Coupled Fixed Point Theorems with Monotone Property in Soft b-Metric Space

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Abstract

In the present paper first we define Soft Metric Space, b-Soft Metric Space, Monotone property and α-Monotone property and then we establish two results. In the first result we prove coupled soft fixed point theorem in ordered soft b-metric space with monotone property. In the second theorem establish coupled soft co-incidence fixed point theorem for mapping satisfying generalized contractive conditions with α-monotone property in an ordered soft b-metric space. We also prove the validity of theorem 2.2 with example.

Mathematical Subject Classification: 47H10, 54E50

Keywords: Soft point, soft metric space, soft contractive mapping, fixed point theorem, monotone property, α-monotone property

1 Introduction and Preliminaries

Metric fixed point theory is an essential part of mathematics because of its
applications in different areas like variational and linear inequalities, improvement and approximation theory. Ali et al. [9], Agraval et al. ([14], [15], [16]), Pathak et al. [7] and many authors (see [1], [2], [3], [8], [13]) established fixed theorems in different spaces like partially ordered metric space, Metric space, Manger space, Banach space, generalized Banach space etc.

A concept of soft theory as new mathematical tool for dealing with uncertainties is discussed in 1999 by Molodtsov [6]. A soft set is a collection of approximate descriptions of an object this theory has rich potential applications. On soft set theory many structures contributed by many researchers (see [5], [10], [12]). Shabir and Naz [11] were studied about soft topological spaces. In these studies, the concept of soft point is explained by different techniques. Later a different concept of soft point introduced by Das and Samanta ([18-19]) using a different notion of soft metric space and investigated some basic properties of these spaces. Recently Wadkar et.al [4] proved fixed point results in soft metric space.

In this paper we first prove soft coupled fixed point theorem in ordered soft b-metric space using monotone property for contractive condition. In the next theorem we prove the coupled soft coincidence fixed point theorem for mapping satisfying generalized contractive conditions with \( \alpha \)-monotone property in an ordered soft b-metric space.

**Definition 1.1:** Let \( X \) and \( E \) are respectively an initial inverse set and a parameter set. A soft set over \( X \) is pair denoted by \((Y,E)\) if and only if \( Y \) is a mapping from \( E \) into the set of all subsets of the set \( X \). That is \( Y : E \rightarrow P(X) \), where \( P(X) \) is the power set of \( X \).

**Definition 1.2:** The intersection of two soft sets \((A,Y),(B,Z)\) over \( X \) is a soft set denoted by \((I,C)\) over \( X \) and is given by \((I,C) = (A \cap B) \cap (Y \cap Z)\) where \( C = A \cap B \) and \( \forall \varepsilon \in C, I(\varepsilon) = Y(\varepsilon) \cap Z(\varepsilon) \).

**Definition 1.3:** The union of two soft sets \((A,Y),(B,Z)\) over \( X \) is the soft set \((I,C)\) where \( C = A \cup B \) and for all \( k \) in \( C \),

\[
I(k) = \begin{cases} 
Y(k), & \text{if } k \text{ is an element of } A \setminus B; \\
Z(k), & \text{if } k \text{ is an element of } B \setminus A; \\
Y(k) \cup Z(k), & \text{if } k \text{ is an element of } A \cap B.
\end{cases}
\]

This is denoted by \((I,C) = (A,Y) \cup (B,Z)\).

**Definition 1.4:** A soft set \((Y,A)\) over \( X \) is said to be a null soft set denoted by \( \Phi \) if \( Y(k) = \emptyset \), for all \( k \) in \( A \).

**Definition 1.5:** For all \( k \in A \), if \( Y(k) = X \) then \((Y,A)\) is called an absolute soft set over \( X \).

**Definition 1.6:** The difference of two soft sets \((F,E)\) and \((G,E)\) over \( X \) is a soft set \((H,E)\) over \( X \), denoted by \((F,E) \setminus (G,E)\), is defined as \( H(x) = F(x) \setminus G(x), \forall x \in E \).

**Definition 1.7:** The complement of soft set \((Y,A)\) is denoted by \((Y,A)^{c}\) and is defined as \((Y,A)^{c} = (Y^{c},A)\), where \( Y^{c} : A \rightarrow P(X) \) is a mapping given by \( Y^{c}(\beta) = X - Y(\beta) \), for all \( \beta \).
Definition 1.8: Let R be the set of real numbers and B(R) be the collection of all nonempty bounded subsets of R and E taken as a set of parameters. Then a mapping \( Y : E \to B(R) \) is called a soft real set, and it is denoted by \((Y, E)\).

Definition 1.9: For two soft real numbers \( \tilde{u} \) and \( \tilde{v} \) the following condition holds:

i. \( \tilde{u} \preceq \tilde{v} \) if \( \tilde{u}(s) \preceq \tilde{v}(s) \), for all \( s \in E \);

ii. \( \tilde{u} \succeq \tilde{v} \) if \( \tilde{u}(s) \succeq \tilde{v}(s) \), for all \( s \in E \);

iii. \( \tilde{u} < \tilde{v} \) if \( \tilde{u}(s) < \tilde{v}(s) \), for all \( s \in E \);

iv. \( \tilde{u} > \tilde{v} \) if \( \tilde{u}(s) > \tilde{v}(s) \), for all \( s \in E \).

Definition 1.10: A soft set \((P, E)\) over \( X \) is said to be a soft point if there is exactly one \( s \in E \) such that \( P(s) = \{x\} \) for some \( x \in X \) and \( \forall s' \in E / \{s\} \). It will be denoted by \( x \).

Definition 1.11: Two soft points \( \tilde{x}_i \) and \( \tilde{y}_j \) are said to be equal if \( i = j \) and \( P(i) = P(j) \) i.e. \( x = y \). Hence \( \tilde{x}_i \neq \tilde{y}_j \) \( \iff \) \( x \neq y \) or \( i \neq j \).

Definition 1.12: A mapping \( \tilde{\rho} : SP(\tilde{X}) \times SP(\tilde{X}) \to R(E)^* \) be soft mapping on \( \tilde{X} \) such that:

- \( SM1. \) for all \( \tilde{x}_s, \tilde{y}_s \in \tilde{X} \), \( \tilde{\rho}(\tilde{x}_s, \tilde{y}_s) \geq 0 \)
- \( SM2. \) \( \tilde{\rho}(\tilde{x}_s, \tilde{y}_s) = 0 \), if and only if \( \tilde{x}_s = \tilde{y}_s \)
- \( SM3. \) for all \( \tilde{x}_s, \tilde{y}_s \in \tilde{X} \), \( \tilde{\rho}(\tilde{x}_s, \tilde{y}_s) = \tilde{\rho}(\tilde{y}_s, \tilde{x}_s) \)
- \( SM4. \) for all \( \tilde{x}_s, \tilde{y}_s, \tilde{z}_s \in \tilde{X} \), \( \tilde{\rho}(\tilde{x}_s, \tilde{z}_s) \leq \tilde{\rho}(\tilde{x}_s, \tilde{y}_s) + \tilde{\rho}(\tilde{y}_s, \tilde{z}_s) \).

The soft set \( \tilde{X} \) with a soft metric \( \tilde{\rho} \) on \( \tilde{X} \) is called a soft metric space and denoted by \((\tilde{X}, \tilde{\rho}, E)\).

Definition 1.13: Let us consider a soft metric \( (\tilde{X}, \tilde{\rho}, E) \) and \( \alpha \) be a non negative soft real number. The soft open ball with center at \( \tilde{x}_s \) and radius \( \tilde{\alpha} \) is given by

\[
B(\tilde{x}_s, \tilde{\alpha}) = \{ \tilde{y}_s \in \tilde{X} : \tilde{\rho}(\tilde{x}_s, \tilde{y}_s) \leq \tilde{\alpha} \} \subseteq SP(\tilde{X});
\]

and the soft closed ball with center at \( \tilde{x}_s \) and radius \( \tilde{\alpha} \) is given by

\[
\overline{B}(\tilde{x}_s, \tilde{\alpha}) = \{ \tilde{y}_s \in \tilde{X} : \tilde{\rho}(\tilde{x}_s, \tilde{y}_s) \leq \tilde{\alpha} \} \subseteq SP(\tilde{X}).
\]

Definition 1.14: A sequence \( \{\tilde{x}_n\} \) of soft points in soft metric space \( (\tilde{X}, \tilde{\rho}, E) \) is said to be convergent in \( (\tilde{X}, \tilde{\rho}, E) \) if there is a soft point \( \tilde{y}_\mu \in \tilde{X} \) such that \( \tilde{\rho}(\tilde{x}_n, \tilde{y}_\mu) \to 0 \) as \( n \to \infty \). That is for every \( \tilde{\varepsilon} > 0 \), selected arbitrary, there is a natural number \( N = N(\tilde{\varepsilon}) \) such that \( \tilde{\rho}(\tilde{x}_n, \tilde{y}_\mu) < \tilde{\varepsilon} \), whenever \( n > N \).

Definition 1.15: Let \( (\tilde{X}, \tilde{\rho}, E) \) be a soft metric space, then the sequence \( \{\tilde{x}_{ij}\} \) of soft points in \( (\tilde{X}, \tilde{\rho}, E) \) is said to be a Cauchy sequence in \( \tilde{X} \), if corresponding to every \( \tilde{\varepsilon} \geq 0 \), \( \exists m \in N \) such that \( \tilde{\rho}(\tilde{x}_i, \tilde{x}_j) \leq \tilde{\varepsilon}, \forall i, j \geq m \), i.e. \( \tilde{\rho}(\tilde{x}_i, \tilde{x}_j) \to 0 \) as \( i, j \to \infty \).
**Definition 1.16:** The soft metric space \((\tilde{X}, \tilde{\rho}, E)\) is called complete, if every Cauchy sequence in \(\tilde{X}\) converges to some point of \(\tilde{X}\).

**Definition 1.17:** Let \((\tilde{X}, \tilde{\rho}, E)\) be a soft metric space. A function \((f, \phi): (\tilde{X}, \tilde{\rho}, E) \to (\tilde{X}, \tilde{\rho}, E)\) is called a soft contraction mapping if there exist, a soft real number \(\alpha \in R, 0 \leq \alpha < 1\) such that for every point \(\tilde{x}_\lambda, \tilde{y}_\mu \in SP(X)\) we have
\[
\tilde{\rho}(f(\phi)(\tilde{x}_\lambda), (f, \phi)(\tilde{y}_\mu)) \leq \alpha \tilde{\rho}(\tilde{x}_\lambda, \tilde{y}_\mu)
\]

**Definition 1.18:** Let \((\tilde{X}, \leq)\) be a partially ordered soft set. \(S: X \times X \to X\) be a self map. One can say that S has the mixed monotone property if \(S(x_\lambda, y_\mu)\) is monotone non-decreasing in \(x_\lambda\) and is monotone non-increasing in \(y_\mu\). That is for all \(x_\lambda^1, x_\lambda^2\), \(y_\mu^1 \leq y_\mu^2\) \(\Rightarrow S(x_\lambda^1, y_\mu^1) \leq S(x_\lambda^2, y_\mu^2)\), for any \(y_\mu \in X\) and for all \(y_\mu^1, y_\mu^2\), \(y_\mu^1 \geq y_\mu^2\) \(\Rightarrow S(x_\lambda, y_\mu^1) \geq S(x_\lambda, y_\mu^2)\) for any \(x_\lambda \in X\).

**Definition 1.19:** Consider partially ordered soft set \((\tilde{X}, \leq)\). Let \(S: X \times X \to X\) and \(\alpha: X \to X\) be two mappings. We say that S has mixed \(\alpha\)-monotone property if S is monotone \(\alpha\) non-decreasing in its first argument and is monotone \(\alpha\) non-increasing in its second argument.

i.e. for all \(x_\lambda^1, x_\lambda^2 \in X\), \(\alpha x_\lambda^1 \leq x_\lambda^2 \Rightarrow S(x_\lambda^1, y_\mu) \leq S(x_\lambda^2, y_\mu)\) for any \(y_\mu \in X\) and
for all \(y_\mu^1, y_\mu^2 \in X\), \(\alpha y_\mu^1 \geq y_\mu^2 \Rightarrow S(x_\lambda, y_\mu^1) \geq S(x_\lambda, y_\mu^2)\) for any \(x_\lambda \in X\).

**Definition 1.20:** An element \((x_\lambda, y_\mu) \in X \times X\) is said to be a coupled fixed point of mapping \(S: X \times X \to X\) if \(S(x_\lambda, y_\mu) = x_\lambda\) and \(S(y_\mu, x_\lambda) = y_\mu\).

**Definition 1.21:** An element \((x_\lambda, y_\mu) \in X \times X\) is called
a) a coupled coincidence soft point of mapping \(S: X \times X \to X\) and \(\alpha: X \to X\) if \(S(x_\lambda, y_\mu) = x_\lambda\) and \(S(y_\mu, x_\lambda) = \alpha y_\mu\).

b) a common coupled soft fixed point of mapping \(S: X \times X \to X\) and \(\alpha: X \to X\) if \(x_\lambda = \alpha x_\lambda = S(x_\lambda, y_\mu)\) and \(y_\mu = \alpha y_\mu = S(y_\mu, x_\lambda)\).

**Definition 1.22:** For a non empty set X, the mappings \(S: X \times X \to X\) and \(\alpha: X \to X\) are said to be commutative, if for all \(x_\lambda, y_\mu \in X\) we have \(\alpha(S(x_\lambda, y_\mu)) = S(\alpha x_\lambda, \alpha y_\mu)\).

**Definition 1.23:** Let X be a non empty soft set and \(s \geq 1\) be a given real number. A function \(\tilde{\rho}: X \times X \to R^+\) be a function such that:

**SM1.** for all \(\tilde{x}_\lambda, \tilde{y}_\mu \in \tilde{X}\), \(\tilde{\rho}(\tilde{x}_\lambda, \tilde{y}_\mu) \leq \tilde{b}\);

**SM2.** \(\tilde{\rho}(\tilde{x}_\lambda, \tilde{y}_\mu) = \tilde{b}\), if and only if \(\tilde{x}_\lambda = \tilde{y}_\mu\);

**SM3.** for all \(\tilde{x}_\lambda, \tilde{y}_\mu \in \tilde{X}\), \(\tilde{\rho}(\tilde{x}_\lambda, \tilde{y}_\mu) = \tilde{\rho}(\tilde{y}_\mu, \tilde{x}_\lambda)\);

**SM4.** for all \(\tilde{x}_\lambda, \tilde{y}_\mu, \tilde{z}_\mu \in \tilde{X}\), \(\tilde{\rho}(\tilde{x}_\lambda, \tilde{y}_\mu, \tilde{z}_\mu) \leq s(\tilde{\rho}(\tilde{x}_\lambda, \tilde{y}_\mu) + \tilde{\rho}(\tilde{y}_\mu, \tilde{z}_\mu)).\)
Then soft set $\tilde{X}$ with a soft metric $\tilde{\rho}$ on $\tilde{X}$ is called a soft b-metric space and denoted by $(\tilde{X}, \tilde{\rho}, E)$.

2 Main Results

Let $(X, \leq)$ be a partially ordered soft set and $\tilde{\rho}$ be a soft metric on X such that $(X, \tilde{\rho}, E)$ is a complete soft b-metric space. Consider a product $(X, \tilde{\rho}, E) \times (X, \tilde{\rho}, E)$ with the following partial order. For all $(x_{\lambda}, y_{\mu}) \in (X, \tilde{\rho}, E) \times (X, \tilde{\rho}, E)$, we have $(u_{\lambda}, v_{\mu}) \leq (x_{\lambda}, y_{\mu}) \iff x_{\lambda} \geq u_{\lambda}, y_{\mu} \leq v_{\mu}$.

**Theorem 2.1:** Let $(X, \tilde{\rho}, E, \leq)$ be a partially ordered complete soft b- metric space and let S be a continuous mapping having the mixed monotone property such that for all $(x_{\lambda}, y_{\mu})$ in $(X, \tilde{\rho}, E)$ and $a \in \left[0, \frac{1}{s}\right]$ with $s \geq 1$ we have

$$\tilde{\rho}(S(x_{\lambda}, y_{\mu}), S(u_{\lambda}, v_{\mu})) \leq a \max \left\{ \tilde{\rho}(x_{\lambda}, S(u_{\lambda}, v_{\mu})), \tilde{\rho}(y_{\mu}, S(v_{\lambda}, u_{\mu})) \right\}$$

Then S has a coupled soft fixed point in soft b- metric space $(X, \tilde{\rho}, E)$.

**Proof:** Choose $x_{\lambda}^0, y_{\mu}^0 \in (X, \tilde{\rho}, E) \times (X, \tilde{\rho}, E)$ and set $x_{\lambda}^1 = S\left(x_{\lambda}^0, y_{\mu}^0\right)$; $y_{\mu}^1 = S\left(y_{\mu}^0, x_{\lambda}^0\right)$

In general $x_{\lambda}^{n+1} = S\left(x_{\lambda}^n, y_{\mu}^n\right)$ and $y_{\mu}^{n+1} = S\left(y_{\mu}^n, x_{\lambda}^n\right)$

with $x_{\lambda}^0 \leq S\left(x_{\lambda}^0, y_{\mu}^0\right) = x_{\lambda}^1$ and $y_{\mu}^0 \geq S\left(y_{\mu}^0, x_{\lambda}^0\right) = y_{\mu}^1$

By iterative process above, we have $x_{\lambda}^2 = S\left(x_{\lambda}^1, y_{\mu}^1\right)$ and $y_{\mu}^2 = S\left(y_{\mu}^1, x_{\lambda}^1\right)$

Therefore $x_{\lambda}^2 = S\left(x_{\lambda}^1, y_{\mu}^1\right) = S\left(S\left(x_{\lambda}^0, y_{\mu}^0\right), S\left(y_{\mu}^0, x_{\lambda}^1\right)\right) = S^2\left(x_{\lambda}^0, y_{\mu}^0\right)$ and $y_{\mu}^2 = S\left(y_{\mu}^1, x_{\lambda}^1\right) = S\left(S\left(y_{\mu}^0, x_{\lambda}^0\right), S\left(x_{\lambda}^0, y_{\mu}^0\right)\right) = S^2\left(y_{\mu}^0, x_{\lambda}^0\right)$

Due to the mixed monotone property of S we obtain

$x_{\lambda}^2 = S^2\left(x_{\lambda}^0, y_{\mu}^0\right) = S\left(x_{\lambda}^1, y_{\mu}^1\right) \geq S\left(x_{\lambda}^0, y_{\mu}^0\right) = x_{\lambda}^1$ and $y_{\mu}^2 = S^2\left(y_{\mu}^0, x_{\lambda}^0\right) = S\left(y_{\mu}^1, x_{\lambda}^1\right) \leq S\left(y_{\mu}^0, x_{\lambda}^0\right) = y_{\mu}^1$

In general we have for $n \in N$

$x_{\lambda}^{n+1} = S^{n+1}\left(x_{\lambda}^0, y_{\mu}^0\right) = S^{n}\left(x_{\lambda}^1, y_{\mu}^1\right)$ and $y_{\mu}^{n+1} = S^{n+1}\left(y_{\mu}^0, x_{\lambda}^0\right) = S^{n}\left(y_{\mu}^1, x_{\lambda}^1\right)$

It is obvious that

$x_{\lambda}^0 \leq S\left(x_{\lambda}^0, y_{\mu}^0\right) = x_{\lambda}^1 \leq S^2\left(x_{\lambda}^0, y_{\mu}^0\right) = x_{\lambda}^2 \leq \ldots \leq S^s\left(x_{\lambda}^0, y_{\mu}^0\right) = x_{\lambda}^s \leq \ldots$ (4)

$y_{\mu}^0 \geq S\left(y_{\mu}^0, x_{\lambda}^0\right) = y_{\mu}^1 \geq S\left(y_{\mu}^1, x_{\lambda}^1\right) = y_{\mu}^2 \geq \ldots \geq S^s\left(y_{\mu}^0, x_{\lambda}^0\right) = y_{\mu}^s \geq \ldots$

Thus by mathematical induction principal we have for $n \in N$

$x_{\lambda}^0 \leq x_{\lambda}^1 \leq x_{\lambda}^2 \leq \ldots \leq x_{\lambda}^n \leq x_{\lambda}^{n+1} \ldots \ldots \ldots \ldots$ and $y_{\mu}^0 \geq y_{\mu}^1 \geq y_{\mu}^2 \geq \ldots \geq y_{\mu}^n \geq y_{\mu}^{n+1} \ldots \ldots \ldots \ldots$
Thus we have by condition (1) that
\[
\hat{\rho}(x_{n_{1}}^{n_{1}}, x_{n_{2}}^{n_{2}}) = \hat{\rho}\left(S(x_{n_{1}}^{n_{1}}, y_{\mu_{1}}^{n_{1}}), S(x_{n_{2}}^{n_{2}}, y_{\mu_{2}}^{n_{2}})\right) \\
\leq a \max \left\{ \hat{\rho}\left(x_{n_{1}}^{n_{1}}, S(x_{n_{1}}^{n_{1}}, y_{\mu_{1}}^{n_{1}})\right) + \hat{\rho}\left(y_{\mu_{1}}^{n_{1}}, S(y_{\mu_{1}}^{n_{1}}, x_{n_{1}}^{n_{1}})\right), \hat{\rho}\left(x_{n_{2}}^{n_{2}}, S(x_{n_{2}}^{n_{2}}, y_{\mu_{2}}^{n_{2}})\right) + \hat{\rho}\left(y_{\mu_{2}}^{n_{2}}, S(y_{\mu_{2}}^{n_{2}}, x_{n_{2}}^{n_{2}})\right) \right\} \\
\leq a \max \left\{ \hat{\rho}\left(x_{n_{1}}^{n_{1}}, x_{n_{1}}^{n_{1}}\right) + \hat{\rho}\left(y_{\mu_{1}}^{n_{1}}, y_{\mu_{1}}^{n_{1}}\right), \hat{\rho}\left(x_{n_{2}}^{n_{2}}, x_{n_{2}}^{n_{2}}\right) + \hat{\rho}\left(y_{\mu_{2}}^{n_{2}}, y_{\mu_{2}}^{n_{2}}\right) \right\}.
\]

(5)

Similarly since \( y_{n_{1}}^{n_{1}} \geq y_{n_{1}}^{n_{1}} \) and \( x_{n_{1}}^{n_{1}} \leq x_{n_{1}}^{n_{1}} \), from (1) we have
\[
\hat{\rho}(y_{n_{1}}^{n_{1}}, y_{n_{1}}^{n_{1}}) \leq a \hat{\rho}(x_{n_{1}}^{n_{1}}, y_{n_{1}}^{n_{1}}) + \hat{\rho}(y_{n_{1}}^{n_{1}}, y_{n_{1}}^{n_{1}}) \\
\] (6)

Adding (5) & (6) we get
\[
\hat{\rho}(x_{n_{1}}^{n_{1}}, x_{n_{1}}^{n_{1}}) + \hat{\rho}(y_{n_{1}}^{n_{1}}, y_{n_{1}}^{n_{1}}) \leq a \left[ \hat{\rho}(x_{n_{1}}^{n_{1}}, x_{n_{1}}^{n_{1}}) + \hat{\rho}(y_{n_{1}}^{n_{1}}, y_{n_{1}}^{n_{1}}) \right] + a \left[ \hat{\rho}(x_{n_{1}}^{n_{1}}, x_{n_{1}}^{n_{1}}) + \hat{\rho}(y_{n_{1}}^{n_{1}}, y_{n_{1}}^{n_{1}}) \right] \\
\leq 2a \left[ s \hat{\rho}(x_{n_{1}}^{n_{1}}, x_{n_{1}}^{n_{1}}) + s \hat{\rho}(x_{n_{1}}^{n_{1}}, y_{n_{1}}^{n_{1}}) \right] + 2a \left[ s \hat{\rho}(x_{n_{1}}^{n_{1}}, x_{n_{1}}^{n_{1}}) + s \hat{\rho}(x_{n_{1}}^{n_{1}}, y_{n_{1}}^{n_{1}}) \right] \\
\leq 2as \left[ \hat{\rho}(x_{n_{1}}^{n_{1}}, x_{n_{1}}^{n_{1}}) + \hat{\rho}(y_{n_{1}}^{n_{1}}, y_{n_{1}}^{n_{1}}) \right] + 2as \left[ \hat{\rho}(x_{n_{1}}^{n_{1}}, x_{n_{1}}^{n_{1}}) + \hat{\rho}(y_{n_{1}}^{n_{1}}, y_{n_{1}}^{n_{1}}) \right] \\
= \frac{2as}{1-2as} \left[ \hat{\rho}(x_{n_{1}}^{n_{1}}, x_{n_{1}}^{n_{1}}) + \hat{\rho}(y_{n_{1}}^{n_{1}}, y_{n_{1}}^{n_{1}}) \right]
\] 

Adding these we get
\[
\hat{\rho}(x_{n_{1}}^{n_{1}}, x_{n_{1}}^{n_{1}}) + \hat{\rho}(y_{n_{1}}^{n_{1}}, y_{n_{1}}^{n_{1}}) \leq \frac{2as}{1-2as} \left[ \hat{\rho}(x_{n_{1}}^{n_{1}}, x_{n_{1}}^{n_{1}}) + \hat{\rho}(y_{n_{1}}^{n_{1}}, y_{n_{1}}^{n_{1}}) \right]
\]

Let us denote \( h = \frac{2as}{1-2as} \) and \( \hat{\rho}(x_{n_{1}}^{n_{1}}, x_{n_{1}}^{n_{1}}) + \hat{\rho}(y_{n_{1}}^{n_{1}}, y_{n_{1}}^{n_{1}}) \) by \( d_{n} \) then \( d_{n} \leq hd_{n-1} \).

Similarly it can be proved that \( d_{n-1} \leq h^{2}d_{n-2} \)

Therefore \( d_{n} \leq h^{n-1}d_{2} \leq \cdots \leq h^{n}d_{n} \).

This implies that \( \lim_{n \to \infty} d_{n} = 0 \). Thus \( \lim_{n \to \infty} d(x_{n_{1}}^{n_{1}}, x_{n_{1}}^{n_{1}}) = \lim_{n \to \infty} d(y_{n_{1}}^{n_{1}}, y_{n_{1}}^{n_{1}}) = 0 \)

For each \( m \geq n \), by (7) and repeat the application of triangle inequality that we obtain that
\[
\hat{\rho}(x_{n_{1}}^{n_{1}}, x_{n_{1}}^{n_{1}}) \leq s^{n-1} \hat{\rho}(x_{n_{1}}^{n_{1}}, x_{n_{1}}^{n_{1}}) + s^{n-2} \hat{\rho}(x_{n_{1}}^{n_{1}}, x_{n_{1}}^{n_{1}}) + \cdots + s^{n-m} \hat{\rho}(x_{n_{1}}^{n_{1}}, x_{n_{1}}^{n_{1}})
\]

Adding these we get
\[
\hat{\rho}(x_{n_{1}}^{n_{1}}, x_{n_{1}}^{n_{1}}) \leq s^{n} \hat{\rho}(x_{n_{1}}^{n_{1}}, x_{n_{1}}^{n_{1}}) + s^{n} \hat{\rho}(x_{n_{1}}^{n_{1}}, x_{n_{1}}^{n_{1}}) + \cdots + s^{n} \hat{\rho}(x_{n_{1}}^{n_{1}}, x_{n_{1}}^{n_{1}})
\]

Therefore \( \{x_{n_{1}}^{n_{1}}\} \) and \( \{y_{n_{1}}^{n_{1}}\} \) are Cauchy sequences. Since \((X, \hat{\rho}, E)\) is complete b-soft metric space. There exist \( x_{\lambda_{1}}, y_{\mu_{1}} \) in \((X, \hat{\rho}, E)\) such that \( \lim_{n \to \infty} x_{n_{1}}^{n_{1}} = x_{\lambda_{1}} \) and

\[
\lim_{n \to \infty} y_{n_{1}}^{n_{1}} = y_{\mu_{1}} \). Thus by taking limit \( n \to \infty \), in equation (4) we get
Coupled fixed point theorems with monotone property

\[ x_\lambda = \lim_{n \to \infty} x_{\lambda_n} = \lim_{n \to \infty} S \left( x_{\lambda_n-1}^{n-1}, y_{\mu_n-1}^{n-1} \right) = S \left( x_{\lambda_n}, y_{\mu_n} \right) \]
\[ y_{\mu_n} = \lim_{n \to \infty} y_{\mu_n} = \lim_{n \to \infty} S \left( x_{\lambda_n-1}^{n-1}, y_{\mu_n-1}^{n-1} \right) = S \left( y_{\mu_n}, x_{\lambda_n} \right) \]

Therefore \( x_\lambda = S(x_\lambda, y_{\mu_n}) \) & \( y_{\mu_n} = S(y_{\mu_n}, x_\lambda) \). Thus \( S \) has coupled soft fixed point in \( (X, \tilde{\rho}, E) \).

In the next theorem we prove coupled soft coincidence fixed point theorem for mapping satisfying generalized contractive conditions with \( \alpha \)-monotone property in an ordered soft b-metric space.

**Theorem 2.2**: Let \( ((X, \tilde{\rho}, E), \leq) \) be a partially ordered set and \( \tilde{\rho} : (X, \tilde{\rho}, E) \times (X, \tilde{\rho}, E) \to R \) be a soft b-metric defined on \( X \) with coefficient \( s \geq 1 \). Let \( \alpha : (X, \tilde{\rho}, E) \to (X, \tilde{\rho}, E) \) and \( S : (X, \tilde{\rho}, E) \times (X, \tilde{\rho}, E) \to (X, \tilde{\rho}, E) \) be two mappings such that

\[ \tilde{\rho} \left( S(x_\lambda, y_{\mu_n}), S(u_\alpha, v_{\beta_n}) \right) + \tilde{\rho} \left( S(y_{\mu_n}, x_\lambda), S(v_{\beta_n}, u_\alpha) \right) \leq k \left\{ \tilde{\rho}(\alpha x_\lambda, \alpha u_\alpha) + \tilde{\rho}(\alpha y_{\mu_n}, \alpha v_{\beta_n}) \right\} \]

for some \( k \in \left[ 0, \frac{1}{s} \right) \) and for all \( x_\lambda, y_{\mu_n}, u_\alpha, v_{\beta_n} \in (X, \tilde{\rho}, E) \) with \( \alpha x_\lambda \geq u_\alpha \) and \( \alpha y_{\mu_n} \leq v_{\beta_n} \). We further assume the following hypothesis

1. \( S(X \times X) \subseteq \alpha(X) \)
2. \( \alpha(X) \) is complete
3. \( \alpha \) is continuous and commute with \( S \)
4. \( S \) has the mixed \( \alpha \)-monotone property on \( X \) and
5. either \( S \) is continuous or \( X \) has the following property
   a. if a non decreasing sequence and \( \{ x_{\lambda_n} \} \to x_\lambda \) then \( \{ x_{\lambda_n} \} \leq x_\lambda \).
   b. if a non increasing sequence and \( \{ y_{\mu_n} \} \to y_{\beta_n} \) then \( \{ y_{\mu_n} \} \geq y_{\beta_n} \).

if there exist two elements \( x_0^0, y_0^0 \) in \((X, \tilde{\rho}, E)\) with \( \alpha x_0^0 \leq S(x_0^0, y_0^0) \) and \( \alpha y_0^0 \geq S(y_0^0, x_0^0) \), then \( S \) and \( \alpha \) have unique a coupled soft fixed point that is there exist a unique \( x_\lambda \in (X, \tilde{\rho}, E) \) such that \( x_\lambda = S(x_\lambda, x_\lambda) = \alpha x_\lambda \).

**Proof**: Let \( x_{\lambda_0}, y_{\mu_0} \in (X, \tilde{\rho}, E) \) be such that \( \alpha x_{\lambda_0} \leq S(x_{\lambda_0}, y_{\mu_0}) \) and \( \alpha y_{\mu_0} \geq S(y_{\mu_0}, x_{\lambda_0}) \). Since \( S(X \times X) \subseteq \alpha(X) \), we can choose \( x_{\lambda_1}^1, y_{\mu_1}^1 \in (X, \tilde{\rho}, E) \) such that \( \alpha x_{\lambda_1}^1 = S(x_{\lambda_0}, y_{\mu_0}) \) and \( \alpha y_{\mu_1}^1 = S(y_{\mu_0}, x_{\lambda_0}) \). Again since \( S(X \times X) \subseteq \alpha(X) \), we can choose \( x_{\lambda_2}^2, y_{\mu_2}^2 \in (X, \tilde{\rho}, E) \) such that \( \alpha x_{\lambda_2}^2 = S(x_{\lambda_1}^1, y_{\mu_1}^1) \) and \( \alpha y_{\mu_2}^2 = S(y_{\mu_1}^1, x_{\lambda_1}^1) \).

Continuing this process we can construct two sequences \( \{ x_{\lambda_n}^n \} \) and \( \{ y_{\mu_n}^n \} \) in \( X \) such that

\[ \alpha x_{\lambda_n}^{n+1} = S(x_{\lambda_n}^n, y_{\mu_n}^n) \]
\[ \alpha y_{\mu_n}^{n+1} = S(y_{\mu_n}^n, x_{\lambda_n}^n) \]

for all \( n \geq 0 \). Now we will prove that for all \( n \geq 0 \)

\[ \alpha x_{\lambda_n}^n \leq \alpha x_{\lambda_n}^{n+1} \]
\[ \alpha y^n_{n^{2}} \geq \alpha y^n_{n^{2}+1} \]  

(12)

We shall use the mathematical law of induction. Let \( n = 0 \). Since \( \alpha x_{1} \leq S(x_{0}, y_{1}) \), \( \alpha y_{n} \geq S(y_{n}, x_{0}) \) and \( \alpha x_{1} = S(x_{1}, y_{1}) \), \( \alpha y_{n} = S(y_{n}, x_{n}) \), we have \( \alpha x_{1} \leq \alpha x_{1} \) and \( \alpha y_{n} \geq \alpha y_{n} \) that is (11) and (12) holds for all \( n = 0 \). We assume that (11) and (12) holds for some \( n > 0 \). As \( S \) has the mixed monotone property and \( \alpha x_{2} \leq \alpha x_{2} \) and \( \alpha y_{n} \geq \alpha y_{n} \) we get

\[ \alpha x_{n^{2}+1} = S(x_{n^{2}+1}, y_{n^{2}+1}) \leq S(x_{n^{2}+1}, y_{n^{2}+1}) \text{ and } S(y_{n^{2}+1}, x_{n^{2}+1}) = \alpha y_{n^{2}+1}. \]

Also for the same reason we have

\[ \alpha x_{n^{2}+1} = S(x_{n^{2}+1}, y_{n^{2}+1}) \geq S(x_{n^{2}+1}, y_{n^{2}+1}) \text{ and } S(y_{n^{2}+1}, x_{n^{2}+1}) = \alpha y_{n^{2}+1}. \]

From (9) and (10) we obtain \( \alpha x_{n^{2}+1} \leq \alpha x_{n^{2}+2} \) and \( \alpha y_{n^{2}+1} \geq \alpha y_{n^{2}+2} \). Thus by mathematical induction we conclude that (11) and (12) holds for all \( n \geq 0 \). Continuing this process one can easily verify that

\[ \alpha x_{1} \leq \alpha x_{2} \leq \ldots \leq \alpha x_{n^{2}} \leq \alpha x_{n^{2}+1} \leq \ldots \]  

(13)

\[ y_{n} \geq \alpha y_{n} \geq \alpha y_{n} \geq \ldots \geq \alpha y_{n} \geq \alpha y_{n} \geq \ldots \]  

(14)

Now if \( \{x_{n^{2}}, y_{n^{2}}\} = \{x_{n^{2}}, y_{n^{2}}\} \), then \( S \) and \( \alpha \) have coupled soft coincidence point.

So assume \( \{x_{n^{2}}, y_{n^{2}}\} \neq \{x_{n^{2}}, y_{n^{2}}\} \) for all \( n \geq 0 \) i.e. we assume that either

\[ \alpha x_{n^{2}+1} = S(x_{n^{2}+1}, y_{n^{2}+1}) \neq \alpha x_{n} \text{ or } \alpha y_{n^{2}+1} = S(y_{n^{2}+1}, x_{n^{2}+1}) \neq \alpha y_{n}. \]

Again

\[ \rho(\alpha x_{n^{2}}, \alpha x_{n^{2}+1}) + \rho(\alpha y_{n^{2}}, \alpha y_{n^{2}+1}) = \rho(S(x_{n^{2}+1}, y_{n^{2}+1}), S(x_{n^{2}+1}, y_{n^{2}+1})) \]

\[ \leq k \left[ \rho(\alpha x_{n^{2}}, \alpha x_{n^{2}}) + \rho(\alpha y_{n^{2}}, \alpha y_{n^{2}}) \right] \]

Now let \( \rho(\alpha x_{n^{2}}, \alpha x_{n^{2}+1}) + \rho(\alpha y_{n^{2}}, \alpha y_{n^{2}+1}) = d_{n} \) then \( d_{n} \leq k d_{n-1} \)  

(15)

Again let \( m \) and \( n \) be two positive integer such that \( m > n \) then we can write

\[ \rho(\alpha x_{n^{2}}, \alpha x_{n^{3}+3}) \leq s \rho(\alpha x_{n^{2}}, \alpha x_{n^{3}+2}) + s \rho(\alpha x_{n^{2}}, \alpha x_{n^{3}+1}) + s \rho(\alpha x_{n^{2}}, \alpha x_{n^{3}+1}) + s \rho(\alpha x_{n^{2}}, \alpha x_{n^{3}+1}) + s \rho(\alpha x_{n^{2}}, \alpha x_{n^{3}+1}) + s \rho(\alpha x_{n^{2}}, \alpha x_{n^{3}+1}) + s \rho(\alpha x_{n^{2}}, \alpha x_{n^{3}+1}) + s \rho(\alpha x_{n^{2}}, \alpha x_{n^{3}+1}) \]

Similarly,

\[ \rho(\alpha y_{n^{2}}, \alpha y_{n^{3}+3}) \leq s \rho(\alpha y_{n^{2}}, \alpha y_{n^{3}+2}) + s \rho(\alpha y_{n^{2}}, \alpha y_{n^{3}+1}) + s \rho(\alpha y_{n^{2}}, \alpha y_{n^{3}+1}) + s \rho(\alpha y_{n^{2}}, \alpha y_{n^{3}+1}) \]

Therefore

\[ \rho(\alpha x_{n^{2}}, \alpha x_{n^{3}+3}) + \rho(\alpha y_{n^{2}}, \alpha y_{n^{3}+3}) \leq s d_{n} + s^{2} d_{n+1} + s^{3} d_{n+2} + \ldots + s^{m} d_{n} \leq k^{n} d_{0} + k^{n+1} d_{0} + \ldots + k^{m-n} d_{0} \]

\[ \leq k^{n} d_{0} \frac{1}{1-sk} \to 0; \text{ as } n \to \infty \]

Hence \( \{x_{n^{2}}\} \) and \( \{y_{n^{2}}\} \) are two Cauchy sequences in \( \alpha X \) and \( \alpha X \) is complete.
Thus there exist two soft point say \( x_\lambda, y_\mu \) in \( X \) such that \( \{ x^{n}_{\lambda_n} \} = x_\lambda = \xi \) and \( \{ y^{n}_{\mu_n} \} \rightarrow y_\mu \rightarrow \eta \) as \( n \rightarrow \infty \). Hence \( S \) is complete and so
\[
\alpha \left( x^{n+1}_{\lambda_n}, y^{n}_{\mu_n} \right) = \alpha \left( S \left( x^{n}_{\lambda_n}, y^{n}_{\mu_n} \right) \right) = S \left( \alpha x^{n}_{\lambda_n}, \alpha y^{n}_{\mu_n} \right) \quad (\because \text{S and } \alpha \text{ are commutative})
\]
\[
\Rightarrow \quad \alpha (\xi) = S(\xi, \eta) \quad (\because \text{S and } \alpha \text{ are continuous})
\]
Similarly we can show that \( \alpha (\eta) = S(\eta, \xi) \). Thus \((\eta, \xi)\) is point of coincidence for \( S \) and \( \alpha \). Again let \( 5b \) holds, by \((13)\) we get that \( \{ x^{n}_{\lambda_n} \} \) is a non decreasing sequence and \( x^{n}_{\lambda_n} \rightarrow \xi \) , therefore \( x^{n}_{\lambda_n} \leq \xi \) for all \( n \). Similarly by \((14)\) we get that \( \{ y^{n}_{\mu_n} \} \) is a non increasing sequence and \( y^{n}_{\mu_n} \rightarrow \eta \), so \( y^{n}_{\mu_n} \geq \eta \) for all \( n \), then
\[
\rho(\alpha(\xi), S(\xi, \eta)) \leq s \rho(\alpha(\xi), \alpha x^{n+1}_{\lambda_n}) + s \rho(\alpha x^{n+1}_{\lambda_n}, S(\xi, \eta)) + s \rho(\alpha y^{n+1}_{\mu_n}, S(\xi, \eta))
\]
\[
= s \rho(\alpha(\xi), \alpha x^{n+1}_{\lambda_n}) + s \rho(\alpha \left( S \left( x^{n}_{\lambda_n}, y^{n}_{\mu_n} \right) \right), S(\xi, \eta))
\]
\[
= s \rho(\alpha(\xi), \alpha x^{n+1}_{\lambda_n}) + s \rho(\alpha \left( S \left( x^{n}_{\lambda_n}, y^{n}_{\mu_n} \right) \right), S(\xi, \eta))
\]
\[
\leq \rho(\alpha(\xi), \alpha x^{n+1}_{\lambda_n}) + s \rho(\alpha \left( S \left( x^{n}_{\lambda_n}, y^{n}_{\mu_n} \right) \right), S(\xi, \eta)) + s \rho(\alpha \left( S \left( x^{n}_{\lambda_n}, y^{n}_{\mu_n} \right) \right), S(\xi, \eta))
\]
\[
\leq \rho(\alpha(\xi), \alpha x^{n+1}_{\lambda_n}) + sk \left\{ \rho(\alpha x^{n+1}_{\lambda_n}, \alpha \xi) + s \rho(\alpha y^{n+1}_{\mu_n}, \alpha \eta) \right\}
\]
\[
\Rightarrow \quad 2 \rho(\alpha(\xi), \alpha \eta) \leq 2k \rho(\alpha(\xi), \alpha \eta) \quad \Rightarrow \quad \rho(\alpha(\xi), \alpha \eta) \leq k \rho(\alpha(\xi), \alpha \eta)
\]
Since \( \alpha \) is continuous, \( \alpha x^{n}_{\lambda_n} \rightarrow \alpha \xi \) and \( \alpha y^{n}_{\mu_n} \rightarrow \alpha \eta \) and hence the right hand side of equation \((16)\) becomes zero as \( n \rightarrow \infty \). Thus \( \alpha(\xi) = S(\xi, \eta) \), similarly we can show that \( \alpha(\eta) = S(\eta, \xi) \). Again
\[
\rho(\alpha(\xi), \alpha \eta) + \rho(\alpha(\eta), \alpha \xi) = \rho(\alpha \left( S \left( \xi, \eta \right) \right), S(\xi, \eta)) + \rho(S(\eta, \xi), S(\xi, \eta)) \leq k \left\{ \rho(\alpha(\xi), \alpha \eta) + \rho(\eta, \alpha \xi) \right\}
\]
\[
\Rightarrow \quad 2 \rho(\alpha(\xi), \alpha \eta) \leq 2k \rho(\alpha(\xi), \alpha \eta) \quad \Rightarrow \quad \rho(\alpha(\xi), \alpha \eta) \leq k \rho(\alpha(\xi), \alpha \eta)
\]
Since \( k < \frac{1}{s} \); \( \rho(\alpha(\xi), \alpha \eta) = 0 \). Thus \( \alpha \xi = \alpha \eta \), Hence \( S(\xi, \eta) = \alpha(\xi) = \alpha(\eta) = S(\eta, \xi) \)

Finally
\[
\rho(\xi, \alpha \xi) \leq s \rho(\xi, \alpha x^{n+1}_{\lambda_n}) + s \rho(\alpha x^{n+1}_{\lambda_n}, \alpha \xi) \leq s \rho(\xi, \alpha x^{n+1}_{\lambda_n}) + s \rho(\alpha x^{n+1}_{\lambda_n}, \alpha \xi)
\]
In the same manner
\[
\rho(\eta, \alpha \eta) \leq s \rho(\eta, \alpha y^{n+1}_{\mu_n}) + s \rho(\alpha y^{n+1}_{\mu_n}, \alpha \eta) \leq s \rho(\eta, \alpha y^{n+1}_{\mu_n}) + s \rho(\alpha y^{n+1}_{\mu_n}, \alpha \eta)
\]
Therefore
\[
\rho(\xi, \alpha \xi) + \rho(\eta, \alpha \eta) \leq s \rho \left( \xi, \alpha x^{n+1}_{\lambda_n} \right) + s \rho \left( \eta, \alpha y^{n+1}_{\mu_n} \right)
\]
\[
= s \rho \left( S(x^{n}_{\lambda_n}, y^{n}_{\mu_n}), S(\xi, \eta) \right) + s \rho \left( S(y^{n}_{\mu_n}, x^{n}_{\lambda_n}), S(\eta, \xi) \right)
\]
\[
\leq s \rho \left( \xi, \alpha x^{n+1}_{\lambda_n} \right) + s \rho \left( \eta, \alpha y^{n+1}_{\mu_n} \right) + s \rho \left( \alpha y^{n}_{\mu_n}, \xi \right) + s \rho \left( \alpha y^{n}_{\mu_n}, \eta \right)
\]
Thus
\[
\left( 1 - k s \right) \left[ \rho(\xi, \alpha \xi) + \rho(\eta, \alpha \eta) \right] \leq s \rho \left( \xi, \alpha x^{n+1}_{\lambda_n} \right) + s \rho \left( \eta, \alpha y^{n+1}_{\mu_n} \right) + s \rho \left( \alpha y^{n}_{\mu_n}, \xi \right) + s \rho \left( \alpha y^{n}_{\mu_n}, \eta \right) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty
\]
Thus \( \rho(\xi, \alpha \xi) = 0 = \rho(\eta, \alpha \eta) \)
\[
\Rightarrow \quad \xi = \alpha \xi \quad \text{and} \quad \eta = \alpha \eta
\]
\[
\Rightarrow \quad \alpha x_{\lambda} = S(x_{\lambda}, x_{\lambda}) = x_{\lambda}
\]
This means that $S$ and $\alpha$ have a common soft fixed point.

**Example 2.1:** Let $(X, \rho, E) = R = X$ and we define $\hat{\rho} : X \times X \to R$ by $\hat{\rho}(x, y) = \left( \frac{|x - y| - y}{2} \right)^2$; for all $x, y \in (X, \rho, E)$. Then $(X, \hat{\rho}, E)$ is a soft b-metric space with the coefficient $s = 2^2$. Let us define $S : (X, \rho, E) \times (X, \rho, E) \to (X, \hat{\rho}, E)$ as $S(x, y) = \left( \frac{|x - y|}{10}, \text{ if } x \geq y, \right)$ for all $x, y \in (X, \hat{\rho}, E)$ and let $\alpha : (X, \hat{\rho}, E) \to (X, \hat{\rho}, E)$ be defined by $\alpha(x) = x$. Then $S(x, y) \leq \alpha(x)$. We shall show that for all $(x, y, z, \lambda, \mu, \rho) \in (X, \hat{\rho}, E)$

$$\hat{\rho}(S(x, y), S(u, v)) + \hat{\rho}(S(y, z), S(v, u)) \leq \frac{1}{2} \{ \hat{\rho}(\alpha x, \alpha u) + \hat{\rho}(\alpha y, \alpha v) \}.$$  

Now we have

**Case I:** If $y > x$ and $v > u$ then

$$\hat{\rho}(S(x, y), S(u, v)) + \hat{\rho}(S(y, z), S(v, u)) \leq \hat{\rho}(0, 0) + \hat{\rho}\left( y - x, v - u \right) \leq \frac{1}{400} \left( |x - u| + |y - v| \right)^2.$$  

Hence it follows that

$$\hat{\rho}(S(x, y), S(u, v)) + \hat{\rho}(S(y, z), S(v, u)) \leq k \{ \hat{\rho}(\alpha x, \alpha u) + \hat{\rho}(\alpha y, \alpha v) \}.$$  

**Case II:** If $y > x$ and $u \geq v$ then

$$\hat{\rho}(S(x, y), S(u, v)) + \hat{\rho}(S(y, z), S(v, u)) \leq \hat{\rho}(0, 0) + \hat{\rho}\left( y - x, v - u \right) \leq \frac{1}{400} \left( |x - u| + |y - v| \right)^2.$$  

Hence it follows that

$$\hat{\rho}(S(x, y), S(u, v)) + \hat{\rho}(S(y, z), S(v, u)) \leq k \{ \hat{\rho}(\alpha x, \alpha u) + \hat{\rho}(\alpha y, \alpha v) \}.$$
Coupled fixed point theorems with monotone property

Case III: If \( x_\lambda \geq y_\mu \) and \( u_\lambda \geq v_\mu \) then

\[
\rho \left( S(\lambda, x_\lambda), S(\lambda, u_\lambda) \right) + \rho \left( S(\mu, y_\mu), S(\mu, v_\mu) \right) \leq \left( \frac{x_\lambda - y_\mu}{10}, \frac{u_\lambda - v_\mu}{10} \right) + \rho(0,0)
\]

\[
\leq \left( \frac{x_\lambda - y_\mu - u_\lambda + v_\mu}{20} \right)^2 \leq \frac{1}{400} \left( \left| x_\lambda - u_\lambda \right|^2 + \left| y_\mu - v_\mu \right|^2 \right).
\]

\[
\frac{1}{2} \left\{ \rho(\lambda, x_\lambda, \mu, x_\lambda) + \rho(\mu, y_\mu, v_\mu) \right\} \leq \frac{1}{2} \left( \rho \left( \frac{x_\lambda - u_\lambda}{2}, \frac{y_\mu - v_\mu}{2} \right) \right)
\]

\[
\leq \frac{1}{2} \left( \left( \frac{x_\lambda - u_\lambda}{4} \right)^2 + \left( \frac{y_\mu - v_\mu}{4} \right)^2 \right) \leq \frac{1}{32} \left( \left| x_\lambda - u_\lambda \right|^2 + \left| y_\mu - v_\mu \right|^2 \right).
\]

Hence it follows that

\[
\rho \left( S(\lambda, x_\lambda, y_\mu), S(\lambda, x_\lambda, u_\lambda) \right) + \rho \left( S(\mu, v_\mu, y_\mu), S(\mu, v_\mu, u_\lambda) \right) \leq k \left\{ \rho(\lambda, x_\lambda, \mu, x_\lambda) + \rho(\mu, y_\mu, v_\mu) \right\}.
\]

Case IV: If \( x_\lambda \geq y_\mu \) and \( v_\mu \leq u_\lambda \) then

\[
\rho \left( S(\lambda, x_\lambda, y_\mu), S(\lambda, u_\lambda, v_\mu) \right) + \rho \left( S(\mu, y_\mu, x_\lambda), S(\mu, u_\lambda, v_\mu) \right) \leq \frac{1}{2} \left( \rho \left( \frac{x_\lambda - y_\mu}{10}, 0 \right) + \rho \left( 0, \frac{u_\lambda - v_\mu}{10} \right) \right)
\]

\[
\leq \frac{x_\lambda - y_\mu}{20} + \frac{y_\mu - u_\lambda}{20} \leq \frac{1}{400} \left( \left| x_\lambda - u_\lambda \right|^2 + \left| y_\mu - v_\mu \right|^2 \right).
\]

\[
\frac{1}{2} \left\{ \rho(\lambda, x_\lambda, \mu, x_\lambda) + \rho(\mu, y_\mu, v_\mu) \right\} \leq \frac{1}{2} \left( \rho \left( \frac{x_\lambda - u_\lambda}{2}, \frac{y_\mu - v_\mu}{2} \right) \right)
\]

\[
\leq \frac{1}{2} \left( \left( \frac{x_\lambda - u_\lambda}{4} \right)^2 + \left( \frac{y_\mu - v_\mu}{4} \right)^2 \right) \leq \frac{1}{32} \left( \left| x_\lambda - u_\lambda \right|^2 + \left| y_\mu - v_\mu \right|^2 \right);
\]

\[
\leq \frac{1}{32} \left( \left| x_\lambda - u_\lambda + y_\mu - v_\mu \right|^2 + \left( \left| y_\mu - u_\lambda \right| + \left| y_\mu - v_\mu \right| \right) \right);
\]

\[
\leq \frac{1}{32} \left( 2 \left( \left| x_\lambda - y_\mu \right|^2 + \left| u_\lambda - y_\mu \right|^2 \right) + \left( \left| y_\mu - u_\lambda \right| + \left| y_\mu - v_\mu \right| \right) \right);
\]

\[
\leq \frac{1}{8} \left( \left( 2 \left| x_\lambda - y_\mu \right|^2 + \left| u_\lambda - y_\mu \right|^2 + \left| v_\mu - u_\lambda \right|^2 \right) \right).
\]

Hence it follows that

\[
\rho \left( S(\lambda, x_\lambda, y_\mu), S(\lambda, x_\lambda, u_\lambda) \right) + \rho \left( S(\mu, v_\mu, y_\mu), S(\mu, v_\mu, u_\lambda) \right) \leq k \left\{ \rho(\lambda, x_\lambda, \mu, x_\lambda) + \rho(\mu, y_\mu, v_\mu) \right\}.
\]

Hence all the condition of the theorem 2.2 are satisfied, S and \( \alpha \) have coupled coincidence point.

Conclusions: In this paper the investigations concerning the existence of coupled soft fixed point theorem for a contractive condition with monotone property and \( \alpha \)-monotone property in an ordered soft b-metric space are established. Example is also given in the support of established results. These results can be extended to any directions and can also be extended to fixed point theory of non-expansive multi-valued mappings. These proved results leads to different directions and aspect of soft metric fixed point theory.
Acknowledgements. The authors would like their sincere thanks to the editor and the anonymous referees for their valuable comments and useful suggestions in improving the article.

References


Coupled fixed point theorems with monotone property

https://doi.org/10.1016/j.camwa.2008.11.009


Received: February 19, 2017; Published: April 8, 2017