On the Finite Sums of Reciprocal Lucas Numbers

Younseok Choo

Department of Electronic and Electrical Engineering
Hongik University
2639 Sejong-Ro, Sejong, 30016, Korea

Abstract

In this paper we derive three identities related to the finite sums of reciprocal Lucas numbers.

Mathematics Subject Classification: 11B39, 11B37

Keywords. Lucas numbers; reciprocal; finite sum; floor function

1 Introduction

As is well known, the Fibonacci numbers $F_n$ are generated from the recurrence relation $F_n = F_{n-1} + F_{n-2}$ ($n \geq 2$) with initial conditions $F_0 = 0$, $F_1 = 1$.

Recently Ohtsuka and Nakamura [5] found interesting properties of the Fibonacci numbers and proved Theorem 1.1 below, where $\lfloor \cdot \rfloor$ is the floor function, and $\mathbb{N}_e$ ($\mathbb{N}_o$, respectively) denotes the set of positive even (odd, respectively) integers.

Theorem 1.1 For the Fibonacci numbers $F_n$, the following identities hold:

$$\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \rfloor = \begin{cases} F_n - F_{n-1}, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ F_n - F_{n-1} - 1, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o, \end{cases} \quad (1)$$

$$\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \rfloor = \begin{cases} F_{n-1}F_n - 1, & \text{if } n \geq 2 \text{ and } n \in \mathbb{N}_e; \\ F_{n-1}F_n, & \text{if } n \geq 1 \text{ and } n \in \mathbb{N}_o. \end{cases} \quad (2)$$
Theorem 1.1 was extended in [2], [3], [7]. On the other hand, Wang and Wen [6] investigated the finite sums of Fibonacci numbers, and considerably extended Theorem 1.1.

In this paper, motivated by the work of Wang and Wen [6], we study the sums of reciprocal Lucas numbers \( L_n \), which are generated from the recurrence relation \( L_n = L_{n-1} + L_{n-2} \) \((n \geq 2)\) with initial conditions \( L_0 = 2, L_1 = 1 \). We derive three identities related to the finite sums of reciprocal Lucas numbers.

2 Results

Lemma 2.1 Let \( m \geq n \) and \( n \geq r \). Then
(a) \( L_n L_{m+1} - L_{n+1} L_m = (-1)^n(2L_{m-n+1} - L_{m-n}) \).
(b) \( L_n^2 - L_{n-r} L_{n+r} = (-1)^{n-r}(L_r - 2L_{r+1})F_r \).
(c) \( L_{n-1} L_n + L_{n+1} L_{n+2} = 5(-1)^{n-1} + 3L_{n-1} L_n \).

Proof. (a) and (b) are special cases of [1, Corollary 2.2]. For (c), we have

\[
L_{n-1} L_n + L_{n+1} L_{n+2} = (L_{n+1} - L_n)L_n + L_{n+1}(2L_n + L_{n-1})
= -L_n^2 + L_{n-1} L_{n+1} + 3L_n L_{n+1}
= 5(-1)^{n-1} + 3L_{n-1} L_n.
\]

Using the expression \( L_n = \alpha^n + \beta^n \) [4], where \( \alpha \) and \( \beta \) are roots of the equation \( x^2 - x - 1 = 0 \), Lemma 2.2 below can be easily proved.

Lemma 2.2 \( L_n L_{n+r} = L_{2n+r} + (-1)^n L_r \) for \( n \geq 0 \) and \( r \geq 0 \).

Using Lemma 2.2, we can easily prove Lemma 2.3 below.

Lemma 2.3 (a) \( L_{n-2} L_{n-1} L_n > 5L_{2n-1} \) for \( n \geq 6 \) and \( n \in \mathbb{N} \).
(b) \( L_{2n-1} > (L_{n-2} + 1)L_n \) for \( n \geq 3 \).
(c) \( 10L_{3n-2} > L_{n-2} L_n L_{n+1} \) for \( n \geq 2 \) and \( n \in \mathbb{N} \).

Proposition 2.1 If \( n = 3 \) or \( n \geq 5 \), then

\[
\sum_{k=n}^{2n} \frac{1}{L_k} < \frac{1}{L_{n-2}}.
\]

Proof. By Lemma 2.1(a), we have

\[
\frac{1}{L_{n-2}} - \frac{1}{L_{n-1}} - \frac{1}{L_n} = \frac{L_{n-3} L_n - L_{n-2} L_{n-1}}{L_{n-2} L_{n-1} L_n} = \frac{5(-1)^{n-1}}{L_{n-2} L_{n-1} L_n},
\]
and
\[
\sum_{k=n}^{2n} \frac{1}{L_k} = \frac{1}{L_{n-2}} - \frac{1}{L_{2n-1}} + 5 \sum_{k=n}^{2n} \frac{(-1)^k}{L_{k-2}L_{k-1}L_k}.
\]

If \( n \geq 6 \) and \( n \in \mathbb{N}_e \), then
\[
\sum_{k=n}^{2n} \frac{(-1)^k}{L_{k-2}L_{k-1}L_k} < \frac{1}{L_{n-2}L_{n-1}L_n},
\]
and so
\[
\sum_{k=n}^{2n} \frac{1}{L_k} < \frac{1}{L_{n-2}} - \frac{1}{L_{2n-1}} + \frac{5}{L_{n-2}L_{n-1}L_n} < \frac{1}{L_{n-2}},
\]
where the last inequality follows from Lemma 2.3(a).

If \( n \geq 3 \) and \( n \in \mathbb{N}_o \), then
\[
\sum_{k=n}^{2n} \frac{(-1)^k}{L_{k-2}L_{k-1}L_k} < 0.
\]

Hence
\[
\sum_{k=n}^{2n} \frac{1}{L_k} < \frac{1}{L_{n-2}} - \frac{1}{L_{2n-1}} < \frac{1}{L_{n-2}},
\]
and the proof is completed.

**Proposition 2.2** If \( n \geq 2 \), then
\[
\sum_{k=n}^{2n} \frac{1}{L_k} > \frac{1}{L_{n-2} + 1}.
\]

**Proof.** By Lemma 2.1(a), we have
\[
\frac{1}{L_{n-2} + 1} \frac{1}{L_{n-1} + 1} \frac{1}{L_n} = \frac{L_{n-3}L_n - L_{n-2}L_{n-1} - L_n}{(L_{n-2} + 1)(L_{n-1} + 1)L_n} = \frac{5(-1)^{n-1} - L_n - 1}{(L_{n-2} + 1)(L_{n-1} + 1)L_n},
\]
and
\[
\sum_{k=n}^{2n} \frac{1}{L_k} = \frac{1}{L_{n-2} + 1} - \frac{1}{L_{2n-1} + 1} + \sum_{k=n}^{2n} \frac{5(-1)^k + L_k + 1}{(L_{k-2} + 1)(L_{k-1} + 1)L_k}.
\]
Assume that \( n \in \mathbb{N}_e \). Since (4) holds for \( n = 2 \), let \( n \geq 4 \). Then
\[
\sum_{k=n}^{2n} \frac{(-1)^k}{(L_{k-2} + 1)(L_{k-1} + 1)L_k} > \frac{1}{(L_{2n-2} + 1)(L_{2n-1} + 1)L_{2n}},
\]
and so
\[
\sum_{k=n}^{2n} \frac{1}{L_k} > \frac{1}{L_{n-2} + 1} - \frac{1}{L_{2n-1} + 1} + \frac{L_n + 1}{(L_{n-2} + 1)(L_{n-1} + 1)L_n}
\]
\[
> \frac{1}{L_{n-2} + 1} - \frac{1}{L_{2n-1} + 1} + \frac{1}{(L_{n-2} + 1)(L_{n-1} + 1)} > \frac{1}{L_{n-2} + 1},
\]
where the last inequality follows from Lemma 2.3(b).
Assume that \( n \in \mathbb{N}_o \). It is easily seen that (4) holds for \( n = 3 \) and \( n = 5 \). Hence, let \( n \geq 7 \). Since \( L_n - 4 > L_{n-1} + 1 \), then
\[
\sum_{k=n}^{2n} \frac{1}{L_k} > \frac{1}{L_{n-2} + 1} - \frac{1}{L_{2n-1} + 1} + \frac{L_n - 4}{(L_{n-2} + 1)(L_{n-1} + 1)L_n}
\]
\[
> \frac{1}{L_{n-2} + 1} - \frac{1}{L_{2n-1} + 1} + \frac{1}{(L_{n-2} + 1)L_n} > \frac{1}{L_{n-2} + 1},
\]
where the last inequality is a result of Lemma 2.3(b), and the proof is completed.

From Proposition 2.1 and Proposition 2.2, we obtain the following result.

**Theorem 2.3** If \( n = 3 \) or \( n \geq 5 \), then
\[
\left\lfloor \left( \sum_{k=n}^{2n} \frac{1}{L_k} \right)^{-1} \right\rfloor = L_{n-2}.
\]
\[ (5) \]

**Proposition 2.4** If \( n \geq 4 \) with \( n \in \mathbb{N}_e \) and \( m \geq n + 1 \), then
\[
\sum_{k=n}^{m} \frac{1}{L_k} < \frac{1}{L_{n-2} - 1}. \]
\[ (6) \]

**Proof.** Consider
\[
\frac{1}{L_{n-2} - 1} - \frac{1}{L_n} - \frac{1}{L_{n+1}} - \frac{1}{L_n} = \frac{X_1}{(L_{n-2} - 1)L_nL_{n+1}(L_n - 1)},
\]
where
\[
X_1 = L_nL_{n+1}(L_n - L_{n-2}) - (L_n + L_{n+1})(L_{n-2} - 1)(L_n - 1).
\]
By Lemma 2.1(a),

\[ X_1 = L_n L_{n+1}(L_{n+2} - L_{n+1}) - L_n L_{n+1}(L_n - L_{n-1}) \]
\[ - (L_n + L_{n+1})(L_n - L_{n-1})(L_{n+2} - L_{n+1}) + L_{n+2}(L_{n-2} + L_n - 1) \]
\[ = (L_{n-1} L_{n+1} - L_n^2) L_{n+2} + (L_n L_{n+2} - L_{n+1}^2) L_{n-1} + L_{n+2}(L_{n-2} + L_n - 1) \]
\[ = 5(-1)^n (L_{n+2} - L_{n+1}) + L_{n+2}(L_{n-2} + L_n - 1), \]

and so
\[ \frac{1}{L_n} + \frac{1}{L_{n+1}} = \frac{1}{L_{n-2}} - \frac{1}{L_n} + \frac{6 - L_{n-2} - L_n}{(L_{n-2} - 1)L_n L_{n+1}(L_n - 1)}. \]

Since \((6 - L_{n-2} - L_n)L_{n+2} - 5L_{n-1} < 0\) for \(n \geq 4\), then
\[ \frac{1}{L_n} + \frac{1}{L_{n+1}} < \frac{1}{L_{n-2}} - \frac{1}{L_n}. \]

If \(m \geq n + 1\) and \(m \in \mathbb{N}_e\), then
\[ \sum_{k=n}^{m} \frac{1}{L_k} < \frac{1}{L_{n-2} - 1} - \frac{1}{L_{m-2} - 1} + \frac{1}{L_m - 1} < \frac{1}{L_{n-2} - 1}. \]

If \(m \geq n + 1\) and \(m \in \mathbb{N}_o\), then
\[ \sum_{k=n}^{m} \frac{1}{L_k} < \frac{1}{L_{n-2} - 1} - \frac{1}{L_{m-1} - 1} < \frac{1}{L_{n-2} - 1}, \]

and the proof is completed.

**Proposition 2.5** If \(n \geq 2\) with \(n \in \mathbb{N}_e\) and \(m \geq 3n\), then
\[ \sum_{k=n}^{m} \frac{1}{L_k} > \frac{1}{L_{n-2}}. \]  

**Proof.** By Lemma 2.1(a),
\[ \frac{1}{L_{n-2}} - \frac{2}{L_n} - \frac{1}{L_{n+1}} = \frac{L_n L_{n+1} - 2L_{n-2} L_{n+1} - L_{n-2} L_n}{L_{n-2} L_n L_{n+1}} \]
\[ = \frac{L_{n-1} L_{n+1} - L_{n-2} L_{n+2}}{L_{n-2} L_n L_{n+1}} \]
\[ = \frac{10(-1)^{n-1}}{L_{n-2} L_n L_{n+1}}. \]

Then
\[ \frac{1}{L_n} + \frac{1}{L_{n+1}} = \frac{1}{L_{n-2}} - \frac{1}{L_n} + \frac{10}{L_{n-2} L_n L_{n+1}}, \]
and
\[ \sum_{k=n}^{3n} \frac{1}{L_k} = \left( \frac{1}{L_n} + \frac{1}{L_{n+1}} + \cdots + \frac{1}{L_{3n-2}} + \frac{1}{L_{3n-1}} \right) + \frac{1}{L_{3n}} > \frac{1}{L_{n-2}} - \frac{1}{L_{3n-2}} + \frac{10}{L_{n-2}L_nL_{n+1}} + \frac{1}{L_{3n}} > \frac{1}{L_{n-2}}, \]
where the last inequality follows from Lemma 2.3(c), and the proof is completed.

**Proposition 2.6** If \( n \geq 3 \) with \( n \in \mathbb{N}_o \) and \( m \geq n + 1 \), then
\[ \sum_{k=n}^{m} \frac{1}{L_k} < \frac{1}{L_{n-2}}. \]  

**Proof.** From the proof of Proposition 2.5,
\[ \frac{1}{L_{n-2}} - \frac{2}{L_n} - \frac{1}{L_{n+1}} = \frac{10(-1)^{n-1}}{L_{n-2}L_nL_{n+1}}. \]
If \( n \in \mathbb{N}_o \), then
\[ \frac{1}{L_n} + \frac{1}{L_{n+1}} = \frac{1}{L_{n-2}} - \frac{1}{L_n} - \frac{10}{L_{n-2}L_nL_{n+1}} < \frac{1}{L_{n-2}} - \frac{1}{L_n}. \]
If \( m \geq n + 1 \) and \( m \in \mathbb{N}_e \), then
\[ \sum_{k=n}^{m} \frac{1}{L_k} < \frac{1}{L_{n-2}} - \frac{1}{L_{m-1}} < \frac{1}{L_{n-2}}. \]
If \( m \geq n + 1 \) and \( m \in \mathbb{N}_o \), then
\[ \sum_{k=n}^{m} \frac{1}{L_k} < \frac{1}{L_{n-2}} - \frac{1}{L_{m-2}} + \frac{1}{L_m} < \frac{1}{L_{n-2}}, \]
and the proof is completed.

From Proposition 2.2, Proposition 2.4–Proposition 2.6, we obtain Theorem 2.7 below.

**Theorem 2.7** For the Lucas numbers \( L_n \), the following identities hold:
\[ \left( \sum_{k=n}^{m} \frac{1}{L_k} \right)^{-1} = \begin{cases} L_{n-2} - 1, & \text{if } n \geq 4 \text{ with } n \in \mathbb{N}_e \text{ and } m \geq 3n; \\ L_{n-2}, & \text{if } n \geq 3 \text{ with } n \in \mathbb{N}_o \text{ and } m \geq 2n. \end{cases} \]
Proposition 2.8 If \( n \geq 2 \) with \( n \in \mathbb{N}_e \) and \( m \geq n + 1 \), then

\[
\sum_{k=n}^{m} \frac{1}{L_k^2} < \frac{1}{L_{n-1}L_n + 1}.
\]  

Proof. Consider

\[
\frac{1}{L_{n-1}L_n + 1} - \frac{1}{L_n} - \frac{1}{L_{n+1}L_{n+2} + 1} = \frac{(L_n^2 + L_{n+1}^2)Y_1}{(L_{n-1}L_n + 1)L_n^2L_{n+1}^2(L_{n+1}L_{n+2} + 1)},
\]

where, by the identity \( L_{n+1}L_{n+2} - L_{n-1}L_n = L_n^2 + L_{n+1}^2 \),

\[
Y_1 = \frac{L_n^2}{L_{n+1}} - L_{n-1}L_nL_{n+1}L_{n+2} - L_{n-1}L_n - L_{n+1}L_{n+2} - 1.
\]

Using Lemma 2.1(a),(c), we have

\[
Y_1 = L_nL_{n+1}(L_nL_{n+1} - L_{n-1}L_{n+2}) - (L_{n-1}L_n + L_{n+1}L_{n+2}) - 1 = 5(-1)^nL_nL_{n+1} - 5(-1)^{n-1} - 1.
\]

If \( n \in \mathbb{N}_e \), then \( Y_1 = 2L_nL_{n+1} + 4 \), and

\[
\frac{1}{L_n^2} + \frac{1}{L_{n+1}^2} = \frac{1}{L_{n-1}L_n + 1} - \frac{1}{L_{n+1}L_{n+2} + 1} - \frac{(L_n^2 + L_{n+1}^2)(2L_nL_{n+1} + 4)}{(L_{n-1}L_n + 1)L_n^2L_{n+1}^2(L_{n+1}L_{n+2} + 1)} < \frac{1}{L_{n-1}L_n + 1} - \frac{1}{L_{n+1}L_{n+2} + 1}.
\]

If \( m \geq n + 1 \) and \( m \in \mathbb{N}_e \), then

\[
\sum_{k=n}^{m} \frac{1}{L_k^2} < \frac{1}{L_{n-1}L_n + 1} - \frac{1}{L_{m-1}L_m + 1} + \frac{1}{L_m^2} < \frac{1}{L_{n-1}L_n + 1}.
\]

If \( m \geq n + 1 \) and \( m \in \mathbb{N}_o \), then

\[
\sum_{k=n}^{m} \frac{1}{L_k^2} = \frac{1}{L_{n-1}L_n + 1} - \frac{1}{L_mL_{m+1} + 1} < \frac{1}{L_{n-1}L_n + 1},
\]

and the proof is completed.

Proposition 2.9 If \( n \geq 2 \) with \( n \in \mathbb{N}_e \) and \( m \geq 2n + 1 \), then

\[
\sum_{k=n}^{m} \frac{1}{L_k^2} > \frac{1}{L_{n-1}L_n + 2}.
\]  

Proof. Consider

\[
\frac{1}{L_{n-1}L_n + 2} - \frac{1}{L_n^2} - \frac{1}{L_{n+1}^2} - \frac{1}{L_{n+1}L_{n+2} + 2} = \frac{(L_n^2 + L_{n+1}^2)Y_2}{(L_{n-1}L_n + 2)L_n^2L_{n+1}^2(L_{n+1}L_{n+2} + 2)},
\]
where, by Lemma 2.1(a),(c),
\[ Y_2 = L_nL_{n+1}(L_nL_{n+1} - L_{n-1}L_{n+2}) - 2(L_nL_n + L_{n+1}L_{n+2}) - 4 \]
\[ = 5(-1)^nL_nL_{n+1} - 2[5(-1)^{n-1} + 3L_nL_{n+1}] - 4. \]

If \( n \in \mathbb{N}_e \), then \( Y_2 = -L_nL_{n+1} + 6 \). Since \( L_nL_{n+1} - 6 - L_{n-1}L_n - 2 = L_n^2 - 8 > 0 \) for \( n \geq 2 \), we obtain
\[
\frac{1}{L_n^2 + 1} + \frac{1}{L_{n+1}^2} = \frac{1}{L_n-1L_n+2} - \frac{1}{L_{n+1}L_{n+2}+2} + \frac{(L_n^2 + L_{n+1}^2)(L_nL_{n+1} - 6)}{(L_n-1L_n+2)L_{n+1}^2 + 2L_nL_{n+1}L_{n+2} + 2) + \frac{L_n^2 + L_{n+1}^2}{L_n^2L_{n+1}(L_nL_{n+1} + 2)}.
\]

and
\[
\sum_{k=n}^{2n+1} \frac{1}{L_k^2} > \frac{1}{L_n-1L_n+2} - \frac{1}{L_{n+1}L_{n+2}+2} + \frac{L_n^2 + L_{n+1}^2}{L_n^2L_{n+1}(L_nL_{n+1} + 2)}.
\]

Using Lemma 2.2, we have
\[
(L_n^2 + L_{n+1}^2)(L_{2n+1}^2 + 2) - L_n^2L_{n+1}^2(L_nL_{n+1} + 2)
= (L_n^2 + L_{n+1}^2)(L_n^3L_{n+1} + 2L_nL_{n+1} - L_{n+1}^2) - L_n^2L_{n+1}^2(L_nL_{n+1} + 2)
= L_n^4L_{n+1}^2 - 2L_n^2L_{n+1}^3 + 2L_n^3L_{n+1}^2 - 3L_n^2L_{n+1}^2
\geq 2L_n^4 - 3L_n^2L_{n+1}^2 + 2L_nL_{n+1}^3 + 2L_n^3L_{n+1}
= L_n^2[L_n^2(4L_n-1) - L_n^2] + 2L_n^3L_{n+1}^2 + 2L_n^3L_{n+1}
> 0,
\]

which, in turn, implies that
\[
\sum_{k=n}^{2n+1} \frac{1}{L_k^2} > \frac{1}{L_n-1L_n+2},
\]
and the proof is completed.

**Proposition 2.10** If \( n \geq 3 \) with \( n \in \mathbb{N}_o \) and \( m \geq n+1 \), then
\[
\sum_{k=n}^{m} \frac{1}{L_k^2} < \frac{1}{L_n-1L_n-2}. \tag{12}
\]

**Proof.** Consider
\[
\frac{1}{L_n-1L_n} - \frac{1}{L_n^2} - \frac{1}{L_{n+1}L_{n+2}} - 2 = \frac{(L_n^2 + L_{n+1}^2)Y_3}{(L_n-1L_n-2)L_n^2L_{n+1}(L_nL_{n+1} + 2)},
\]
where, by Lemma 2.1(a),(c),
\[ Y_3 = L_n L_{n+1}(L_n L_{n+1} - L_{n-1} L_{n+2}) + 2(L_{n-1} L_n + L_{n+1} L_{n+2}) - 4 = 5(-1)^n L_n L_{n+1} + 2[(-1)^{n-1} + 3 L_n L_{n+1}] - 4. \]

If \( n \in \mathbb{N}_0 \), then \( Y_3 = L_n L_{n+1} + 6 \) and
\[
\frac{1}{L_n^2} + \frac{1}{L_{n+1}^2} = \frac{1}{L_n-1 L_n-2} - \frac{1}{L_{n+1} L_{n+2}-2} - \frac{(L_n^2 + L_{n+1}^2)(L_n L_{n+1} + 6)}{(L_n-1 L_n-2) L_n^2 L_{n+1}^2 (L_{n+1} L_{n+2}-2)} < \frac{1}{L_n-1 L_n-2} - \frac{1}{L_{n+1} L_{n+2}-2}. 
\]

If \( m \geq n+1 \) and \( m \in \mathbb{N}_e \), then
\[
\sum_{k=n}^m \frac{1}{T_k^2} < \frac{1}{L_n-1 L_n-2} - \frac{1}{L_{m-1} L_m+1} + \frac{1}{L_m^2} < \frac{1}{L_n-1 L_n-2}. 
\]

If \( m \geq n+1 \) and \( m \in \mathbb{N}_o \), then
\[
\sum_{k=n}^m \frac{1}{T_k^2} < \frac{1}{L_n-1 L_n-2} - \frac{1}{L_{m-1} L_m+2} < \frac{1}{L_n-1 L_n-2}, 
\]
and the proof is completed.

**Proposition 2.11** If \( n \geq 3 \) with \( n \in \mathbb{N}_0 \) and \( m \geq 2n \), then
\[
\sum_{k=n}^m \frac{1}{T_k^2} > \frac{1}{L_n-1 L_n-1}. \tag{13}
\]

**Proof.** Consider
\[
\frac{1}{L_n-1 L_n-1} - \frac{1}{L_n^2} - \frac{1}{L_{n+1}^2} = \frac{1}{L_{n+1} L_{n+2}-2} - \frac{(L_n^2 + L_{n+1}^2) Y_4}{(L_n-1 L_n-1) L_n^2 L_{n+1}^2 (L_{n+1} L_{n+2}-2)},
\]
where, by Lemma 2.1(a),(c),
\[
Y_4 = L_n L_{n+1}(L_n L_{n+1} - L_{n-1} L_{n+2}) + (L_{n-1} L_n + L_{n+1} L_{n+2}) - 1 = 5(-1)^n L_n L_{n+1} + 5(-1)^{n-1} + 3 L_n L_{n+1} - 1.
\]

If \( n \in \mathbb{N}_0 \), then \( Y_4 = -2L_n L_{n+1} + 4 \). Since \( 2L_n L_{n+1} - 4 \geq 3(L_{n-1} L_n - 1) \) for \( n \geq 2 \), we have
\[
\frac{1}{L_n^2} + \frac{1}{L_{n+1}^2} = \frac{1}{L_n-1 L_n-1} - \frac{1}{L_{n+1} L_{n+2}-1} + \frac{(L_n^2 + L_{n+1}^2)(2L_n L_{n+1} - 4)}{(L_n-1 L_n-1) L_n^2 L_{n+1}^2 (L_{n+1} L_{n+2}-1)} > \frac{1}{L_n-1 L_n-1} - \frac{1}{L_{n+1} L_{n+2}-1} + \frac{3(L_n^2 + L_{n+1}^2)}{L_n^2 L_{n+1}^2 (L_{n+1} L_{n+2}-1)}. 
\]
and so
\[
\sum_{k=n}^{2n} \frac{1}{L_k^2} > \frac{1}{L_{n-1}L_n - 1} - \frac{1}{L_{2n}L_{2n+1} - 1} + \frac{3(L_n^2 + L_{n+1}^2)}{L_n^2L_{n+1}^2(L_{n+1}L_{n+2} - 1)}.
\]

Using Lemma 2.2, we can show that
\[
3(L_n^2 + L_{n+1}^2)(L_{2n}L_{2n+1} - 1) > L_n^2L_{n+1}^2(L_{n+1}L_{n+2} - 1),
\]
from which we obtain
\[
\sum_{k=n}^{2n} \frac{1}{L_k^2} > \frac{1}{L_{n-1}L_n - 1},
\]
and the proof is completed.

**Theorem 2.12** For the Lucas numbers $L_n$, the following identities hold:

\[
\left( \sum_{k=n}^{m} \frac{1}{L_k^2} \right)^{-1} = \begin{cases} 
L_{n-1}L_n + 1, & \text{if } n \geq 2 \text{ with } n \in \mathbb{N}_e \text{ and } m \geq 2n + 1; \\
L_{n-1}L_n - 2, & \text{if } n \geq 1 \text{ with } n \in \mathbb{N}_o \text{ and } m \geq 2n.
\end{cases}
\]

(14)

**Proof.** (14) clearly holds for $n = 1$. For the cases $n \geq 2$, the results follow from Proposition 2.8–Proposition 2.11.

**References**


Received: May 25, 2017; Published: June 7, 2017