n, f_{n+r}, f_{n+s}, f_{n+v}, f_{n+v1}, f_{n+1}]^T,$$

$$U(x) = [U_0(x), U_1(x), ..., U_6(x)]^T,$$

and $M$ is a matrix, $T$ denotes the transpose and $U(x) = x^j, j = 0, 1, ..., 6$ are basis functions.

**Proof:** The proof is the same as in [21] with slight notational modifications. The $k$-step LMM is obtained from theorem (2.1) after some manipulation and expressed in the form

$$Y(x) = \sum_{j=0}^{m-1} \alpha_j(x)y_{n+j} + h^4 \sum_{j=0}^{w-1} \beta_j(x)f_{n+j}$$  \hspace{0.5cm} (5)

where the $\alpha_j(x)'s$ and $\beta_j(x)'s$ are the continuous coefficients which are determined at the points of evaluation.
3 Specification of the Method

In this section, we set out to construct the hybrid BNTM. We choose $m = 4$, $w = 6$ and the number of off-grid points $q = 4$. The BNTM is then specified by evaluating (4) at $x = x_{n+1}$ and $x = x_{n+c}$, $c = s, r, v, v_1$, where \( \{s, r, v, v_1\} = \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\} \). The coefficients of BNTM are given in tabular form as

<table>
<thead>
<tr>
<th>$y_n$</th>
<th>$y_{n+\frac{1}{5}}$</th>
<th>$y_{n+\frac{2}{5}}$</th>
<th>$y_{n+\frac{3}{5}}$</th>
<th>$y_{n+1}$</th>
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<td>$162000000$</td>
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</tr>
</tbody>
</table>


4 Analysis of the methods

4.1 Block form

The BNTM can be given in block form as follows:

\[
A^{(0)}Y_{\mu} = A^{(1)}Y_{\mu-1} + h^4(B^{(1)}F_{\mu-1} + B^{(0)}F_{\mu}) \tag{6}
\]
where \( \mu = 1, \ldots, N \), \( n = 0, \ldots, N - 1 \), \( A^{(i)}, B^{(i)}, i = 0, 1 \) are matrices whose entries are given by the coefficients represented in tabular form above and \( A^{(0)} \) is 20 by 20 identity matrix. We also define the vectors \( Y_\mu \), \( Y_{\mu-1} \), \( F_\mu \), and \( F_{\mu-1} \) as follows:

\[
Y_\mu = (y_{n+\frac{1}{5}}, y_{n+\frac{2}{5}}, y_{n+\frac{3}{5}}, y_{n+\frac{4}{5}}, y_{n+1}, y_{n+\frac{1}{5}}, y_{n+\frac{2}{5}}, y_{n+\frac{3}{5}}, y_{n+\frac{4}{5}}, y_{n+1}, y_{n+\frac{1}{5}}, y_{n+\frac{2}{5}}, y_{n+\frac{3}{5}}, y_{n+\frac{4}{5}}, y_{n+1})^T,
\]

\[
F_\mu = (f_{n+\frac{1}{5}}, f_{n+\frac{2}{5}}, f_{n+\frac{3}{5}}, f_{n+\frac{4}{5}}, f_{n+1}, f_{n+\frac{1}{5}}, f_{n+\frac{2}{5}}, f_{n+\frac{3}{5}}, f_{n+\frac{4}{5}}, f_{n+1}, f_{n+\frac{1}{5}}, f_{n+\frac{2}{5}}, f_{n+\frac{3}{5}}, f_{n+\frac{4}{5}}, f_{n+1})^T,
\]

\[
Y_{\mu-1} = (y_{n-\frac{1}{5}}, y_{n-\frac{2}{5}}, y_{n-\frac{3}{5}}, y_{n-\frac{4}{5}}, y_{n}, y_{n-\frac{1}{5}}, y_{n-\frac{2}{5}}, y_{n-\frac{3}{5}}, y_{n-\frac{4}{5}}, y_{n}, y_{n-\frac{1}{5}}, y_{n-\frac{2}{5}}, y_{n-\frac{3}{5}}, y_{n-\frac{4}{5}}, y_{n})^T,
\]

\[
F_{\mu-1} = (f_{n-\frac{1}{5}}, f_{n-\frac{2}{5}}, f_{n-\frac{3}{5}}, f_{n-\frac{4}{5}}, f_{n}, f_{n-\frac{1}{5}}, f_{n-\frac{2}{5}}, f_{n-\frac{3}{5}}, f_{n-\frac{4}{5}}, f_{n}, f_{n-\frac{1}{5}}, f_{n-\frac{2}{5}}, f_{n-\frac{3}{5}}, f_{n-\frac{4}{5}}, f_{n})^T.
\]

The additional \( f_{n+i} \) for \( i = (\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}) \) introduced are to augment the zero entries of the vector notations.

### 4.2 Local truncation error and Order

We define the local truncation error of the BNTM using (6) as

\[
L[Y(x); h] = A^{(0)}Y_\mu - A^{(1)}Y_{\mu-1} - h^4(B^{(1)}F_{\mu-1} + B^{(0)}F_\mu),
\]

(7)

where \( L[Y(x); h] \) is a linear difference operator and impose that \( Y(x_{n+1}) = y_{n+1} = y(x_{n} + jh), Y(x_{n+c_j}) = y_{n+c_j} = y(x_{n} + c_j h), y'_{n+1} = Y'(x)|_{x=x_{n+1}} = y'(x_{n} + jh), y'_{n+c_j} = Y'(x)|_{x=x_{n+c_j}} = y'(x_{n} + c_j h), y''_{n+1} = Y''(x)|_{x=x_{n+1}} = y''(x_{n} + jh), y''_{n+c_j} = Y''(x)|_{x=x_{n+c_j}} = y''(x_{n} + c_j h), f_{n+1} = y'''(x_{n} + jh), f_{n+c_j} = y'''(x_{n} + c_j h), j = 1, \ldots, q. \) Suppose that \( Y(x) \) is sufficiently differentiable, then, the expansion of \( L[Y(x); h] \) about point \( x \) using Taylor series gives \( L[Y(x); h] \) and \( Y^{(i)}(x) = C_i Y(x) + C_1 h Y'(x) + \ldots + C_p h^p Y^p(x) + \ldots + C_p^4 h^p Y^{p+4}(x) + \ldots \)

where \( C_i, i = 0, 1, \ldots \) are column vectors whose entries comprise the error constants.

**Definition**

The BNTM has algebraic order at least \( p \geq 1 \) provided there exists a constant \( C_{p+4} \neq 0 \) such that the local truncation error \( E_\mu \) satisfies \( \| E_\mu \| = C_{p+4} h^{p+4} + O(h^{p+5}) \), where \( \| \cdot \| \) is the maximum norm.

According to this definition, the local truncation error constants \( C_{p+4} \) of \( (y_{n+k}, h y'_{n+k})^T \)

for BNTM are given as

where \( C_0 = C_1 = C_2 = \ldots C_p = \ldots C_{p+3} = 0 \).

The order \( p \) of the BNTM has been obtained from the computation of the local truncation error constant as six (6).

### 4.3 Consistency, Zero-stability and Convergence of BNTM

The consistency of the method is established by the fact that the order of BNTM is greater than one (see Jator [20], Henrici [17]).

The zero-stability of a numerical method reveals the behavior of the method with a given value of \( h > 0 \) i.e. the stability of the difference system in the limit as \( h \) tends to zero. Thus, as \( h \to 0 \), (7) tends to the difference system

\[ A^{(1)}Y_{\mu-1} - A^{(0)}Y_\mu = 0 \]

whose first characteristic polynomial is given by

\[ \rho(R) = \det(RA^{(0)} - A^{(1)}) \]

(8)

The block BNTM has its \( A^{(0)} \) and \( A^{(1)} \) as 20 by 20 matrices where \( A^{(0)} \) is identity matrix and \( A^{(1)} \) has its fifth, tenth, fifteenth and twentieth columns as

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{5} & \frac{2}{3} & \frac{3}{5} & \frac{4}{3} & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{50} & \frac{2}{3} & \frac{3}{5} & \frac{4}{3} & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{750} & \frac{375}{2} & \frac{3}{5} & \frac{4}{3} & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}^T
\]

respectively while other entries are zero.

Substituting \( A^{(0)} \) and \( A^{(1)} \) in (8), we obtain \( \rho(R) = R^{16} (R^4 - 1) \).

According to Fatunla ([13], [14]), the method is zero-stable since \( \rho(R) = 0 \) satisfies \( |R_j| \leq 1 \), \( j = 1 \) and for those roots with \( |R_j| = 1 \), the multiplicity does not exceed four.

**Theorem 4.1**

The necessary and sufficient condition for a linear multi-step method to be convergent are consistency and zero-stability (see Dahlquist [11]).

According to this theorem, BNTM is convergent since it is zero stable and consistent.
5 Implementation of BNTM

We implement the BNTM using a written code in Mathematica 10.0 enhanced by the features \textit{NSolve} \cite{190} for linear problems and \textit{FindRoot} \cite{190} for nonlinear problems respectively. In what follows, we summarize how BNTM is applied to solve initial value problems (IVPs) in a block-by-block fashion as well as applied to solve boundary value problems (BVPs) via a block unification technique.

5.1 IVPs-Block-by-block algorithm

- Step 1: Choose $N, h = (x_N - x_0)/N$, on the partition $Q_N$.
- Step 2: Using (7), $n = 0, \mu = 1$, for BNTM, solve for the values of

$$\left[y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18}, y_{19}, y_{20}\right]^T$$

simultaneously on the sub-interval $[x_0, x_1]$, as $y_0, y'_0, y''_0$ and $y'''_0$ are known from the IVP (1).
- Step 3: Next, for $n = 1, \mu = 2$ the values of

$$\left[y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18}, y_{19}, y_{20}\right]^T$$

are simultaneously obtained over the sub-interval $[x_1, x_2]$, as $y_1, y'_1, y''_1$ and $y'''_1$ are known from the previous block.
- Step 4: The process is continued for $n = 2, \ldots, N-1$ and $\mu = 3, \ldots, N$ to obtain the numerical solution to (1) on the sub-intervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{N-1}, x_N]$.

5.2 BVPs-Block unification algorithm

- Step 1: Choose $N, h = (x_N - x_0)/N$, on the partition $Q_N$.
- Step 2: Using (7), $n = 0, \mu = 1$, for BNTM, generate the variables

$$\left[y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18}, y_{19}, y_{20}\right]^T$$

on the interval $[x_0, x_1]$ and do not solve yet.
- Step 3: Next, for $n = 1, \mu = 2$ generate the variables
\[\begin{bmatrix} y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13}, y_{14}, y_{15} \end{bmatrix}^T\]

on the sub-interval \([x_1, x_2]\), and do not solve yet.

- Step 4: The process is continued for \(n = 2, \ldots, N - 1\) and \(\mu = 3, \ldots, N\) until all the variables on the sub-intervals \([x_0, x_1], [x_1, x_2], \ldots, [x_{N-1}, x_N]\) are obtained.

- Step 5: Create a single block matrix equation by the unification of all the blocks generated in Step 2 and Step 3 on \(Q_N\).

- Step 6: Solve the single block matrix equation to simultaneously obtain all the solutions for (1) on the entire \([x_0, x_N]\).

6 Numerical Examples

In this section, we give some numerical examples to illustrate the accuracy of the method. We find the absolute error of the approximate solution as \(|y - y(x)|\). We investigate the effectiveness and accuracy of the proposed BNTM by solving three IVPs and three BVPs test problems solved by different existing methods. For each example considered, we find the absolute error \(|y(x) - y_n(x)|\) of the approximate solution.

6.1 IVPs

Example 1: Consider the linear fourth order problem (see Jator [20])
\[y^{iv} = y'' + y' + 2y, \quad y(0) = y'(0) = y''(0) = 0, \quad y'''(0) = 30, \quad 0 \leq t \leq 2\]
whose theoretical solution is \(y(t) = 2e^{2t} - 5e^{-t} + 3\cos t - 9\sin t\).

This problem was solved by Yap and Ismail [34], Awoyemi [4], Jator [20] adopting block hybrid collocation method (BHCM4), multiderivative collocation method in (Awoyemi), finite difference method (Jator). We solve this problem using the proposed method, BNTM for \(0 < t < 2\) and compare the absolute error of our result at \(t = 2\) with these existing methods and the Adams Bashforth-Adams Moulton method (Adams) as shown in Table 1. BNTM compares favourably well with these existing methods.

Example 2: Consider the nonlinear fourth order problem (see Awoyemi [4])
\[y^{iv} = (y')^2 - yy''' - 4t^2 + e^t(1 + t^2 - 4t), \quad y(0) = y'(0) = 1, \quad y''(0) = 3, \quad y'''(0) = \]

\[y''''(0) =\]
Example 3: This is an application problem from Ship Dynamics which was stated by Wu [33] when a sinusoidal wave of frequency $\Omega$ passes along a ship or offshore structure, the resultant fluid actions vary with time $t$. In a particular case study by Wu et al. [33], the fourth order problem is defined as

$$y^{iv} + 3y'' + y(2 + \varepsilon \cos(\Omega t)) = 0, \quad y(0) = 1, \quad y'(0) = y''(0) = y'''(0) = 0, \quad t > 0$$

where $\varepsilon = 0$ for the existence of the theoretical solution, $y(t) = 2 \cos t - \cos(t\sqrt{2})$. The theoretical solution is undefined when $\varepsilon \neq 0 = 0$ (see Twizell [30]).

Tables 2 and 3 show the performance comparison of results between the BNTM
and the existing Yap and Ismail block hybrid collocation method [34], Adams method, Jator finite difference method [20] and Awoyemi multi-derivative collocation method [4]. The superiority of BNTM which is of lower order to the order 8 of BHCM4 is established as it is more accurate than the existing methods compared with.

### 6.2 BVPs

**Example 4:** We consider the following nonlinear boundary value problem in 
\([0, 1]\), (see [28], [29], [10]).
\[
\begin{align*}
y^{(iv)}(x) &= (y(x))^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48 \\
y(0) &= 0, \quad y'(0) = 0 \\
y(1) &= 1, \quad y'(1) = 1
\end{align*}
\]
with exact solution \( y(x) = x^5 - 2x^4 + 2x^2 \).
This problem was solved by Noor and Mohyud-Din ([28],[29]) using variational method (NMD method) and Costabile and Napoli [10] (HBVP method). We compare the result of our method, BNTM with their results as shown in Table 4.
It is obvious from the numerical results in Table 4 that BNTM is more accurate.

**Example 5** (see [28], [10])
\[
\begin{align*}
y^{(iv)}(t) &= y(t) + y''(t) + e^t(t - 3), t \in [0, 1] \\
y(0) &= 1, \quad y'(0) = 0 \\
y(1) &= 0, \quad y'(1) = -e
\end{align*}
\]
Exact solution is \( y(t) = (1-t)e^t \).
Table 5 compares the results of NMD, HBVP and BNTM methods. Its second and third columns show respectively the error in the NMD and HBVP methods.
Table 4: Numerical Results for Example 4

<table>
<thead>
<tr>
<th>t</th>
<th>NMD</th>
<th>HBVP</th>
<th>BNTM</th>
</tr>
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<tbody>
<tr>
<td>0.1</td>
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<td>$7.35 \times 10^{-16}$</td>
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Table 5: Numerical Results for Example 5

<table>
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<tr>
<th>t</th>
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<th>HBVP (degree 15)</th>
<th>HBVP (degree 8)</th>
<th>BNTM</th>
</tr>
</thead>
<tbody>
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<td>$1.69 \times 10^{-15}$</td>
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</table>

Example 6 (see [28], [10])

\[
\begin{align*}
\frac{d^4y}{dt^4}(t) &= \sin t + \sin^2 t - (y''(t))^2, \quad t \in [0, 1] \\
y(0) &= 0, \ y'(0) = 1 \\
y(1) &= \sin(1), \ y'(1) = \cos(1)
\end{align*}
\]

with exact solution \( y(t) = \sin(t) \)

It was solved by Noor and Mohyud-Din (see [28], [29]) and Costabile and Napoli [10] taking \( h = 0.1 \) by using NMD and HBVP methods of approximating polynomials of degrees 11 and 8 respectively. We also solved for the same step size with our method, BNTM and the absolute errors at different points are shown in Table 6. The superiority of BNTM is established numerically.
7 Conclusion

The derivation and implementation of an efficient hybrid BNTM have been considered. The method which directly solves IVPs and BVPs circumvents the requirement of starting values and predictors which are inherent in predictor-corrector methods as its implementation is in block by block manner. Numerical solutions of six test problems, IVPs and BVPs inclusive show the accuracy and efficiency of the method. In our future research, attention will be given to extending this method to solve partial differential equations.

References


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