On a Hilbert-Type Integral Inequality in the Whole Plane with the Exponential Function

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Abstract

By introducing the exponential function as the interval variable and obtaining the weight functions, a new Hilbert-type integral inequality in the whole plane with a best possible constant factor expressed by the beta function is given. The equivalent forms and a few particular cases are considered.

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1 Introduction

In 1925, Hardy [1] gave the following Hardy-Hilbert’s integral inequality by introducing one pair of conjugate exponents \((p, q)\) \((\frac{1}{p} + \frac{1}{q} = 1)\): If \(p > 1, f(x), g(y) \geq 0, 0 < \int_{0}^{\infty} f^p(x)dx < \infty\) and \(0 < \int_{0}^{\infty} g^q(y)dy < \infty\), then

\[
\int_{0}^{\infty} \int_{0}^{\infty} \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_{0}^{\infty} f^p(x)dx\right)^{\frac{1}{p}} \left(\int_{0}^{\infty} g^q(y)dy\right)^{\frac{1}{q}},
\]

with the best possible constant factor \(\frac{\pi}{\sin(\pi/p)}\). For \(p = q = 2\), (1) reduces to the well known Hilbert’s integral inequality. (1) as well as Hilbert’s integral inequality are important in analysis and its applications (cf. [2], [3], [4]).

In recent years, a number of extensions of (1) were given by Yang [4]. Noticing that inequalities (1) is with a homogenous kernel of degree \(-1\), in 2009, a survey of the study of Hilbert-type inequalities with the homogeneous kernels of degree negative numbers and some parameters is given by [5]. Recently, some inequalities with the homogenous kernels of degree 0 and non-homogenous kernels have been studied by [6]-[10]. All of the above integral inequalities are built in the quarter plane.

In 2007, Yang [11] first gave a Hilbert-type integral inequality in the whole plane as follows:

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x)g(y)}{(1+e^{x+y})\lambda}dxdy < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left(\int_{-\infty}^{\infty} e^{-\lambda x}f^2(x)dx \int_{-\infty}^{\infty} e^{-\lambda y}g^2(y)dy\right)^{\frac{1}{2}},
\]

where, the constant factor \(B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)(\lambda > 0)\) is the best possible, and

\[
B(u, v) := \int_{0}^{\infty} \frac{t^{u+1}dt}{(1+t)^{u+v}} = \int_{0}^{1} \frac{t^{v-1}dt}{(1-t)^{1-u}} (u, v > 0)
\]

is the beta function (cf. [12]). Since then, He et al. [13]-[17] also provided some Hilbert-type integral inequalities in the whole plane.

In this paper, by introducing the exponential function as the interval variable and obtaining the weight functions, a new Hilbert-type integral inequality in the whole plane with the best possible constant factor

\[
k_{\beta}(\lambda) := 2 \left| B(1 - \lambda - \beta, \frac{\lambda}{2}) - B(1 - \lambda - \beta, \frac{\lambda}{2} + \beta) \right|,
\]

similar to (2) is given as follows:

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|e^{\beta x} - e^{\beta y}|}{e^{x} - e^{y}\lambda+\beta} f(x)g(y)dxdy < k_{\beta}(\lambda) \left(\int_{-\infty}^{\infty} e^{-\lambda x}f^2(x)dx \int_{-\infty}^{\infty} e^{-\lambda y}g^2(y)dy\right)^{\frac{1}{2}},
\]
On a Hilbert-type integral inequality

\((\beta \in (-1, 0) \cup (0, 1), \max\{0, -2\beta\} < \lambda < 1 - \beta)\). Moreover, a general form of (4) with multi-parameters and the equivalent forms are considered.

## 2 Equivalent inequalities

**Definition 1.** Suppose that \(\mathbb{R} = (-\infty, \infty), a, b \in \mathbb{R} \setminus \{0\}, \beta \in (-1, 0) \cup (0, 1), \lambda_1, \lambda_2 > \max\{0, -\beta\}, \lambda_1 + \lambda_2 = \lambda < 1 - \beta\). We define weight functions \(\omega(\lambda_1, y)\) and \(\varpi(\lambda_2, x)\) as follows:

\[
\omega(\lambda_1, y) = e^{b\lambda_1 y} \int_{-\infty}^{\infty} \frac{|e^{\beta(ax+by)} - 1|}{|e^{ax+by} - 1|^{1+\beta}} e^{a\lambda_1 x} dx \quad (y \in \mathbb{R}),
\]

\[
\varpi(\lambda_1, x) = e^{a\lambda_1 x} \int_{-\infty}^{\infty} \frac{|e^{\beta(ax+by)} - 1|}{|e^{ax+by} - 1|^{1+\beta}} e^{b\lambda_1 y} dy \quad (x \in \mathbb{R}).
\]

**Lemma 1.** As regards the assumptions of Definition 1, we set

\[
k(\lambda_1) := \left| \sum_{i=1}^{2} (B(1 - \lambda - \beta, \lambda_i) - B(1 - \lambda - \beta, \lambda_i + \beta)) \right|.
\]

Then for \(y, x \in \mathbb{R}\), we have

\[
\omega(\lambda_1, y) = \frac{1}{|a|} k(\lambda_1) \in \mathbb{R}_+ = (0, \infty),
\]

\[
\varpi(\lambda_1, x) = \frac{1}{|b|} k(\lambda_1) \in \mathbb{R}_+.
\]

**Proof.** Setting \(u = e^{ax+by}\) in (5) and (6), it follows that

\[
\omega(\lambda_1, y) = e^{b\lambda_1 y} \int_{-\infty}^{\infty} \frac{|u^\beta - 1|}{|u - 1|^{1+\beta}} e^{a\lambda_1 (\ln u - by)} \frac{1}{|a| u} du
\]

\[
= \frac{1}{|a|} \int_{0}^{\infty} \frac{|u^\beta - 1|}{|u - 1|^{1+\beta}} u^{\lambda_1 - 1} du,
\]

\[
\varpi(\lambda_1, x) = e^{a\lambda_1 x} \int_{-\infty}^{\infty} \frac{|u^\beta - 1|}{|u - 1|^{1+\beta}} e^{a\lambda_1 (\ln u - ax)} \frac{1}{|b u} du
\]

\[
= \frac{1}{|b|} \int_{0}^{\infty} \frac{|u^\beta - 1|}{|u - 1|^{1+\beta}} u^{\lambda_1 - 1} du.
\]

For \(-1 < \beta < 0\), in view of (3), we find

\[
0 < \int_{0}^{\infty} \frac{|u^\beta - 1|}{|u - 1|^{1+\beta}} u^{\lambda_1 - 1} du
\]

\[
= \int_{0}^{1} \frac{u^{\lambda_1 + 1} - u^{\lambda_1 - 1}}{(1 - u)^{1+\beta}} du + \int_{1}^{\infty} \frac{u^{\lambda_1 - 1} - u^{\beta + \lambda_1 - 1}}{(u - 1)^{1+\beta}} du
\]
for \(0 < \beta < 1\), by (3), we still find

\[
0 < \int_0^\infty \frac{|u^\beta - 1|}{|u - 1|^{\lambda + \beta}} u^{\lambda_1 - 1} du
= \int_0^1 \frac{u^{\lambda_1 - 1} - u^{\lambda_1 + \beta - 1}}{(1 - u)^{\lambda + \beta}} du + \int_1^\infty \frac{u^{\lambda_1 + \beta - 1} - u^{\lambda_1 - 1}}{(u - 1)^{\lambda + \beta}} du
= \int_0^1 \frac{u^{\lambda_1 - 1} - u^{\lambda_1 + \beta - 1}}{(1 - u)^{1 - (1 - \lambda - \beta)}} du + \int_0^1 \frac{v^{\lambda_1 - 1} - v^{\lambda_1 + \beta - 1}}{(1 - v)^{1 - (1 - \lambda - \beta)}} dv
= \sum_{i=1}^2 (B(1 - \lambda - \beta, \lambda_i) - B(1 - \lambda - \beta, \lambda_i + \beta)) < \infty.
\]

Hence, we have (8) and (9). \(\square\)

**Theorem 1.** Suppose that \(p > 1, \frac{1}{p} + \frac{1}{q} = 1, a, b \in \mathbb{R} \setminus \{0\}, \beta \in (-1, 0) \cup (0, 1), \lambda_1, \lambda_2 > \max\{0, -\beta\}, \lambda_1 + \lambda_2 = \lambda < 1 - \beta, k(\lambda_1) \in \mathbb{R}_+\) is indicated by (7), \(f(x), g(y)\) are non-negative measurable functions in \((-\infty, \infty)\), satisfying

\[
0 < \int_{-\infty}^\infty \left(\frac{f(x)}{e^{a\lambda_1 x}}\right)^p dx < \infty \quad \text{and} \quad 0 < \int_{-\infty}^\infty \left(\frac{g(y)}{e^{b\lambda_1 y}}\right)^q dy < \infty.
\]

We have the following equivalent inequalities:

\[
I := \int_{-\infty}^\infty \int_{-\infty}^\infty \left|\frac{e^{\beta(ax+by)} - 1}{e^{ax+by} - 1}\right|^{\lambda + \beta} f(x)g(y) dx dy
< \frac{k(\lambda_1)}{|b|^{\frac{1}{p}} |a|^{\frac{1}{q}}} \left[ \int_{-\infty}^\infty \left(\frac{f(x)}{e^{a\lambda_1 x}}\right)^p dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^\infty \left(\frac{g(y)}{e^{b\lambda_1 y}}\right)^q dy \right]^{\frac{1}{q}}, \quad (10)
\]

\[
J := \left\{ \int_{-\infty}^\infty e^{b\lambda_1 y} \left[ \int_{-\infty}^\infty \left|\frac{e^{\beta(ax+by)} - 1}{e^{ax+by} - 1}\right|^{\lambda + \beta} f(x) dx \right]^p dy \right\}^{\frac{1}{p}}
< \frac{k(\lambda_1)}{|b|^{\frac{1}{p}} |a|^{\frac{1}{q}}} \left[ \int_{-\infty}^\infty \left(\frac{f(x)}{e^{a\lambda_1 x}}\right)^p dx \right]^{\frac{1}{p}}. \quad (11)
\]
Proof. By Hölder’s inequality with weight (cf. [18]) and (5), we have

\[
\int_{-\infty}^{\infty} \frac{|e^{\beta(ax+by)} - 1|}{e^{ax+by} - 1} f(x) dx \\
= \int_{-\infty}^{\infty} \frac{|e^{\beta(ax+by)} - 1|}{|e^{ax+by} - 1|^{\lambda+\beta}} \left( \frac{e^{b\lambda_1 y/p}}{e^{a\lambda_1 x/q}} \right) \left( \frac{e^{a\lambda_1 x/q}}{e^{b\lambda_1 y/p}} \right) f(x) dx \\
\leq \left[ \int_{-\infty}^{\infty} \frac{|e^{\beta(ax+by)} - 1|}{e^{ax+by} - 1} e^{b\lambda_1 y} f^p(x) dx \right]^{\frac{1}{p}} \\
\times \left[ \int_{-\infty}^{\infty} \frac{|e^{\beta(ax+by)} - 1|}{e^{ax+by} - 1} e^{a\lambda_1 x} \frac{1}{e^{(p-1)a\lambda_1 x}} dx \right]^{\frac{1}{q}}.
\] (12)

If (12) takes the form of equality for a \( y \in (-\infty, \infty) \), then there exists constants \( A \) and \( B \), such that they are not all zero, and

\[
A \frac{e^{b\lambda_1 y}}{e^{(p-1)a\lambda_1 x}} f^p(x) = B \frac{e^{a\lambda_1 x}}{e^{(q-1)b\lambda_1 y}} \ a.e. \ in \ (-\infty, \infty).
\]

We suppose that \( A \neq 0 \) (otherwise, \( B = A = 0 \)). Then it follows that

\[
\left( \frac{f(x)}{e^{a\lambda_1 x}} \right)^p = \frac{B e^{-q\lambda_1 y}}{A} \ a.e. \ in \ (-\infty, \infty),
\]

which contradicts the fact that \( 0 < \int_{-\infty}^{\infty} \left( \frac{f(x)}{e^{a\lambda_1 x}} \right)^p dx < \infty \).

Then by (9), Fubini theorem (cf. [19]) and (6), we have

\[
J < \left( \frac{k(\lambda_1)}{|a|} \right)^\frac{1}{q} \left\{ \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \frac{|e^{\beta(ax+by)} - 1|}{e^{ax+by} - 1} \frac{e^{b\lambda_1 y} f^p(x)}{e^{(p-1)a\lambda_1 x}} dx \right| dy \right\}^{\frac{1}{p}} \\
= \left( \frac{k(\lambda_1)}{|a|} \right)^\frac{1}{q} \left[ \int_{-\infty}^{\infty} \omega(\lambda_1, x) \left( \frac{f(x)}{e^{a\lambda_1 x}} \right)^p dx \right]^{\frac{1}{p}}.
\] (13)

Hence, in view of (9), inequality (11) follows.

By Hölder’s inequality (cf. [18]), we still find

\[
I = \int_{-\infty}^{\infty} \frac{e^{b\lambda_1 y}}{e^{(p-1)a\lambda_1 x}} f^p(x) dx \left( e^{-b\lambda_1 y} g(y) \right) dy \\
\leq J \left[ \int_{-\infty}^{\infty} \left( \frac{g(y)}{e^{b\lambda_1 y}} \right)^q dy \right]^{\frac{1}{q}}.
\] (14)
Then by (11), we have (10). On the other hand, suppose that (10) is valid. Setting
\[
g(y) := e^{\beta y} \left[ \int_{-\infty}^{\infty} \frac{|e^{\beta(x+by)} - 1|}{|e^{ax+by} - 1|^{\lambda+\beta}} f(x) \, dx \right]^{p-1}, \quad y \in \mathbb{R},
\] (15)
it follows that \( J = \int_{-\infty}^{\infty} \left( \frac{g(y)}{e^{\lambda y}} \right)^q \, dy \). By (13), we have \( J < \infty \). If \( J = 0 \), then (11) is trivially valid; if \( 0 < J < \infty \), then by (10), we obtain
\[
\int_{-\infty}^{\infty} \left( \frac{g(y)}{e^{\lambda y}} \right)^q \, dy = J^p = I
\]
\[
< \frac{k(\lambda_1)}{|b|^{1/p} |a|^{1/q}} \left[ \int_{-\infty}^{\infty} \left( \frac{f(x)}{e^{\lambda_1 x}} \right)^p \, dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} \left( \frac{g(y)}{e^{\lambda_1 y}} \right)^q \, dy \right]^{\frac{1}{q}},
\] (16)
\[
J = \left[ \int_{-\infty}^{\infty} \left( \frac{g(y)}{e^{\lambda_1 y}} \right)^q \, dy \right]^{\frac{1}{p}} < \frac{k(\lambda_1)}{|b|^{1/p} |a|^{1/q}} \left[ \int_{-\infty}^{\infty} \left( \frac{f(x)}{e^{\lambda_1 x}} \right)^p \, dx \right]^{\frac{1}{p}}.
\] (17)

Hence, we have (11), which is equivalent to (10). □

## 3 Best possible constant factor

**Theorem 2.** As regard the assumptions of Theorem 1, the constant factor
\[
k(\lambda_1) = \frac{k(\lambda_1)}{|b|^{1/p} |a|^{1/q}}
\] in (10) and (11) is the best possible.

In particular, for \( a = b = 1 \) in (10) and (11), we have the following equivalent inequalities with the non-homogeneous kernel and a best possible constant factor \( k(\lambda_1) \):
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|e^{\beta(x+y)} - 1|}{|e^{ax+y} - 1|^{\lambda+\beta}} f(x) g(y) \, dx \, dy
\]
\[
< k(\lambda_1) \left[ \int_{-\infty}^{\infty} \left( \frac{f(x)}{e^{\lambda_1 x}} \right)^p \, dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} \left( \frac{g(y)}{e^{\lambda_1 y}} \right)^q \, dy \right]^{\frac{1}{q}},
\] (18)
\[
\left\{ \int_{-\infty}^{\infty} \left( \frac{f(x)}{e^{\lambda_1 x}} \right)^p \, dx \right\}^{\frac{1}{p}}
\]
\[
< k(\lambda_1) \left[ \int_{-\infty}^{\infty} \left( \frac{f(x)}{e^{\lambda_1 x}} \right)^p \, dx \right]^{\frac{1}{p}}.
\] (19)

**Proof.** We set \( E_a = \{ x \in \mathbb{R}; ax \geq 0 \} \), wherefrom \( E_{(-b)} = \{ x \in \mathbb{R}; (-b)x \geq 0 \} = \{ y \in \mathbb{R}; by \leq 0 \} \). For any \( n \in \mathbb{N} \), we define functions \( \tilde{f}_n(x) \) and \( \tilde{g}_n(y) \) as
follows:

\[
\tilde{f}_n(x) := \begin{cases} 
e^{a(\lambda_1 - \frac{1}{pm})x}, & x \in E_a, \\ 0, & x \in \mathbb{R} \setminus E_a \end{cases}, \quad \tilde{g}_n(y) := \begin{cases} 
e^{b(\lambda_1 + \frac{1}{pm})y}, & y \in E_{(b)}, \\ 0, & y \in \mathbb{R} \setminus E_{(b)} \end{cases}.
\]

Then we obtain

\[
\tilde{J}_n := \left[ \int_{-\infty}^{\infty} \left( \frac{\tilde{f}_n(x)}{e^{a\lambda_1 x}} \right)^p \ dx \right]^{1/p} \left[ \int_{-\infty}^{\infty} \left( \frac{\tilde{g}_n(y)}{e^{b\lambda_1 y}} \right)^q \ dy \right]^{1/q}
= \left[ \int_{E_a} \left( \frac{e^{a(\lambda_1 - \frac{1}{pm})x}}{e^{a\lambda_1 x}} \right)^p \ dx \right]^{1/p} \left[ \int_{E_{(b)}} \left( \frac{e^{b(\lambda_1 + \frac{1}{pm})y}}{e^{b\lambda_1 y}} \right)^q \ dy \right]^{1/q}
= \left( \int_{E_a} e^{-\frac{a}{n}x} \ dx \right)^{1/p} \left( \int_{E_{(b)}} e^{\frac{b}{n}y} \ dy \right)^{1/q}
= \left( \frac{1}{|a|} \int_{0}^{\infty} e^{-\frac{a}{n}u} \ du \right)^{1/p} \left( \frac{1}{|b|} \int_{-\infty}^{0} e^{\frac{b}{n}v} \ dv \right)^{1/q} = \frac{n}{|a|^{1/p}|b|^{1/q}},
\]

\[
\tilde{I}_n := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|e^{\beta(ax+by)} - 1|}{|e^{ax+by} - 1|^{\lambda+\beta}} \tilde{f}_n(x) \tilde{g}_n(y) \ dx \ dy
= \int_{E_{(b)}} e^{b(\lambda_1 + \frac{1}{pm})y} \left[ \int_{E_a} \frac{|e^{\beta(ax+by)} - 1|}{|e^{ax+by} - 1|^{\lambda+\beta}} e^{a(\lambda_1 - \frac{1}{pm})x} \ dx \right] \ dy.
\]

Setting \( u = e^{ax+by} \) in the above integral, we obtain

\[
\tilde{I}_n = \frac{1}{|a|} \int_{E_{(b)}} e^{b(\lambda_1 + \frac{1}{pm})y} \left[ \int_{e^{by}}^{\infty} \frac{|u^\beta - 1|}{u^{\lambda+\beta} u^{(\lambda_1 - \frac{1}{pm})(\ln u - by)}} \ du \right] \ dy
= \frac{1}{|a|} \int_{E_{(b)}} e^{\frac{b}{n}y} \left[ \int_{e^{by}}^{\infty} \frac{|u^\beta - 1|}{u - 1} u^{(\lambda_1 - \frac{1}{pm})^{-1}} \ du \right] \ dy
= \frac{1}{|ab|} \int_{-\infty}^{0} e^{\frac{v}{n}} \left[ \int_{e^{vy}}^{\infty} \frac{|u^\beta - 1|}{u - 1} u^{(\lambda_1 - \frac{1}{pm})^{-1}} \ du \right] \ dv
= \frac{1}{|ab|} \left[ \int_{-\infty}^{0} e^{\frac{v}{n}} \int_{e^{vy}}^{1} \frac{|u^\beta - 1|}{u - 1} u^{(\lambda_1 - \frac{1}{pm})^{-1}} \ du \ dv + \int_{0}^{\infty} e^{\frac{v}{n}} \int_{1}^{\infty} \frac{|u^\beta - 1|}{u - 1} u^{(\lambda_1 - \frac{1}{pm})^{-1}} \ du \ dv \right]
\]
Remark 1. For \( a = 1, b = -1 \) in (11) and (12), replacing \( e^{\lambda y} g(y) \) by \( g(y) \), we obtain \( 0 < \int_{-\infty}^{\infty} \left( \frac{g(y)}{e^{\lambda_2 y}} \right)^{\frac{q}{p}} dy < \infty \), and the following equivalent inequalities with the homogeneous kernel and the best possible constant factor \( k(\lambda_1) \):

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{|e^{\beta x} - e^{\beta y}|}{|e^{x} - e^{y}|^{1+\beta}} f(x) g(y) dx dy < k(\lambda_1) \left[ \int_{-\infty}^{\infty} \left( \frac{f(x)}{e^{\lambda_1 x}} \right)^{p} dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} \left( \frac{g(y)}{e^{\lambda_2 y}} \right)^{q} dy \right]^{\frac{1}{q}},
\]

(21)

\[
\left\{ \int_{-\infty}^{\infty} e^{p\lambda_2 y} \left[ \int_{-\infty}^{\infty} \frac{|e^{\beta x} - e^{\beta y}|}{|e^{x} - e^{y}|^{1+\beta}} f(x) dx \right] dy \right\}^{\frac{1}{p}} < k(\lambda_1) \left[ \int_{-\infty}^{\infty} \left( \frac{f(x)}{e^{\lambda_1 x}} \right)^{p} dx \right]^{\frac{1}{p}}.
\]

(22)
In particular, for $p = q = 2, \lambda_1 = \lambda_2 = \frac{\lambda}{2}$ in (21), we obtain (4).

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