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Solitary Wave Solutions for Variant Two-Dimensional Zakharov-Kuznetsov Equation

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Abstract

We show the existence, of solitary waves associated to the following equation

$$u_t + \partial_x \triangle u - \mathcal{H}\partial_x^2 u + u u_x = 0,$$

where $\mathcal{H}$ is the Hilbert transform with respect to x.

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1 Introduction

We are interested in the proof of the existence of solitary waves associated to the Zakharov-Kuznetsov equation

$$u_t + \partial_x \triangle u - \mathcal{H}\partial_x^2 u + uu_x = 0, \tag{1}$$

in the space

$$X^{\frac{1}{2}} = \{f \in H^1(\mathbb{R}^2) : D_x^{\frac{1}{2}} f \in L^2(\mathbb{R}^2)\}$$

where $\mathcal{H}$ is the Hilbert transform with respect to x.

The solitary waves, or waves of permanent form, are known to arise in several models as a consequence of the combined effects of the linear and the nonlinear parts.

Using Kato's theory, for instance, it can be proved that the equation (1) is local well-posed in H^s for $s > 3$.

This is a two-dimensional case of the Benjamin-Ono equation

$$\partial_t u + \mathcal{H}\partial_x^2 u + u\partial_x u = 0, \tag{2}$$

which describes certain models in physics about wave propagation in a stratified thin regions . This last equation shares with the equation KdV

$$u_t + u_x + uu_x + u_{xxx} = 0 \tag{3}$$

many interesting properties. For example, they both have infinite conservation laws, they have solitary waves as solutions which are stable and behave like soliton (this last is evidenced by the existence of multisoliton type solutions) see [4]. Also, the local and global well-posedness was proven in the Sobolev spaces context (in low regularity spaces inclusive, see, e.g., [5] and [7])

We should note that the equation (1) is the model of dispersive long wave motion in a weakly nonlinear two-fluid system, where the interface is subject to capillarity and bottom fluid is infinitely deep.

This paper is organized as follows. In Section 2 , we include some preliminaries. In Section 3 we present the proof of existence of solitary waves solution to the equation (1), to this we use minimax theory techniques.

2 Preliminaries

Here we present some results that are used in this paper.

Is easy to see that $X^{\frac{1}{2}}$ is a Hilbert space with the inner product defined by

$$\langle f, g\rangle_{\frac{1}{2}} = \int_{\mathbb{R}^2} \left[fg + f_x g_x + f_y g_y + D_x^{\frac{1}{2}} f D_x^{\frac{1}{2}} g \right] dxdy,$$

and the corresponding norm

$$\|f\|^2_{X^{\frac{1}{2}}} = \int_{\mathbb{R}^2} \left[f^2 + f_x^2 + +f_y^2 + (D_x^{\frac{1}{2}} f)^2 \right] dxdy.$$

Lemma 2.1. *The following embeddding is continuous:*

$$X^{\frac{1}{2}} \hookrightarrow L^3(\mathbb{R}^2).$$

Lemma 2.2. *The embedding $X^{\frac{1}{2}} \hookrightarrow L^3_{loc}(\mathbb{R}^2)$ is compact.*

In other words, if (u_n) is a bounded sequence in $X^{\frac{1}{2}}$ and $R > 0$, there exists a subsequence (u_{n_k}) of (u_n) which converges strongly to u in $L^3(B_R)$.

Hereafter, $B(x, y; R)$ denotes the ball in $\mathbb{R}^2$ of center (x, y) and radius $R > 0$.

Lemma 2.3. *If (u_n) is bounded in $X^{\frac{1}{2}}$ and*

$$\lim_{n\to\infty} \sup_{(x,y)\in\mathbb{R}^2} \int_{B(x,y;R)} |u_n|^2 dxdy = 0, \tag{4}$$

as $n \to \infty$, then $u_n \to 0$ in $L^3(\mathbb{R}^2)$

The proof of the lemmas above follows closely the arguments in [6].

The following lemma gives us a minimax priciple and is an inmediate consequence of Theorem 2.8 in [8]

Lemma 2.4. *Suppose X is a Banach space and $\Phi \in C^1(X, \mathbb{R})$ satisfies the following properties:*

1. *$\Phi(0) = 0$, and there exists $\rho > 0$, such that $\Phi|_{\partial B_\rho(0)} \geq \alpha > 0$.*
2. *There exists $\beta \in X \setminus \overline{B}_\rho(0)$ such that $\Phi(\beta) \leq 0$.*

Let Γ be the set of all paths which connects 0 and β, i.e.,

$$\Gamma = \{g \in C([0, 1], X) \,|\, g(0) = 0, g(1) = \beta\},$$

and

$$c = \inf_{g\in\Gamma} \max_{t\in[0,1]} \Phi(g(t)). \tag{5}$$

Then $c \geq \alpha$ and Φ possesses a Palais-Smale sequence at level c, i.e., there exists a sequence (u_n) such that $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ as $n \to \infty$.

3 Existence of Solitary Waves

If $\phi(x-ct,y)$ is a solitary wave solution solution to (1), then

$$-c\partial_x\phi+\phi\partial_x\phi-\mathcal{H}\partial_x^2\phi+\partial_x\triangle\phi=0. \tag{6}$$

If $\phi\in X^{\frac{1}{2}}$, we can write (6) as

$$-c\phi-\mathcal{H}\partial_x\phi+\triangle\phi+\frac{\phi^2}{2}=0. \tag{7}$$

where the term on the right hand is in $\left(X^{\frac{1}{2}}\right)^*$, the topological dual of $X^{\frac{1}{2}}$. Then ϕ is a critical point of the functional Φ on $X^{\frac{1}{2}}$ defined as

$$\Phi(\phi)=\int_{\mathbb{R}^2}\frac{1}{2}\left(c\phi^2+(D_x^{\frac{1}{2}}\phi)^2+(\partial_x\phi)^2+(\partial_y\phi)^2\right)-\frac{\phi^3}{6}\,dxdy.$$

Let us see that Φ satisfies the conditions of the Lemma 2.4. It is obvious that Φ is a C^1 functional. $\Phi(0)=0$ and, since

$$\Phi(\phi)\geq\frac{\min\{c,1\}}{2}\|\phi\|_{X^{\frac{1}{2}}}-\int_{\mathbb{R}^2}\frac{|\phi|^3}{6}dxdy,$$

by Lemma 2.1, there exist a ρ such that

$$\inf_{\partial B_\rho(0)}\Phi=\alpha>0,$$

which shows 1). Now, for $\vartheta\in\mathbb{R}$ and $u\in X^{\frac{1}{2}}$,

$$\Phi(\vartheta u)=\vartheta^2\left(\Phi(u)+\int_{\mathbb{R}^2}\frac{u^3}{6}\,dxdy\right)-\vartheta^3\int_{\mathbb{R}^2}\frac{u^3}{6}\,dxdy.$$

Then, taking u fixed and ϑ large enough, we have 2) with $\beta=\vartheta u$. So, we have shown the following lemma.

Lemma 3.1. *Let Φ, α and β be defined as above and let Γ and c be defined as Lemma 2.4. Then, there exists a sequence (ϕ_n) such that $\Phi(\phi_n)\to c$ and $\Phi'(\phi_n)\to 0$.*

Now, we can prove the following theorem.

Theorem 3.2. (6) *has nontrivial solutions in $X^{\frac{1}{2}}$.*

Proof. It is enough to show that Φ have non-zero critical points in $X^{\frac{1}{2}}$. By Lemma 3.1, there exists a Palais-Smale sequence (ϕ_n) at level c of Φ. Therefore,

$$c+1 \geq \Phi(\phi_n) - \frac{\langle \Phi'(\phi_n), \phi_n \rangle_{X^{\frac{1}{2}}}}{3} \geq \frac{1}{6}\min\{1,c\}\|\phi_n\|^2_{X^{\frac{1}{2}}},$$

for n big enough. Hence (ϕ_n) is bounded in $X^{\frac{1}{2}}$. Considering that

$$0 < c = \lim_{n\to\infty} \Phi(\phi_n) - \frac{1}{2}\langle \Phi'(\phi_n), \phi_n \rangle_{X^{\frac{1}{2}}} = \lim_{n\to\infty} \frac{1}{12}\int_{\mathbb{R}^2} \phi_n^3\, dxdy,$$

the Lemma 2.3 implies that

$$\delta = \limsup_{n\to\infty} \sup_{(x,y)\in\mathbb{R}^2} \int_{B(x,y;1)} \phi_n^2\, dxdy > 0.$$

Then, passing to a subsequence if necessary, we can assume that there exists a sequence (x_n, y_n) in $\mathbb{R}^2$ such that

$$\int_{B(x_n,y_n;1)} \phi_n^2\, dxdy > \delta/2, \tag{8}$$

for n big enough. Let $\tilde{\phi}_n = \phi_n(\cdot + (x_n, y_n))$. Then, again passing to a subsequence if necessary, we can assume that, for some $\phi \in X^{\frac{1}{2}}$, $\tilde{\phi}_n \rightharpoonup \phi$ in $X^{\frac{1}{2}}$. In view of (8), for n large enough, and Lemma 2.2, $\phi \neq 0$. The Lemma 2.2 and the continuity of the function $u \to u^{p+1}$ from L^{p+2} to $L^{\frac{p+2}{p+1}}$, in any measure space, imply that

$$\langle \Phi'(\phi), w \rangle_{X^{\frac{1}{2}}} = \lim_{n\to\infty} \langle \Phi'(\tilde{\phi}_n), w \rangle_{X^{\frac{1}{2}}} = 0.$$

This shows this theorem. □

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