Some Strong Limit Theorems of Markov Chains in Markovian Environments Indexed by Cayley’s Trees

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Abstract

We prove a strong limit theorem on a function of four variables for Markov chains indexed by a Cayley’s tree in a nonhomogeneous Markovian environment. As corollaries, we also obtain some limit theorems for frequency of ordered couples of random variables in some particular sets.

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1. Introduction

A graph which is connected and doesn’t contain any circuits is called a tree $T$. A vertex can be chosen as root 0 and then vertices following is easy to be numbered. The unique path connecting two different vertices $\alpha$ and $t$ is denoted by $\alpha \overline{t}$. The distance $d(\alpha, t)$ is the number of edges contained in the $\alpha \overline{t}$. $d(0, t)$ can be simplified as $|t|$. $|t| = n$ is equivalent to $t \in L_n$ and $|t| \leq n$ is equivalent to $t \in T(n)$. That a vertex $\beta$ is on the $\overline{0t}$ can be simplified as $\beta \leq t$. A vertex $\gamma$ is denoted by $1_t$ if $\gamma \leq t$ and $d(\gamma, t) = 1$. $s \land t$ expresses a vertex

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that satisfying $s \land t \leq s$, $s \land t \leq t$ and $|s \land t|$ get its maximum. The degree of a vertex is defined to be the number of edges connecting it. A infinite $T$ which every vertex has degree $d + 1$ is called Cayley’s tree and is denoted by $T_{C,d}$. For convenience, we replace $T_{C,d}$ by $T$.

Cogburn [1,2] and Nawrotzki [3,4] gave a general definition of Markov chains in random environments (MCRE). Li et. al.[5] proved a strong limit theorem on a function of four variables for Markov chains in Markovian Environments. Tree-indexed Markov chains in random environments is an emerging research topic. Shi and Yang [6] gave the definition of tree-indexed Markov chains in random environments and proved the existence of them and the equivalent conditions between tree-indexed Markov chains in Markovian environments and tree-indexed Markov chains. Huang [7] proved some limit properties of the harmonic mean of a random transition probability for finite Markov chains indexed by homogeneous trees in a nonhomogeneous Markovian environments with finite state space. Motivated by the work above, in this paper we focus on strong limit properties of Markov chains in Markovian environments indexed by Cayley’s trees. The definition of Markov chains in Markovian environments indexed by Cayley’s trees will be given below.

**Definition 1.** Let $T$ be a Cayley’s tree and $\{X_t, t \in T\}$ be a stochastic process defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which takes values in the finite state space $\mathcal{X}$. Let

$$p = \{p(x), x \in \mathcal{X}\}$$

be a distribution on $\mathcal{X}$ and

$$P_t = (p_t(y|x)), x, y \in \mathcal{X},$$

transition probability matrices on $\mathcal{X}^2$. If, for any vertex $t$,

$$\mathbb{P}(X_t = y|X_{1_t} = x, X_s = x_s, s \land t \leq 1_t)$$

$$= \mathbb{P}(X_t = y|X_{1_t} = x) = p_t(y|x), \forall x, y \in \mathcal{X}$$

(1.3)

$$\mathbb{P}(X_0 = x) = p(x), \forall x \in \mathcal{X}$$

(1.4)

then $\{X_t, t \in T\}$ will be called $\mathcal{X}$-valued nonhomogeneous Markov chains indexed by a Cayley’s tree with initial distribution (1.1) and transition probability matrices (1.2).

**Definition 2.** Let $T$ be a Cayley’s tree, $\Theta$ a finite state space and $\{\xi_t, t \in T\}$ a $\Theta$-valued random field indexed by $T$. if, for any vertex $t \in T$,

$$\mathbb{P}(X_t = y|X_{1_t} = x, X_s = x_s, s \land t \leq 1_t; \xi_t, t \in T)$$

$$= P(y|x, \xi_{1_t}), a.s.,$$

(1.5)

$$\mathbb{P}(X_0 = x|\xi_t, t \in T) = P(x|\xi_0), \forall x \in \mathcal{X}$$

(1.6)
then we call \( \{X_t, t \in T\} \) a Markov chain indexed by tree \( T \) in a random environment \( \{\xi_t, t \in T\} \). The \( \xi_t \)s are called the environmental process or control process indexed by tree \( T \). The probability of going from \( x \) to \( y \) in one step in the \( \theta \)th environment is denoted by \( P(\theta; x, y) \).

In this paper we will assume that \( \{\xi_t, t \in T\} \) is a nonhomogeneous \( T \)-indexed Markov chain in finite state space \( \Theta \) with initial distribution \( \mu = \{\mu(\theta), \theta \in \Theta\} \). We also suppose that the one-step transition probability of going from \( \alpha \) to \( \beta \) for nonhomogeneous \( T \)-indexed Markov chain \( \{\xi_t, t \in T\} \) is \( K_t(\theta, \alpha) \). In this case, \( \{X_t, \xi_t, t \in T\} \) is a Markov chain indexed by \( T \) with initial distribution \( q = \{q(x, \theta), x \in \mathcal{X}, \theta \in \Theta\} \) and one-step transition probability on \( \mathcal{X} \times \Theta \) determined by

\[
\mathbb{P}(x, \theta; y, \alpha) = P(\theta; x, y)K_t(\theta, \alpha),
\]

and it will be called the bichain indexed by a tree.

**2. Main Results**

**Lemma 1.** Let \( \{W_n(\omega)\}_{n \in \mathbb{Z}_+} \) be a sequence of nonnegative random variables satisfying \( \mathbb{E}\{W_n\} \leq c, n \in \mathbb{Z}_+ \), then

\[
\limsup_{n \to \infty} (2n+1)^{-1} \log W_n(\omega) \leq 0 \quad \text{a.e..} \quad (2.1)
\]

The lemma can be verified easily.

**Theorem 1.** Let \( \{X_t, \xi_t, t \in T\} \) be a Markov chain indexed by \( T \) with initial distribution

\[
g_0 = \{g_0(x, \theta), x \in \mathcal{X}, \theta \in \Theta\}
\]

and one-step transition probability

\[
Q_t(x, \theta; y, \alpha), x, y \in \mathcal{X}, \theta, \alpha \in \Theta,
\]

and let \( \{g_t(\cdot, \cdot, \cdot, \cdot)\}_{t \in T} \) be a class of real functions on \( (\mathcal{X} \times \Theta)^2 \). If there exists a real number \( 0 < \gamma < \infty \) such that, for any \( y \in \mathcal{X}, \alpha \in \Theta \),

\[
\limsup_{n \to \infty} \max_{x, \theta} \frac{1}{|T(n)|} \sum_{t \in T(n) \setminus \{0\}} g_t^2(x, \theta, y, \alpha)Q_t(x, \theta; y, \alpha)e^{\gamma|g_t(x, \theta, y, \alpha)|} = C_\gamma(y, \alpha) < +\infty.
\]

(2.4)

Then

\[
\lim_{n \to \infty} \frac{1}{|T(n)|} \sum_{t \in T(n) \setminus \{0\}} \{g_t(X_t, \xi_t, X_t, \xi_t) - \mathbb{E}[g_t(X_t, \xi_t, X_t, \xi_t) | \mathcal{F}_{n-1}]\} = 0 \quad \text{a.e.} \quad (2.5)
\]

where \( \mathcal{F}_n = \sigma(T(n)), n \in \mathbb{Z}_+; \mathcal{F}_0 = \{\emptyset, \Omega\} \).
Proof. Let $s$ be a nonzero real number. Construct a likelihood ratio, for fixed $(y, \alpha) \in \mathcal{X} \times \Theta$,

$$
\Lambda_n(s, \omega) = \frac{e^{s \sum_{t \in T^{(n)} \setminus \{0\}} g_t(X_t, \xi_t, y, \alpha) 1_{(y)}(X_t) 1_{(\alpha)}(\xi_t)}}{\prod_{t \in T^{(n)} \setminus \{0\}} [1 + (e^{sg} - 1)Q_t(X_1, \xi_1; y, \alpha)]}, \quad n = 1, 2, \ldots \quad (2.6)
$$

where $1_A(\cdot)$ is used throughout as the indicator function of set $A$.

For convenience, we replace $g_t(X_t, \xi_t, y, \alpha)$ and $Q_t(X_1, \xi_1, y, \alpha)$ by $g_t$ and $Q_t$ resp.. Notice that

$$
\limsup_{n \to \infty} \left( \log \Lambda_n(s, \omega) \right) \leq 0 \quad a.e., \text{ i.e. there exists a set } D(s; y, \alpha) \in \mathcal{F} \text{ with } \mathbb{P}(D(s; y, \alpha)) = 1
$$

and

$$
\mathbb{E} \Lambda_n(s, \omega) = 1.
$$

According to lemma 1 and $|T^{(n)}| = 2^{(n+1)} - 1$, we can conclude that $\limsup_{n \to \infty} \frac{\log \Lambda_n(s, \omega)}{|T^{(n)}|} \leq 0 \quad a.e., \text{ i.e. there exists a set } D(s; y, \alpha) \in \mathcal{F} \text{ with } \mathbb{P}(D(s; y, \alpha)) = 1
$$

and

$$
\limsup_{n \to \infty} \frac{\log \Lambda_n(s, \omega)}{|T^{(n)}|} \leq 0, \quad \omega \in D(s; y, \alpha). \quad (2.7)
$$
Notice that

\[
\frac{1}{|T(n)|} \log \Lambda_n(s, \omega) = \frac{s}{|T(n)|} \sum_{t \in T(n) \setminus \{0\}} g_t 1_{(y)}(X_t) 1_{(\alpha)}(\xi_t) - \frac{1}{|T(n)|} \sum_{t \in T(n) \setminus \{0\}} \log[1 + (e^{sg_t} - 1)Q_t].
\]

(2.8)

We have by Eq. (2.7) and Eq. (2.8)

\[
\limsup_{n \to \infty} \left\{ \frac{s}{|T(n)|} \sum_{t \in T(n) \setminus \{0\}} g_t 1_{(y)}(X_t) 1_{(\alpha)}(\xi_t) - \frac{1}{|T(n)|} \sum_{t \in T(n) \setminus \{0\}} \log[1 + (e^{sg_t} - 1)Q_t] \right\} \leq 0, \ \omega \in D(s; y, \alpha).
\]

(2.9)

Let \(s > 0\), Eq. (2.9) can be changed to

\[
\limsup_{n \to \infty} \frac{1}{|T(n)|} \left\{ \sum_{t \in T(n) \setminus \{0\}} g_t 1_{(y)}(X_t) 1_{(\alpha)}(\xi_t) - \frac{1}{s} \sum_{t \in T(n) \setminus \{0\}} \log[1 + (e^{sg_t} - 1)Q_t] \right\} \leq 0, \ \omega \in D(s; y, \alpha).
\]

(2.10)

Inequalities \(\log(1+x) \leq x, (x > -1), e^x - 1 - x \leq x^2 e^{|x|}\) are applied in Equation above, then

\[
\limsup_{n \to \infty} \frac{1}{|T(n)|} \left[ \sum_{t \in T(n) \setminus \{0\}} g_t 1_{(y)}(X_t) 1_{(\alpha)}(\xi_t) - \sum_{t \in T(n) \setminus \{0\}} g_t Q_t \right] \leq \limsup_{n \to \infty} \frac{1}{|T(n)|} \sum_{t \in T(n) \setminus \{0\}} \left\{ \frac{1}{s} \log[1 + (e^{sg_t} - 1)Q_t] - g_t Q_t \right\} \leq \limsup_{n \to \infty} \frac{1}{|T(n)|} \sum_{t \in T(n) \setminus \{0\}} \frac{Q_t}{s} (e^{sg_t} - 1 - sg_t) \leq s \limsup_{n \to \infty} \frac{1}{|T(n)|} \sum_{t \in T(n) \setminus \{0\}} g_t^2 Q_t e^{sg_t}, \ \omega \in D(s; y, \alpha).
\]

(2.10)

If \(0 < s < \gamma\), we have by Eqs. (2.4) and (2.10) that

\[
\limsup_{n \to \infty} \frac{1}{|T(n)|} \sum_{t \in T(n) \setminus \{0\}} g_t 1_{(y)}(X_t) 1_{(\alpha)}(\xi_t) - g_t Q_t \leq sC_\gamma(y, \alpha), \ \omega \in D(s; y, \alpha).
\]

(2.11)
Choosing $s_m \in (0, \gamma), m = 1, 2, \cdots$, such that $s_m \to 0$ (as $m \to 0$) and let $D^{(1)}(y, \alpha) = \bigcap_{m=1}^{\infty} D(s_m; y, \alpha)$, then by Eq. (2.11), for any $m$, we have

$$\limsup_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)} \setminus \{0\}} [g_t 1_{\{y\}}(X_t) 1_{\{\alpha\}}(\xi_t) - g_t Q_t] \leq s_m C_\gamma(y, \alpha), \ \omega \in D^{(1)}(y, \alpha).$$

(2.12)

Since $s_m \to 0$ (as $m \to \infty$), we have by Eq. (2.12)

$$\limsup_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)} \setminus \{0\}} [g_t 1_{\{y\}}(X_t) 1_{\{\alpha\}}(\xi_t) - g_t Q_t] \leq 0, \ \omega \in D^{(1)}(y, \alpha).$$

(2.13)

If $-\gamma < s < 0$, Eq. (2.9) can be changed to

$$\liminf_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)} \setminus \{0\}} [g_t 1_{\{y\}}(X_t) 1_{\{\alpha\}}(\xi_t) - g_t Q_t] \geq s \liminf_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)} \setminus \{0\}} g_t^2 Q_t e^{sg_t} \geq s C_\gamma(y, \alpha), \ \omega \in D(s; y, \alpha).$$

(2.14)

Choosing $s_m \in (-\gamma, 0), m = 1, 2, \cdots$, such that $s_m \to 0$ (as $m \to 0$), and let $D^{(2)}(y, \alpha) = \bigcap_{m=1}^{\infty} D(s_m; y, \alpha)$, then by Eq. (2.14), for any $m$, we have

$$\liminf_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)} \setminus \{0\}} [g_t 1_{\{y\}}(X_t) 1_{\{\alpha\}}(\xi_t) - g_t Q_t] \geq 0, \ \omega \in D^{(2)}(y, \alpha).$$

(2.15)

Let $D(y, \alpha) = D^{(1)}(y, \alpha) \cap D^{(2)}(y, \alpha)$. We have by Eqs. (2.13) and (2.14)

$$\lim_{n \to \infty} \frac{1}{|T^{(n)}|} \sum_{t \in T^{(n)} \setminus \{0\}} g_t[1_{\{y\}}(X_t) 1_{\{\alpha\}}(\xi_t) - Q_t] = 0, \ \omega \in D(y, \alpha).$$

(2.16)

Let $D = \bigcap_{y \in \mathcal{Y}} \bigcap_{\alpha \in \Theta} D(y, \alpha)$, since

$$g_t(X_{1t}, \xi_{1t}, X_t, \xi_t) - \mathbb{E}[g_t(X_{1t}, \xi_{1t}, X_t, \xi_t)|X_{1t}, \xi_{1t}] = \sum_{y \in \mathcal{Y}} \sum_{\alpha \in \Theta} g_t(X_{1t}, \xi_{1t}, y, \alpha) 1_{\{y\}}(X_t) 1_{\{\alpha\}}(\xi_t) - \sum_{y \in \mathcal{Y}} \sum_{\alpha \in \Theta} g_t(X_{1t}, \xi_{1t}, y, \alpha) Q_t(X_{1t}, \xi_{1t}; y, \alpha)$$

$$= \sum_{y \in \mathcal{Y}} \sum_{\alpha \in \Theta} g_t(X_{1t}, \xi_{1t}, y, \alpha) [1_{\{y\}}(X_t) 1_{\{\alpha\}}(\xi_t) - Q_t(X_{1t}, \xi_{1t}; y, \alpha)], \ \omega \in D. \quad (2.17)$$

and $\mathbb{P}(D) = 1$, Eq. (2.5) follows from Eqs. (2.16) and (2.17) immediately. □
Corollary 1. Let \( \{X_t, \xi_t; t \in T\} \) be defined as in theorem 1. Let \( T_n(y, \alpha) \) be the number of times that \((y, \alpha)\) appears in the set \( \{(X_t, \xi_t); t \in T^{(n)} \backslash \{0\}\} \), i.e. \( T_n(y, \alpha) = \sum_{t \in T^{(n)} \backslash \{0\}} 1_{\{y\}}(X_t)1_{\{\alpha\}}(\xi_t) \), then

\[
\lim_{n \to \infty} \frac{1}{T^{(n)}} \left[ T_n(y, \alpha) - \sum_{t \in T^{(n)} \backslash \{0\}} Q_t(X_t, \xi_t; y, \alpha) \right] = 0 \text{ a.e. and in } \mathcal{L}_1.
\]

Proof. For fixed \( y \in \mathcal{X}, \alpha \in \Theta \), put \( g_t(x, \theta, z, \omega) = 1_{\{y\}}(z)1_{\{\alpha\}}(\omega), t \in T^{(n)} \backslash \{0\} \) in Theorem 1, then we have

\[
\frac{1}{T^{(n)}} \sum_{t \in T^{(n)} \backslash \{0\}} \{g_t(X_t, \xi_t, X_t, \xi_t) - \sum_{z \in \mathcal{X}} \sum_{\omega \in \Theta} g_t(X_t, \xi_t, z, \omega)Q_t(X_t, \xi_t; z, \omega)\}
\]

\[
= \frac{1}{T^{(n)}} \sum_{t \in T^{(n)} \backslash \{0\}} \{1_{\{y\}}(X_t)1_{\{\alpha\}}(\xi_t) - \sum_{z \in \mathcal{X}} \sum_{\omega \in \Theta} 1_{\{y\}}(z)1_{\{\alpha\}}(\omega)Q_t(X_t, \xi_t; z, \omega)\}
\]

\[
= \frac{T_n(y, \alpha)}{T^{(n)}} - \frac{1}{T^{(n)}} \sum_{t \in T^{(n)} \backslash \{0\}} Q_t(X_t, \xi_t; y, \alpha).
\]

It is easy to see know that \( g_t(x, \theta, z, \omega) = 1_{\{y\}}(z)1_{\{\alpha\}}(\omega) \) satisfy condition (2.4) and \( |T^{(n)}|^{-1} \left[ T_n(y, \alpha) - \sum_{t \in T^{(n)} \backslash \{0\}} Q_t(X_t, \xi_t; y, \alpha) \right] \) are uniformly integrable. Hence, it follows from Theorem 1 directly.

Similarly, we can obtain the following corollaries.

Corollary 2. Let \( T_n(\alpha) \) be the number of times that \( \alpha \) appears in the set \( \{\xi_t; t \in T^{(n)} \backslash \{0\}\} \). Then

\[
\lim_{n \to \infty} \frac{1}{T^{(n)}} \left[ T_n(\alpha) - \sum_{t \in T^{(n)} \backslash \{0\}} K_t(\xi_t, \alpha) \right] = 0 \text{ a.e. and in } \mathcal{L}_1.
\]

Corollary 3. Let \( T_n(\theta, \alpha) \) be the number of times that \((\theta, \alpha)\) appears in the set \( \{(\xi_t, \xi_t); t \in T^{(n)} \backslash \{0\}\} \). Then

\[
\lim_{n \to \infty} \frac{1}{T^{(n)}} \left[ T_n(\theta, \alpha) - \sum_{t \in T^{(n)} \backslash \{0\}} 1_{\{\theta\}}(\xi_t)K_t(\theta, \alpha) \right] = 0 \text{ a.e. and in } \mathcal{L}_1.
\]

Corollary 4. Let \( T_n(y) \) be the number of times that \( y \) appears in the set \( \{X_t; t \in T^{(n)} \backslash \{0\}\} \). Then

\[
\lim_{n \to \infty} \frac{1}{T^{(n)}} \left[ T_n(y) - \sum_{t \in T^{(n)} \backslash \{0\}} P(y \mid X_t, \xi_t) \right] = 0 \text{ a.e. and in } \mathcal{L}_1.
\]

Corollary 5. Let \( T_n(x, y) \) be the number of times that \((x, y)\) appears in the set \( \{(X_t, X_t); t \in T^{(n)} \backslash \{0\}\} \). Then

\[
\lim_{n \to \infty} \frac{1}{T^{(n)}} \left[ T_n(x, y) - \sum_{t \in T^{(n)} \backslash \{0\}} 1_{\{x\}}(X_t)P(y \mid x, \xi_t) \right] = 0 \text{ a.e. and in } \mathcal{L}_1.
\]
Corollary 6. Let \( T_n(x, \theta, \alpha) \) be the number of times that \((x, \theta, \alpha)\) appears in the set \(\{(X_{1t}, \xi_{1t}, \xi_t); t \in T^{(n)}\}\). Then
\[
\lim_{n \to \infty} \frac{1}{T^{(n)}} |T_n(x, \theta, \alpha) - \sum_{t \in T^{(n)} \setminus \{0\}} 1_{\{x\}}(X_{1t}) 1_{\{\xi_{1t}\}} K_t(\theta, \alpha)| = 0 \text{ a.e. and in } \mathcal{L}_1.
\]

Corollary 7. Let \( T_n(\theta, y, \alpha) \) be the number of times that \((\theta, y, \alpha)\) appears in the set \(\{\xi_{1t}, X_t, \xi_t); t \in T^{(n)}\}\). Then
\[
\lim_{n \to \infty} \frac{1}{T^{(n)}} |T_n(\theta, y, \alpha) - \sum_{t \in T^{(n)} \setminus \{0\}} 1_{\{\xi_{1t}\}} Q_t(\theta, y, \alpha)| = 0 \text{ a.e. and in } \mathcal{L}_1.
\]

Corollary 8. Let \( T_n(x, y, \alpha) \) be the number of times that \((x, y, \alpha)\) appears in the set \(\{(X_{1t}, X_t, \xi_t); t \in T^{(n)}\}\). Then
\[
\lim_{n \to \infty} \frac{1}{T^{(n)}} |T_n(x, y, \alpha) - \sum_{t \in T^{(n)} \setminus \{0\}} 1_{\{x\}}(X_{1t}) Q_t(x, \xi_{1t}, y, \alpha)| = 0 \text{ a.e. and in } \mathcal{L}_1.
\]

Corollary 9. Let \( T_n(x, \theta, y) \) be the number of times that \((x, \theta, y)\) appears in the set \(\{(X_{1t}, \xi_{1t}, X_t); t \in T^{(n)}\}\). Then
\[
\lim_{n \to \infty} \frac{1}{T^{(n)}} |T_n(x, \theta, y) - \sum_{t \in T^{(n)} \setminus \{0\}} 1_{\{x\}}(X_{1t}) 1_{\{\xi_{1t}\}} P(\theta, y, x)| = 0 \text{ a.e. and in } \mathcal{L}_1.
\]

Corollary 10. Let \( T_n(x, \theta; y, \alpha) \) be the number of times that \((x, \theta; y, \alpha)\) appears in the set \(\{(X_{1t}, \xi_{1t}, X_t, \xi_t); t \in T^{(n)} \setminus \{0\}\}\). Then
\[
\lim_{n \to \infty} \frac{1}{T^{(n)}} |T_n(x, \theta; y, \alpha) - \sum_{t \in T^{(n)} \setminus \{0\}} 1_{\{x\}}(X_{1t}) 1_{\{\xi_{1t}\}} Q_t(x, \theta; y, \alpha)| = 0 \text{ a.e. and in } \mathcal{L}_1.
\]

References


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