Moore-Penrose Inverses of Operators in Hilbert $C^*$-Modules

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Abstract

For two given bounded adjointable operators $T$ and $S$ between Hilbert $C^*$-modules, it is well known that an operator Moore-Penrose inverse exists iff the operator has closed range. In this paper, we give some formulas for the Moore-Penrose inverses of products $TS$.

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1 Introduction and preliminaries

Throughout the paper, we assume that $E$, $F$, $G$ are Hilbert $A$-modules, where $A$ is a $C^*$-algebra. For an operator $T$, let $R(T)$ be the range of $T$ and $K(T)$ be the kernel of $T$. The notation $[T, S]$ denotes the commutator $TS - ST$ of $T$ and $S$. We abbreviate Moore-Penrose inverse to MP-inverse.
For two given invertible operator $T$, $S$ in Hilbert $C^*$-modules, the equality $(TS)^{-1} = S^{-1}T^{-1}$ is called the reverse order law. If $T$ and $S$ are invertible operator then the reverse order law effective but this case does not validate for the MP-inverse in general. The problem first studied by Greville [5] and then reconsidered by Bouldin [1] and Izumino [6]. A number of researchers discussed the problem such that reverse order law holds from different angles [2, 3, 4, 8, 15]. We also refer to another interesting Sharifi and Bonakdar [15] of this type.

In this paper we continue and supplement this research by using the space decompositions and operator matrix representations in $C^*$-modules. We specialize the investigations to the MP-inverses of $TS$ and give some formulas for the MP-inverses of $TS$.

The notion of $C^*$-module has been presented by Kaplansky [7] and Paschke [13]. A Hilbert $C^*$-module $E$ is right $A$-module with an inner product $\langle \cdot , \cdot \rangle : E \times E \rightarrow A$ satisfying

1. $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$,
2. $\langle x, ya \rangle = \langle x, y \rangle a$,
3. $\langle y, x^* \rangle = \langle x, y \rangle$,
4. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$,

for all $x, y, z \in E$, $a \in A$, $\alpha, \beta \in C$ and such that $E$ is complete corresponding to $\|x\| = \sqrt{\|\langle x, x \rangle\|}$.

Hilbert $C^*$-module can be regard as a generalization of the Hilbert space. In spite of this promotion is natural, some basic properties of Hilbert spaces are not applicable in Hilbert $C^*$-modules. Therefore, when studying a certain problem in $C^*$-modules, it is useful to find conditions to gain the results homologous to those for Hilbert spaces.

The elements of $L(E,F)$ is said to be bounded adjointable operators, if there is an operator $T^* : F \rightarrow E$ satisfying $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ for each $x \in E$ and $y \in F$. The operator $T$ is selfadjoint if $T = T^*$ and $T \in L(E,F)$ is $MP$-invertible if there is $X \in L(F,E)$ such that

$$TXT = T, \quad XTX = X, \quad (TX)^* = TX, \quad (XT)^* = XT.$$ 

There is at most one element $X$ satisfying the above formula, if $T \in L(E,F)$ is MP-invertible, then the unique $X$ is called $MP$-inverse of $T$. In symbols this is denoted by $T^+$. Xu and Sheng [16] show that an operator permits a MP-inverses iff the operator has closed range. Moreover, Sharifi [14] show that $TS$ has closed range iff the kernel of $T$ is orthogonally complemented with the range of $S$, iff the kernel of $S^*$ is orthogonally complemented with the range of $T^*$.

If a Hilbert $A$-submodule $W$ of a Hilbert $A$-module $E$ and its orthogonal complement $W^\perp$ yield $E = W \oplus W^\perp$, then $W$ is orthogonal complemented. We associate the matrix representation of an operator $T \in L(E,F)$ with respect
to some natural decompositions of $C^*$-modules. If $E = K \oplus K^\perp$, $F = H \oplus H^\perp$ then operator $T$ has the following matrix form

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$$

where $T_1 \in L(K, H)$, $T_2 \in L(K^\perp, H)$, $T_3 \in L(K, H^\perp)$, $T_4 \in L(K^\perp, H^\perp)$.

## 2 Main results

First, we need following three auxiliary results.

**Lemma 2.1.** [9] Let $T \in L(E, F)$ and $R(T)$ be closed. Then $T$ has the following matrix form:

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T^*) \\ K(T) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(T) \\ K(T^*) \end{bmatrix},$$

where $T_1$ is invertible. Moreover

$$T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T) \\ K(T^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(T^*) \\ K(T) \end{bmatrix}.$$  

**Lemma 2.2.** [15] Let $E_1$, $E_2$ be closed submodules of $E$ and $F_1$, $F_2$ be closed submodules of $F$ such that $E = E_1 \oplus E_2$ and $F = F_1 \oplus F_2$. If $T \in L(E, F)$ has closed range, then $T$ has the following matrix form:

(i)

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(T) \\ K(T^*) \end{bmatrix}.$$  

Moreover

$$T^\dagger = \begin{bmatrix} T_1^* D^{-1} & 0 \\ T_2^* D^{-1} & 0 \end{bmatrix},$$

where $D = T_1 T_1^* + T_2 T_2^* \in L(\mathcal{R}(T))$ is positive and invertible.

(ii)

$$T = \begin{bmatrix} T_1 & 0 \\ T_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(T^*) \\ K(T) \end{bmatrix} \to \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}.$$  

Moreover

$$T^\dagger = \begin{bmatrix} D^{-1} T_1^* & D^{-1} T_2^* \\ 0 & 0 \end{bmatrix}.$$
where $D = T_1^*T_1 + T_2^*T_2 \in L(R(T^*))$ is positive and invertible.

**Lemma 2.3.** Let $N \in L(E, F)$ have a closed range, and let $M \in L(F)$ be selfadjoint and invertible. Then $R(MN) = R(N)$ iff $[M, NN^\dagger] = 0$.

**Proof.** Suppose $N$ have a closed range, $M$ be selfadjoint and invertible. We consider the orthogonal direct sums $E = R(N^*) + K(N)$ and $F = R(N) + K(N^*)$. Then the operators $N$ and $M$ have the corresponding matrix forms as follows:

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(N^*) \\ K(N) \end{bmatrix} \to \begin{bmatrix} R(N) \\ K(N^*) \end{bmatrix},$$

where $N_1$ is invertible, and

$$M = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} : \begin{bmatrix} R(N) \\ K(N^*) \end{bmatrix} \to \begin{bmatrix} R(N) \\ K(N^*) \end{bmatrix},$$

where $M_3 = M_2^*$. It follows that

$$MN = \begin{bmatrix} M_1N_1 & 0 \\ M_3N_1 & 0 \end{bmatrix} : \begin{bmatrix} R(N^*) \\ K(N) \end{bmatrix} \to \begin{bmatrix} R(N) \\ K(N^*) \end{bmatrix}.$$

Hence, $R(MN) = R(N)$ implies $M_3 = 0$ and $M_2 = 0$, so $M = \begin{bmatrix} M_1 & 0 \\ 0 & M_4 \end{bmatrix}$.

Since $M$ is selfadjoint and invertible, we obtain that $M_1$ and $M_4$ are also selfadjoint and invertible. Since $N^\dagger = \begin{bmatrix} N_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$, we obtain that $MNN^\dagger = NN^\dagger M$ holds.

Conversely, if $M$ is invertible and $MNN^\dagger = NN^\dagger M$, then

$$R(MN) = R(MNN^\dagger) = R(NN^\dagger M) = R(NN^\dagger) = R(N).$$

□

In the following, we give some formulas about $(TS)^\dagger$.

**Theorem 2.4.** Suppose $S \in L(E, F)$, $T \in L(F, G)$, $TS$ have closed ranges. Then $(TS)^\dagger = (T^*TS)^\dagger T^\dagger$ iff $R(TT^*TS) = R(TS)$.

**Proof.** By Lemma (2.1), the operator $S$ and its MP-inverse $S^\dagger$ have the following matrix forms:

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(S^*) \\ K(S) \end{bmatrix} \to \begin{bmatrix} R(S) \\ K(S^*) \end{bmatrix},$$

$$S^\dagger = \begin{bmatrix} S_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(S) \\ K(S^*) \end{bmatrix} \to \begin{bmatrix} R(S^*) \\ K(S) \end{bmatrix}.$$

From Lemma 2.2 it follows that the $T$ and its MP-inverse $T^\dagger$ have the following matrix forms:
\[ T = \begin{bmatrix} T_1 & T_2 \\ 0 & 0 \end{bmatrix} : R(S) \rightarrow R(T) \]

\[ T^\dagger = \begin{bmatrix} T_1^* D^{-1} & 0 \\ T_2^* D^{-1} & 0 \end{bmatrix}, \]

where \( D = T_1 T_1^* + T_2 T_2^* \in L(R(T)) \) is positive and invertible. Then we have the following products

\[ TS = \begin{bmatrix} T_1 S_1 & 0 \\ 0 & 0 \end{bmatrix} \]

and

\[ (TS)^\dagger = \begin{bmatrix} (T_1 S_1)^\dagger & 0 \\ 0 & 0 \end{bmatrix} \]

\[ S^\dagger T^\dagger = \begin{bmatrix} S_1^{-1} T_1^* D^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \]

Notice that

\[ R((T^\dagger TS)^*) = R(S^* T^\dagger T) = S^* R(T^\dagger T) = S^* R(T^*) = R((TS)^*) \]

is closed, so \( R(T^\dagger TS) \) is closed. First, we find the equivalent conditions for our statements.

(1). Let us denote \( C = T^\dagger TS \). Using [[11], Corollary 2.4] we find \( C^\dagger \) as follows

\[ C^\dagger = (C^*C)^\dagger C^* = \begin{bmatrix} (S_1^* T_1^* D^{-1} T_1^* S_1^\dagger) & (S_1^* T_1^* D^{-1} T_1^* S_1^\dagger) \\ 0 & 0 \end{bmatrix}. \]

Now, we can see that \( (TS)^\dagger = (T^\dagger TS)^\dagger T^\dagger \) is equivalent with

\[ (T_1 S_1)^\dagger = (S_1^* T_1^* D^{-1} T_1^* S_1^\dagger) S_1^* T_1^* D^{-1} = (D^{-\frac{1}{2}} T_1 S_1)^\dagger D^{-\frac{1}{2}}. \]

(2). It is obvious that \( TT^* TS = \begin{bmatrix} DT_1 S_1 & 0 \\ 0 & 0 \end{bmatrix} \), so \( R(TT^* TS) = R(TS) \) holds iff \( R(DT_1 S_1) = R(T_1 S_1) \).

(1) \( \Rightarrow \) (2) From the third Moore-Penrose equation for \( (T_1 S_1)^\dagger = (D^{-\frac{1}{2}} T_1 S_1)^\dagger D^{-\frac{1}{2}} \), we see that \( T_1 S_1 (D^{-\frac{1}{2}} T_1 S_1)^\dagger D^{-\frac{1}{2}} \) is selfadjoint. So we have the following equivalents:

\[ T_1 S_1 (D^{-\frac{1}{2}} T_1 S_1)^\dagger D^{-\frac{1}{2}} \text{ is selfadjoint} \]

\[ \Leftrightarrow D^{-\frac{1}{2}} T_1 S_1 (D^{-\frac{1}{2}} T_1 S_1)^\dagger D^{-1} \text{ is selfadjoint} \]

\[ \Leftrightarrow [D, D^{-\frac{1}{2}} T_1 S_1 (D^{-\frac{1}{2}} T_1 S_1)^\dagger] = 0 \]

\[ \Leftrightarrow D^\frac{1}{2} T_1 S_1 (D^{-\frac{1}{2}} T_1 S_1)^\dagger = D^{-\frac{1}{2}} T_1 S_1 (D^{-\frac{1}{2}} T_1 S_1)^\dagger D \]

\[ \Leftrightarrow DT_1 S_1 (D^{-\frac{1}{2}} T_1 S_1)^\dagger = T_1 S_1 (D^{-\frac{1}{2}} T_1 S_1)^\dagger D. \]

Now,

\[ R(DT_1 S_1) = R(DT_1 S_1 (T_1 S_1)^\dagger) = R(T_1 S_1 (T_1 S_1)^\dagger D) = R(T_1 S_1). \]

(2) \( \Rightarrow \) (1) If \( R(DT_1 S_1) = R(T_1 S_1) \), then we apply Lemma 2.3 to obtain

\[ [D, T_1 S_1 (T_1 S_1)^\dagger] = 0. \]
Now, from the previous argument it follows that $T_1 S_1 (D^{-\frac{1}{2}} T_1 S_1)^\dagger D^{-\frac{1}{2}}$ is self-adjoint. Noticed that $(D^{-\frac{1}{2}} T_1 S_1)^\dagger D^{-\frac{1}{2}} T_1 S_1$ is the orthogonal projection onto 

$$R((T_1 S_1)^* D^{-\frac{1}{2}}) = R((T_1 S_1)^*),$$

so $T_1 S_1 (D^{-\frac{1}{2}} T_1 S_1)^\dagger D^{-\frac{1}{2}} T_1 S_1 = T_1 S_1$. Finally, it is not difficult to examine that $(T_1 S_1)^\dagger = (D^{-\frac{1}{2}} T_1 S_1)^\dagger D^{-\frac{1}{2}}$ holds. □

Similarly to Theorem 2.4 we have:

**Theorem 2.5.** Suppose $S \in L(E, F)$, $T \in L(F, G)$. $TS$ have closed ranges. Then $(TS)^\dagger = S^\dagger (T S S^\dagger)^\dagger$ iff $R(S^* S(TS)^*) = R((TS)^*)$.

**Proof.** According to Theorem 2.4, we have the following equivalences:

$$(S^* T^*)^\dagger = (T^*)^\dagger (S^* T^\dagger T)^\dagger \iff R(T T^* S) = R(T S)$$

Now, take $T' = S^*$ and $S' = T^*$,

$$(T' S')^\dagger = S'^\dagger (T' S' S'^\dagger)^\dagger \iff R(S'^* S' S'^* T'^*) = R(S'^* T'^*).$$

□

**Theorem 2.6.** Suppose $S \in L(E, F)$, $T \in L(F, G)$. $TS$ have closed ranges. Then $S^\dagger = (TS)^\dagger T$ iff $R(T^* T S) = R(S)$.

**Proof.** We keep the matrix forms of $T$ and $S$ as in previous theorems.

(1) We can obtain that $S^\dagger = (TS)^\dagger T$ iff $I = (T_1 S_1)^\dagger T_1 S_1$ and $(T_1 S_1)^\dagger T_2 = 0$. Hence, $S^\dagger = (TS)^\dagger T$ is equivalent to the following two conditions: $T_1$ is injective with closed range and $(T_1 S_1)^\dagger T_2 = 0$.

(2) $R(T^* T S) = R(S)$ iff $R(T_1^* T_1 S_1) = R(S_1)$ and $T_2^* T_1 S_1 = 0$. Hence, $R(T^* T S) = R(S)$ is equivalent to the following two conditions: $T_1$ is injective with closed range and $T_1^* T_2 = 0$.

To prove the equivalence (1) \iff (2), we have the following:

$$(T_1 S_1)^\dagger T_2 = 0 \iff R(T_2) \subset K((T_1 S_1)^\dagger) = K((T_1 S_1)^*) \iff (T_1 S_1)^* T_2 = 0 \iff T_1^* T_2 = 0.$$

□

We also prove the following result.

**Theorem 2.7.** Suppose $S \in L(E, F)$, $T \in L(F, G)$. $TS$ have closed ranges. Then $T^\dagger = S(TS)^\dagger$ iff $R(SS^* T^*) = R(T^*)$.

**Proof.** From the Theorem 2.6 it follows that $S^{*\dagger} = T^* (S^* T^*)^\dagger$ iff $R(T^* T S) = R(S)$. Now replace $T^*$ and $S^*$ by $S'$ and $T'$, to obtain theorem holds. □
Remark 2.8. The conditions $R(T^*TS) = R(S)$ and $R(SS^*T^*) = R(T^*)$ taken in Theorem 2.6 and Theorem 2.7 imply the reverse order law $(TS)^\dagger = S^\dagger T^\dagger$ holds, since $S^\dagger T^\dagger = (TS)^\dagger TS(TS)^\dagger = (TS)^\dagger$.

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References


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