Numerical Solution of Fuzzy Delay

Predator-Prey System

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Abstract

The question for developing efficient and accurate fuzzification scheme for fuzzy delay predator-prey system (FDPP) is an object of long-standing interest. We perform a numerical solution of fuzzy delay predator-prey system. The solution procedure is developed by Runge-Kutta method (RK4). The nature of numerical solution are demonstrated via numerical example (FDPP) system to show the convergence and accuracy of the proposed method. The superiority of fuzzy systems over the crisp one is established.

Keywords: Delay differential system, Fuzzy delay predator-prey system (FDPP) system, Runge Kutta method

1 Introduction

The study of population dynamics including (stable, unstable and oscillatory behavior) has become very important since Volterra and Lotka proposed the seminal models of predator-prey models in 1920. Predator-prey models represent the basis of many models used today in the analysis of population dynamics and is one of the most popular in mathematical ecology. The dynamic properties of the predator-prey models which have significant biological background have been paid a great attention. Some studies in the area of predator-prey interaction, that treat population can be extended by including time delay. The time delay is inclu-
ded into population dynamics when the rate of the population is not only a function of the present population but also depends on the pervious population. In 2012, Xu and Li [1] explained the stability of the local Hopf bifurcation for delay predator-prey model with two delays. In 2008, Toaha [2] showed a deterministic and continuous model for predator-prey with delay and constant rates of harvesting and studied the combined effects of harvesting and time delay on the dynamics of predator-prey model.

The use of fuzzy sets may be seen as an effective tool for better understanding of this phenomena. It is therefore not surprising that there is a vast literature dealing with fuzzy differential equations for modeling this kind of perception. In the study of delay differential equations in a fuzzy environment, the term fuzzy delay differential equations is used for referring to differential equations with fuzzy coefficients, delay differential equations with fuzzy initial values or fuzzy boundary values, dealing with functions on the space of fuzzy intervals (see [3], [4]).

In 2009, Mohammed and Bernard [5] explained numerical solutions of delay predator-prey with fuzzy initial populations. They solved this model numerically by fourth-order Runge-Kutta method. In 2015, Barzinji [6] solved the stability of steady state of fuzzy delay predator-prey model, the work shows that the trivial steady state is unstable for all values of delays, meanwhile the semi trivial steady state is locally asymptotically stable for all values of delay under certain conditions.

In this paper, we propose a new way of studying numerical solutions of the fuzzy delay predator-prey population model by Runge-Kutta method.

The organization of this paper is as follows. In Section 2, the basic notation a fuzzy number, fuzzy sets, fuzzy differential equations have been introduced. In Section 3, fuzzy delay predator-prey is introduced. In Section 4, we propose new method (based on the Runge-Kutta method) and described it in details. In Section 5, the applicability of the method is illustrated by an example. Finally, Section 6, concludes the main results with discussions.

2 Preliminary notes

Definition 1 [6]
As mentioned before, a fuzzy set describes mapping from a universal set into [0,1], where every function \( \mu: X \to [0,1] \) is represented as a fuzzy set. A set \( F_1 = \{ x \in R | x \ is \ about \ a_2 \} \) with triangular membership function can be defined as

\[
\mu_{F_1} = \begin{cases} 
\frac{x - a_1}{a_2 - a_1} & x \in [a_1, a_2) \\
1 & x = a_2 \\
\frac{-x + a_3}{a_2 - a_1} & x \in (a_2, a_3] 
\end{cases}
\]
A function being represented by a set of ordered pairs defines any fuzzy set as \( F = \{(x, \mu_F(x)) : x \in X\} \).

**Definition 2 [6]**

A fuzzy number is a function \( u : R \to [0,1] \) which satisfies the following:

1. \( u \) is normal, i.e. \( \exists x_0 \in R \) with \( u(x_0) = 1 \).
2. \( u \) is a convex fuzzy set i.e.
   \[ u(\alpha x + (1 - \alpha) y) \geq \min\{u(x), u(y)\} \forall x, y \in R, \alpha \in [0,1]. \]
3. \( u \) is upper semi-continuous on \( R \).
4. \( \{x \in R : u(x) > 0\} \) is compact with \( \tilde{A} \) denoting the closure of \( A \).

**Definition 3 [6]**

An \( \alpha \)-cut, \( u_\alpha \), is a crisp set containing all elements of the universal set \( X \) possessing a membership function at least to the degree of \( \alpha \). It is expressed as
\[ u_\alpha = \{x \in X : \mu_u(x) \geq \alpha\}. \]
Alternatively, a fuzzy number is defined as,

**Definition 4 [3]**

A fuzzy number \( u \) can completely be determined by any pair \( u = (\overline{u}, \underline{u}) \) of functions \( \overline{u}(\alpha), \underline{u}(\alpha) : [0,1] \to R \) if the following three conditions are satisfied:

1. \( \overline{u}(\alpha) \) is a bounded, monotonic, (non-decreasing) left-continuous function for all \( \alpha \in (0,1] \) and right-continuous for \( \alpha = 0 \).
2. \( \underline{u}(\alpha) \) is a bounded, monotonic, (non-increasing) left-continuous function for all \( \alpha \in (0,1] \) and right-continuous for \( \alpha = 0 \).
3. For all \( \alpha \in (0,1] \) one obtains \( u(\alpha) \geq \underline{u}(\alpha) \).

For every \( u = (\overline{u}, \underline{u}), v = (\overline{v}, \underline{v}) \) and \( k > 0 \) one gets,

1. \( (u + v)(\alpha) = u(\alpha) + v(\alpha) \)
2. \( (\overline{u} + \overline{v})(\alpha) = \overline{u}(\alpha) + \overline{v}(\alpha) \)
3. \( (ku)(\alpha) = ku(\alpha), (ku)(\alpha) = k\overline{u}(\alpha) \)

This set fuzzy numbers with addition and multiplication is denoted by \( E^1 \).

**Definition 5 [3]**

The distance between two fuzzy numbers \( u = (\overline{u}, \underline{u}) \) and \( v = (\overline{v}, \underline{v}) \) yields
\[ d(u, v) = \sup_{\alpha \in [0,1]} \{ \max \{ |u(\alpha) - \underline{v}(\alpha)|, |\overline{u}(\alpha) - \overline{v}(\alpha)| \} \}. \]
Here \( (E^1, d) \) form a complete metric space.

**Definition 6 [3]**

The function \( f : R \to E^1 \) is called a fuzzy function. Now if, for an arbitrary fixed \( \hat{t} \in R \) and \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that \( |t - \hat{t}| < \delta \Rightarrow d(f(t), f(\hat{t})) < \varepsilon \), then \( f \) is said to be continuous. Note that \( d \) is the metric which is defined in Definition 5 (In this article we simply replace \( R \) by \( [t_0, T] \)).
Definition 7 [3]
Let \( u, v \in E^1 \). If there exists \( w \in E^1 \) such that \( u = v + w \) then \( w \) is called the H-difference of \( u, v \) and it is denoted by \( u - v \).

Definition 8 [3]
A function \( f: (a, b) \to E^1 \) is called H-differentiable at \( \hat{t} \in (a, b) \) if, for \( h > 0 \) sufficiently small, there exist the H-differences \( f(\hat{t} + h) - f(\hat{t}), f(\hat{t}) - f(\hat{t} - h) \), and an element \( f'(\hat{t}) \in E^1 \) such that:

\[
0 = \lim_{h \to 0^+} d \left( f(\hat{t} + h), f(\hat{t}) \right) = \lim_{h \to 0^+} d \left( f(\hat{t}), f(\hat{t} - h) \right). 
\]

Then \( f'(\hat{t}) \) is called the fuzzy derivative of \( f \) at \( \hat{t} \).

Consider the first order linear fuzzy initial value differential equation give by:

\[
\dot{x}(t) = Ax(t) + Bx(t - \tau) \quad t \in [t_0, T] \\
\begin{array}{c}
x(t) = x_0 \quad t \in [t_0, T] \\
\end{array}
\]

Where \( x(t) \) and \( x(t - \tau) \in \{x(t): t < T\} \) are \( n \)-dimensional fuzzy functions of \( t \). Every element of the matrices \( A = [a_{ij}]_{n \times m}, a_{ij} \in F(R) \) and \( B = [b_{ij}]_{n \times n}, b_{ij} \in F(R) \) is assumed to be a fuzzy number where \( F(R) \) represents the fuzzy sets on \( R \). The function \( \dot{x}(t) \) is the fuzzy derivatives of \( x(t) \) at \( t \in I \) and \( x_0 \) is a fuzzy number, and the delay parameter \( \tau \) is a known positive number. The \( \alpha \)-cut sets of \( x(t) \) and \( x(t - \tau) \) for \( t \in [t_0, T] \) are as below:

\[
x^k_{\alpha}(t) = [x^k_{\alpha}, x^k_{\alpha}] \quad k = 1,2, \ldots n, \\
x^k_{\alpha}(t - \tau) = [x^k_{\alpha}(t - \tau), x^k_{\alpha}(t - \tau)] \quad k = 1,2, \ldots n.
\]

Let \( x_{\alpha}(t) = [x_{\alpha}(t), \bar{x}_{\alpha}(t)] \) be the solution of the fuzzy delay systems(1) in which the elements of matrices \( A \) and \( B \) are symmetric triangular fuzzy numbers, then (1) can be rewritten with \( x_{\alpha}(t) = [x_{\alpha}(t), \bar{x}_{\alpha}(t)] \) as its solution in the following forms:

\[
\begin{align*}
\dot{x}_{\alpha}(t) &= A_{\alpha}x_{\alpha}(t) + B_{\alpha}x_{\alpha}(t - \tau), \quad t \in [t_0, T] \\
\dot{x}_{\alpha}(t) &= A_{\alpha}x_{\alpha}(t) + B_{\alpha}x_{\alpha}(t - \tau), \quad 0 < \alpha \leq 1 \\
x_{\alpha}(t) &= x_{\alpha 0}, \quad t \in [t_0 - \tau, t_0] \\
\bar{x}_{\alpha}(t) &= \bar{x}_{\alpha 0}.
\end{align*}
\]

Suppose \( (a_{ij})_{\alpha} = [(a_{ij})^-_{\alpha}, (a_{ij})^+_{\alpha}] \), \( A_{\alpha} = [A^-_{\alpha}, A^+_{\alpha}] \) where \( A^-_{\alpha} = [(a_{ij})^-_{\alpha}]_{n \times n} \), \( A^+_{\alpha} = [(a_{ij})^+_{\alpha}]_{n \times n} \), \( (b_{ij})_{\alpha} = [(b_{ij})^-_{\alpha}, (b_{ij})^+_{\alpha}] \), \( B_{\alpha} = [B^-_{\alpha}, B^+_{\alpha}] \) where \( B^-_{\alpha} = [(b_{ij})^-_{\alpha}]_{n \times n}, B^+_{\alpha} = [(b_{ij})^+_{\alpha}]_{n \times n} \).

Barzinji [6] introduce the following theorem that extends from Farahi’s theorem:
Theorem 1 [6]
If $A(\mu, \alpha) = [a_{ij}(\mu, \alpha)]_{n \times n} = (1 - \mu)A^-_\alpha + \mu A^+\alpha$, $B(\mu, \alpha) = [b_{ij}(\mu, \alpha)]_{n \times n} = (1 - \mu)B^-\alpha + \mu B^+\alpha$, for $\mu \in [0,1]$.

Then the solution of (2) is $x_\alpha(t) = [x_\alpha(t), \overline{x}_\alpha(t)]$, if and only if $x_\alpha(t) = [x_\alpha(t), \overline{x}_\alpha(t)]$ is also a solution of the following system:

$$
\begin{align*}
\dot{x}_\alpha(t) &= \bigcup_{\mu=0}^1 C(\mu, \alpha)x_\alpha(t) + \bigcup_{\mu=0}^1 D(\mu, \alpha)x_\alpha(t - \tau), \\
\dot{\overline{x}}_\alpha(t) &= \bigcup_{\mu=0}^1 C(\mu, \alpha)\overline{x}_\alpha(t) + \bigcup_{\mu=0}^1 D(\mu, \alpha)\overline{x}_\alpha(t - \tau), \\
x_\alpha(t) &= x_\alpha(0), \\
\overline{x}_\alpha(t) &= \overline{x}_\alpha(0), \\
n &\in [t_0, T], \\
0 < \alpha \leq 1
\end{align*}
$$

(3)

The entries of matrices $C$ and $D$ are determined from $A(\mu, \alpha)$ and $B(\mu, \alpha)$ as follows:

$$
c_{ij} = \begin{cases} 
e a_{ij}(\mu, \alpha) & a_{ij} \geq 0 \\
 0 & a_{ij} < 0 
\end{cases}
$$

and

$$
d_{ij} = \begin{cases} 
e b_{ij}(\mu, \alpha) & b_{ij} \geq 0 \\
 0 & b_{ij} < 0 
\end{cases}
$$

such that $e$ is the identity operation and $g$ corresponds to a flip about the diagonal, i.e., $ea_{ij}$ means $a_{ij}$ is extended to $\left(\begin{array}{cc} a_{ij} & 0 \\
 0 & a_{ij} \end{array}\right)$, and $ga_{ij}$ means $a_{ij}$ is extended to $\left(\begin{array}{cc} 0 & a_{ij} \\
 a_{ij} & 0 \end{array}\right)$.

3 Two-Dimensional Fuzzy Delay Differential Systems

In this section, first consider a general two-dimensional systems with fuzzy initial values:

$$
\begin{align*}
\dot{x}(t) &= f(x(t), x_t, y(t), y_t), \\
\dot{y}(t) &= g(x(t), x_t, y(t), y_t), \\
x(t_0) &= x_0, \\
y(t_0) &= y_0.
\end{align*}
$$

(4)

Where $f, g$ are fuzzy functions and $x_0, y_0$ are fuzzy numbers. A solution to (4) can be approximated by any numerical method existing in the literature.
4 Numerical Methods

In this section, substituting all the fuzzy functions that introduced in Section 3 by crisp numbers, the new system which contains 4 crisp delay differential equations is in the following form:

\[ \dot{X}(t) = AX(t) + B(X(t)) + C(X(t - \tau)), \quad t \in [t_0, T] \]
\[ X(t) = X_0. \quad t \in [t_0 - \tau, t_0] \]  \hspace{1cm} (5)

Where for each \( t \in [t_0, T] \), 4-dimensional functions \( X(t), X(t - \tau), \dot{X}(t) \) are defined as follows:

\[ X(t) = X = [\bar{x}_t, \tilde{x}_t, \underline{x}_t, \check{x}_t]^T, B(X(t)) = B(X) \\
= [b_{ij}(\bar{x}_t, \tilde{x}_t, \underline{x}_t, \check{x}_t)]^T, \\
C(X(t - \tau)) = C(X_t) = [c_{ij}(\bar{x}_{t_0}, \tilde{x}_{t_0}, \underline{x}_{t_0}, \check{x}_{t_0})]^T, \]

\[ \dot{X}(t) = \dot{X} = [\ddot{x}_t, \dddot{x}_t, \ddddot{x}_t, \ddddddot{x}_t]^T, \quad X_0 = [\bar{x}_{t_0}, \tilde{x}_{t_0}, \underline{x}_{t_0}, \check{x}_{t_0}]^T \in R^4, \quad A = [a_{ij}]_{4 \times 4} \in R^4 \] respectively.

I replace the interval \([t_0, T]\) by a set of discrete equally spaced grid points \( t_0 < t_1 < t_2 < \cdots < t_N = T, \quad h = \frac{T - t_0}{N}, \quad t_i = t_0 + ih, \quad i = 1, 2, \ldots, N \).

Thus the classical 4-step Runge-Kutta method for solving the system of delay differential equations (5) is summarized as follows:

\[ K_0 = h \left( AX_i + B(X(t_i)) + C(X_{it}) \right) \]
\[ K_1 = h(A \left( X_i + \frac{K_0}{2} \right) + B \left( X_i + \frac{K_0}{2} \right) + C(\left( X_i + \frac{K_0}{2} \right) \right) \]
\[ K_2 = h(A \left( X_i + \frac{K_1}{2} \right) + B \left( X_i + \frac{K_1}{2} \right) + C \left( \left( X_i + \frac{K_0}{2} \right) \right) \]
\[ K_3 = h \left( A(X_i + K_2) + B(X_i + K_2) + C \left( \left( X_i + \frac{K_0}{2} \right) + K_2 \right) \right) \]

\[ X_{i+1} = X_i + \frac{1}{6}(K_0 + 2(K_1 + K_2) + K_3), \quad i = 0, 1, \ldots, N - 1, \]

with value \( X(t) = X_0 \) for all \( t \leq t_0 \).

5 Fuzzy Delay Predator-Prey System

In many cases, model (5) presented in Section 4 cannot be solved analytically and therefore I need a numerical scheme to approximate the exact solution. I consider the following delay predator-prey system:
\[
\begin{align*}
\dot{x}(t) &= x(1 - x) - cyx \\
\dot{y}(t) &= -dy + bce^{-d\tau}y(t - \tau)x(t - \tau).
\end{align*}
\] (6)

Then the system (6) is fuzzified using fuzzy symmetric triangular number where \(x(t), y(t)\) are non-negative fuzzy functions. By using the Theorem 1 the system (6) can be written as:

\[
\begin{align*}
\dot{x}_\alpha(t) &= a_1x_\alpha - x_\alpha^2 - cy_\alpha x_\alpha \\
\dot{y}_\alpha(t) &= a_1y_\alpha - y_\alpha^2 - cy_\alpha y_\alpha \\
\dot{y}_\alpha(t) &= -a_2y_\alpha + cbe^{-d\tau}x_\alpha(t - \tau)y_\alpha(t - \tau) \\
\dot{y}_\alpha(t) &= -a_2y_\alpha + cbe^{-d\tau}x_\alpha(t - \tau)y_\alpha(t - \tau).
\end{align*}
\] (7)

The derived system (7) is known as fuzzy delay predator-prey (FDPP) system. Consider the system (7) with parameters, \(b = 0.2, c = 0.5, d_\tau = 1\) with initial condition be

\[
\begin{align*}
(x_\alpha(0), x_\alpha(0)) &= (4 - (1 - \alpha)\sigma_1, 4 + (1 - \alpha)\sigma_1), \\
(y_\alpha(0), y_\alpha(0)) &= (3 - (1 - \alpha)\sigma_2, 3 + (1 - \alpha)\sigma_2).
\end{align*}
\]

For all \(t \leq 0\) over the time interval \([0,5]\).

Figure 1 depicts the solution of this system for \(\alpha = 0.9, N = 10000, \sigma_1 = 0.2, \sigma_2 = 0.5\)

![Figure 1: The evolution of predator over time.](image-url)
Figure 2: The evolution of prey over time

Figure 3: Fuzzy phase plane with direction field (h=0.1)
Conclusions

The present paper proposes a new fuzzification scheme for numerical solution of fuzzy delay predator-prey (FDPP) system. The numerical solution of such FDPP system are obtained using facile Runge-Kutta method in numerical example of FDPP system confirm the nature of numerical solutions. The numerical solution of fuzzy systems is found to be better than those of crisp structures. The excellent features of the results suggest that my novel method of fuzzification may constitute a basis for modeling biological populations and other related problems.

References


Received: May 30, 2017; Published: June 26, 2017