Caristi-type Fixed Point Theorem
of Set-Valued Maps in Metric Spaces

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Abstract

In this article are presented existence of non-negative function forms \( \varphi \) of Caristi’s fixed point theorem for set-valued mapping on a complete metric space that has compact values. In this case, be formed function \( \varphi \) of the set-valued as a support function. By it, the existence fixed point was shown.

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1 Introduction

In [7], S.B. Nadler has introduced the notion of contraction set-valued function by using the Hausdorff metric. By it, he has proved a set-valued version of the well-known Banach contraction principle and states that each closed bounded valued contraction map on a complete metric space has a fixed point. Since then various fixed point results concerning set-valued contractions have appeared. The real generalization of the Nadler’s fixed point theorem was obtained by S. Reich [9], T. Suzuki [10], and A. Latif [5]. All of the result it using the concept of the distance on space of all a subset. In addition, the other
researcher applying Caristi’s fixed point theorem on set-valued mapping such as N. Mizoguchi (see [6]) and J.R Jachymski [4]. They don’t involve Hausdorff metric in the space of all subsets as that discussed by Nadler.

In this article, author’s would be applying Caristi’s fixed point theorem on set-valued maps but unlike that do by the researcher that cited above. We will highlight an existence of function \( \varphi \) that is the main characteristic of Caristi type mapping. Furthermore, various types function \( \varphi \) would be used to prove existence a fixed point of set-valued maps has compact values.

2 Preliminary Notes

Let \( X, Y \) be a metric space and \( \mathcal{P}_0(Y) \) the collection of all non-empty subsets of \( Y \). The mapping \( F : X \rightarrow \mathcal{P}_0(Y) \) is called set-valued maps. A solution \( \bar{x} \) of the inclusion \( \bar{x} \in F(\bar{x}) \) is called a fixed point of \( F \). The graph of a set-valued map \( F \) is defined by

\[
G_r(F) = \{(x, y) \in X \times Y \mid y \in F(x)\}
\]

**Definition 2.1.** Let \( B \) be the unit ball of \( Y \). A set-valued \( F : X \rightarrow \mathcal{P}_0(Y) \) is called upper semi-continuous at \( x_0 \) if, for all \( \epsilon > 0 \), there exists a neighborhood \( N(x_0) \) of \( x_0 \) such that \( F(x) \subset B_{\epsilon}(F(x_0)) \) for all \( x \in N(x_0) \). It is upper semi-continuous on \( X \) if, it is upper semi-continuous at all points \( x \in X \).

We are to introduce the class of functions \( \varphi : X \rightarrow [0, +\infty] \) and to associate them with their domain

\[
\text{Dom}(\varphi) = \{x \in X \mid \varphi(x) < +\infty\} \quad (1)
\]

**Definition 2.2.** A function \( \varphi : X \rightarrow [0, +\infty] \) is strict if \( \text{Dom}(\varphi) \) is non-empty.

**Definition 2.3.** We shall say that a function \( \varphi : X \rightarrow [0, +\infty] \) is lower semi-continuous at \( x_0 \in X \) if for all \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

\[
\varphi(x_0) - \epsilon \leq \varphi(x) \quad (2)
\]

for every \( x \in B_\delta(x_0) \).

Where \( B_\delta(x_0) = \{x \in X \mid d(x_0, x) < \delta\} \) be the ball with center point \( x_0 \) and radius \( \delta > 0 \) of \( X \).

We shall say that \( \varphi \) is lower semi-continuous on \( X \) if it is lower semi-continuous at every point of \( X \). A function is upper semi-continuous if \((-\varphi)\) is lower semi-continuous and continuous if it is lower semi-continuous and upper semi-continuous.
Theorem 2.4. (Caristi)(see [6]) Let \((X,d)\) be a complete metric space and \(F : X \rightarrow \mathcal{P}_0(X)\) be a set-valued maps. Suppose there exists \(\varphi : X \rightarrow [0, +\infty] \) is a strict and lower semi continuous function such that for each \(x \in X\), there exists \(y \in F(x)\) such that
\[
\varphi(y) + d(x, y) \leq \varphi(x). \tag{3}
\]
Then the set-valued map \(F\) has a fixed point.

Since we are discussing fixed-point theorem, we shall proof another result in which \(\varphi\) is no longer assumed to be lower semi-continuous; However, the set-valued map \(F\) must have a closed graph.(see [1])

Theorem 2.5. Let \(X\) be a complete metric space. We consider a set-valued map \(F : X \rightarrow \mathcal{P}_0(X)\) with a closed graph. If there exists a strict function \(\varphi : X \rightarrow [0, +\infty] \) satisfying condition (3), then the set-valued \(F\) has a fixed point.

Proof. By hypothesis we take a point \(x_0 \in \text{Dom}(\varphi)\), there exists \(x_1 \in F(x_0)\) such that
\[
d(x_1, x_0) \leq \varphi(x_0) - \varphi(x_1)
\]
For a point \(x_n \in X\), there exists \(x_{n+1} \in F(x_n)\) such that
\[
d(x_{n+1}, x_1) \leq \varphi(x_n) - \varphi(x_{n+1}) \tag{4}
\]
This implies that the sequence of positive number \(\langle \varphi(x_n) \rangle\) is decreasing; thus, it converges to a number, \(\alpha\). Adding the inequalities (4) from \(n = p\) to \(n = q - 1\), the triangle inequality implies that
\[
d(x_p, x_q) \leq d(x_p, x_{p+1}) + d(x_{p+1}, x_{p+2}) + \cdots + d(x_{q-2}, x_{q-1}) + d(x_{q-1}, x_q) \\
\leq (\varphi(x_p) - \varphi(x_{p+1})) + (\varphi(x_{p+1}) - \varphi(x_{p+2})) + \cdots \\
\cdots + (\varphi(x_{q-2}) - \varphi(x_{q-1})) + (\varphi(x_{q-1}) - \varphi(x_q)) \\
\leq \varphi(x_p) - \varphi(x_q).
\]
Since the term on the right tends to \((\alpha - \alpha) = 0\) as \(p\) and \(q\) tend to infinity, we deduce that the sequence of the \(x_n\) is Cauchy sequence which thus converges to an element \(\bar{x}\) since the space is complete.
Since the pairs \((x_n, x_{n+1})\) belong to the graph of \(F\), which is closed, and converge to the pair \((\bar{x}, \bar{x})\) which thus belongs to the graph of \(F\), the limit point \(\bar{x}\) is fixed point of \(F\). \qed
3 Main Results

In the main result, we present the function \( \varphi \) that defined as the support function of set-valued be given.

Let \((X, d)\) be a metric space with the metric \(d\). For set-valued map \(F : X \to \mathcal{P}_0(X)\) we defined a function \(\varphi_F : X \to [0, +\infty)\) as

\[
\varphi_F(x) = d(x, F(x)) = \inf_{y \in F(x)} d(x, y) \in [0, +\infty]
\]

for all \(x \in X\).

In [3] be state that if \(F : X \to \mathcal{P}_0(X)\) is a upper semi continuous and non empty closed values then \(\varphi_F\) is lower semi continuous. In addition, be showed that if \(F : X \to \mathcal{CB}(X)\) is contraction then \(\varphi_F\) is continuous, where \(\mathcal{CB}(X)\) is a collection all closed bounded subset of \(X\) [4].

**Proposition 3.1.** Suppose \(F : X \to \mathcal{P}_0(X)\) is a set-valued map with properties

\[
\forall y \in F(x), \quad F(y) \subseteq F(x).
\]

If \(F\) is a compact values on \(X\), then the function \(\varphi_F : X \to [0, +\infty)\) in (5) satisfies inequality (3).

**Proof.** Since \(F(x)\) is compact for each \(x \in X\) there exits \(y \in F(x)\) such that \(d(x, y) = d(x, F(x)) = \varphi_F(x)\) is finite.

\[
\varphi_F(y) + d(x, y) = \inf_{z \in F(y)} d(y, z) + d(x, y) \leq d(y, z) + d(x, y),
\]

for each \(z \in F(y)\). According to hypothesis if \(y \in F(x)\) then \(F(y) \subseteq F(x)\). Consequently, for each \(z \in F(y)\) the point \(z\) is belong in \(F(x)\). Taking the infimum with respect to \(z\) on \(F(x)\) so that inequality above to be

\[
\varphi_F(y) + d(x, y) \leq \inf_{z \in F(x)} d(y, z) + d(x, y) = 0 + d(x, y) = d(x, y) = \varphi_F(x).
\]

\[\square\]

**Lemma 3.2.** Let \(F : X \to \mathcal{P}_0(X)\) is set-valued map. If function \(\varphi : X \to [0, +\infty]\) is strict and satisfies inequality (3), then the function \(\varphi_F(x) \leq \varphi(x)\) for each \(x \in X\).

**Proof.** By hypothesis for each \(x \in X\) there exists \(y \in F(x)\) such that

\[
d(x, y) \leq \varphi(x) - \varphi(y) \leq \varphi(x).
\]
It follows that we obtained
\[ \varphi_F(x) = d(x, F(x)) = \inf_{y \in F(x)} d(x, y) \leq d(x, y) \leq \varphi(x) < \infty, \]
for each \( x \in X \).

The following theorem is the main result.

**Theorem 3.3.** Let \( X \) be a complete metric space. We consider a set-valued map \( F : X \rightarrow \mathcal{P}_0(X) \) satisfying (6) and \( F \) compact on \( X \), then \( F \) has a fixed point.

**Proof.** According to Proposition 3.1 the function \( \varphi_F(x) = d(x, F(x)) \) satisfies inequality (3) and \( f_F \) is finite by Lemma 3.2. We then apply Theorem 2.5 to deduce that \( F \) has a fixed point.

We shall proof another result in which \( F \) is no longer assumed to be satisfies (6); However, the set-valued map \( F \) must upper semi-continuous.

**Theorem 3.4.** Let \( X \) be a complete metric space. We consider a set-valued map \( F : X \rightarrow \mathcal{P}_0(X) \) is upper semi-continuous. If \( F \) compact values on \( X \), then \( F \) has a fixed point.

**Proof.** We know that \( \varphi_F(x) = \inf_{y \in F(x)} d(x, y) \). Set valued \( F \) upper semi-continuous at \( x \in X \). Then for each \( \epsilon > 0 \) there exists \( \delta > 0 \) such that
\[ F(t) \subset B_{\epsilon}(F(x)) \]
for each \( t \in B_{\delta}(x) \).
For each \( \epsilon > 0 \) and \( x \in X \) there exists \( y \in F(x) \) such that
\[ d(x, y) < \varphi_F(x) + \epsilon, \]
we obtained
\[ \varphi_F(y) + d(x, y) = d(y, F(y)) + d(x, y) < d(y, F(y)) + f_F(x) + \epsilon \]
Since \( F \) upper semi-continuous, choose \( y \in F(x) \) such that \( y \in B_{\delta}(x) \), then \( F(y) \subset B_{\epsilon}(F(x)) \), it follow that
\[ \varphi_F(y) + d(x, y) < d(y, F(y)) + \varphi_F(x) + \epsilon \]
\[ < \epsilon + \varphi_F(x) + \epsilon \]
\[ < \varphi_F(x) + 2\epsilon \]
Since arbitrary $\epsilon > 0$ we obtained
\[
\varphi_F(y) + d(x, y) \leq \varphi_F(x).
\]
Thus $\varphi_F$ satisfies (3) and it is clear $\varphi_F$ finite. We then apply again Theorem 2.5 to deduce that $F$ has a fixed point. \qed

For set valued $F : X \rightarrow \mathcal{P}_0(X)$ and for some $\varepsilon_0 > 0$ we considering a set $B_{\varepsilon_0}(F(x)) = \{y \in X \mid d(y, F(x)) \leq \varepsilon_0\}$. We associate any $\varepsilon > 0$ with the function $\varphi_\varepsilon : X \rightarrow [0, \infty]$ defined by
\[
\varphi_{\varepsilon_0}(x) = d(x, B_{\varepsilon_0}(F(x)) = \inf\{d(x, y) \mid y \in B_{\varepsilon_0}(F(x))\}
\]
for each $x \in X$.

**Lemma 3.5.** If $\delta, \varepsilon_0 > 0$ and $\delta \leq \varepsilon_0$, then
\[
\varphi_{\varepsilon_0}(x) \leq \varphi_\delta(x) \leq \varphi_F(x).
\]
for all $x \in X$.

**Proof.** Suppose $x \in X$. For $\delta \leq \varepsilon_0$, we obtain sets
\[
F(x) \subset B_\delta(F(x)) \subset B_{\varepsilon_0}(F(x))
\]
We choose element $y \in F(x), x_\delta \in B_\delta(F(x))$ and $x_{\varepsilon_0} \in B_{\varepsilon_0}(F(x))$, we obtained
\[
d(x, x_\delta) \leq d(x, x_{\varepsilon_0}) \leq d(x, y).
\]
If take for all infimum of sets, then
\[
\inf_{x_{\varepsilon_0} \in B_{\varepsilon_0}(F(x))} d(x, x_{\varepsilon_0}) \leq \inf_{x_\delta \in B_\delta(F(x))} d(x, x_\delta) \leq \inf_{y \in F(x)} d(x, y)
\]
Thus proved, $\varphi_{\varepsilon_0}(x) \leq \varphi_\delta(x) \leq \varphi_F(x)$. \qed

We know that $\varphi_\varepsilon(x) = d(x, B_\varepsilon(F(x))) = \inf\{d(x, y) \mid y \in B_\varepsilon(F(x))\}$. For $y \in B_\varepsilon(F(x))$, we defined function by $\varphi_\varepsilon(y) = d(y, B_\varepsilon(F(y))) = \inf\{d(y, z) \mid z \in B_\varepsilon(F(y))\}$. We obtained
\[
\varphi_\varepsilon(x) + \varphi_\varepsilon(y) = \inf\{d(x, y) + d(y, z) \mid y \in B_\varepsilon(F(x)), z \in B_\varepsilon(F(y))\}.
\]

In general, for each $x \in X$, there exists sequence $\langle x_n \rangle_{n=0}^{\infty} \subset X$ such that $x = x_0, x_{n+1} \in B_\varepsilon(F(x_n))$. For $\varepsilon > 0$, we defined function $\varphi_\varepsilon : X \rightarrow [0, +\infty]$ by
\[
\varphi_\varepsilon(x) = \inf \left\{ \sum_{n=0}^{\infty} d(x_n, x_{n+1}) \mid x = x_0, x_{n+1} \in B_\varepsilon(F(x_n)) \right\}.
\]
For $n = 0$, $\varphi_\varepsilon(x) = \inf\{d(x, x_1) \mid x = x_0, x_1 \in B_\varepsilon(G(x))\} = \varphi_{\varepsilon_0}(x)$.
Theorem 3.6. If $\epsilon > 0$ and the function $\varphi_\epsilon : X \rightarrow [0, +\infty]$ with defined as (9), then there exists $x_\epsilon \in B_\epsilon(F(x))$ such that
\[ \varphi_\epsilon(x_\epsilon) + d(x, x_\epsilon) \leq \varphi_\epsilon(x) + \epsilon. \] (10)

Proof. Take $\epsilon > 0$, there exists a sequence $\langle x_n \rangle_{n=0}^\infty \subset X$ such that $x_0 = x$, $x_{n+1} \in B_\epsilon(F(x_n))$ and
\[ \sum_{n=0}^\infty d(x_n, x_{n+1}) \leq \varphi_\epsilon(x) + \epsilon. \]

Now,
\[ d(x, x_1) + \varphi_\epsilon(x_1) \leq d(x, x_1) + \sum_{n=1}^\infty d(x_n, x_{n+1}) \]
\[ = \sum_{n=0}^\infty d(x_n, x_{n+1}) \]
\[ \leq \varphi_\epsilon(x) + \epsilon. \]

Thus we take $x_\epsilon = x_1$ \hfill $\square$

Lemma 3.7. Suppose set-valued $F$ is upper semi-continuous with compact values on $X$. Then for all $\epsilon > 0$ there exist a sub sequence $\langle x_{\epsilon_k} \rangle$ converging to an element $\bar{x} \in F(x)$ such that, for all $\delta > 0$, there exists $k(\delta)$, such that for each $k \geq k(\delta)$,
\[ \varphi_\delta(\bar{x}) + d(x, x_{\epsilon_k}) - d(\bar{x}, x_{\epsilon_k}) \leq f_{\epsilon_k} + \epsilon_k \] (11)

Proof. We fix $x$. Since $F(x)$ is compact, for each $\epsilon > 0$, choose subsequence $x_{\epsilon_k} \in B_{\epsilon_k}(F(x))$ and it is converges to $\bar{x} \in F(x)$. Since $F$ upper semi-continuous, for any $\delta > 0$ there exists an integer $k(\delta) > 0$ such that for each $k > k(\delta)$
\[ B_{\epsilon_k}(F(x_{\epsilon_k})) \subset B_{\delta}(F(\bar{x})). \]

Consequently any sequence $\langle x_n \rangle_{n=0}^\infty$ such that $x_0 = x_{\epsilon_k}, x_{n+1} \in B_{\epsilon_k}(F(x_n))$ may be associated with sequence the same such that $x_0 = \bar{x}, x_{n+1} \in B_{\delta}(F(\bar{x}))$. Thus, taking into account (10),
\[ \varphi_\delta(\bar{x}) \leq d(\bar{x}, x_{\epsilon_k}) + \inf \left\{ \sum_{n=0}^\infty d(x_n, x_{n+1}) \mid x_0 = x_{\epsilon_k}, x_{n+1} \in B_{\epsilon_k}(F(x_n)) \right\} \]
\[ = d(\bar{x}, x_{\epsilon_k}) + f_{\epsilon_k}(x_{\epsilon_k}) \]
\[ \leq d(\bar{x}, x_{\epsilon_k}) - d(x, x_{\epsilon_k}) + \varphi_{\epsilon_k}(x) + \epsilon_k. \] \hfill $\square$
Theorem 3.8. If \( \lim_{\varepsilon \to 0} \varphi_{\varepsilon}(x) = \varphi_0(x) \), then \( \varphi_0 \) satisfies (3) and \( \varphi_0 = \varphi_F \).

**Proof.** If \( k \) tend to infinity then inequality (11) to be
\[
\varphi_\delta(\bar{x}) + d(x, \bar{x}) \leq \varphi_0(x)
\]
and letting \( \delta \) tend to 0, we obtain
\[
\varphi_0(\bar{x}) + d(x, \bar{x}) \leq \varphi_0(x)
\]
where \( \bar{x} \in F(x) \). Thus \( \varphi_0 \) satisfies (3) and following Lemma 3.2 and (8), we have \( \varphi_F(x) \leq \varphi_0(x) \leq \varphi_F(x) \) or \( \varphi_0(x) = \varphi_F(x) \) for each \( x \in X \). 

We now define contraction type for set-valued.

**Definition 3.9.** [1] The set-valued map \( F : X \to P_0(X) \) is called contraction in the sense that there exists \( \lambda \in (0, 1) \) such that
\[
F(y) \subset B(F(x), \lambda)
\]
for all \( x, y \in X \)

**Theorem 3.10.** Let \( X \) be a complete metric space. We consider a set-valued map \( F : X \to P_0(X) \) is contraction in the sense Definition 3.9. If \( F \) upper semi-continuous with compact values, then \( F \) has fixed point.

**Proof.** We using, in this case, the function \( \varphi_0(x) = \lim_{\varepsilon \to 0} \varphi_{\varepsilon}(x) \), for all \( x \in X \). Since set valued \( F \) upper semi-continuous with compact values on \( X \), from Theorem 3.8 we obtained
\[
\varphi_0(y) + d(x, y) \leq \varphi_0(x)
\]
with \( y \in F(x) \). Thus \( \varphi_0 \) satisfies (3). Now we show that \( \varphi_0 \) is a finite function. Take an arbitrary \( x \in X \) and from (13)
\[
\varphi_0(x) = \lim_{\varepsilon \to 0} \varphi_{\varepsilon}(x)
\]
\[
\leq \lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} d(x_n, x_{n+1})
\]
\[
= d(x, x_1) + \lim_{\varepsilon \to 0} \varphi_{\varepsilon}(x_1)
\]
\[
= d(x, x_1) + \varphi_0(x_1) = d(x, x_1) + \varphi_F(x_1)
\]
\[
= d(x, x_1) + d(x_1, F(x_1)) = d(x, x_1) + d(x_1, x_2)
\]
\[
\leq d(x, x_1) + \lambda d(x, x_1) = (1 + \lambda)d(x, x_1) < +\infty
\]

We then apply Theorem 3.3 to deduce that \( G \) has a fixed point. 

\( \square \)
References


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