Generalizations of Caristi-Kirk Theorem in Partial Metric Spaces and Applications

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Abstract

In this paper, some generalizations of Caristi-Kirk fixed point theorem in partial metric spaces are established, and applications of our results are given.

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1 Introduction and preliminaries

Let \((X, d)\) be a complete metric space, and let \(T : X \to X\) be a map.

Caristi-Kirk fixed point theorem states that \(T\) has a fixed point, whenever there exists a lower semicontinuous function \(\phi : X \to \mathbb{R}_+\), where \(\mathbb{R}_+\) denotes the set of all positive real numbers, such that

\[
d(x, Tx) \leq \phi(x) - \phi(Tx) \text{ for all } x \in X.
\]

Since the theorem is proved, many authors gave generalizations and applications of it.

Especially, the authors of [7, 8] obtained functional type Caristi-Kirk theorems in metric spaces.
The author of [1] obtained a generalization of Banach’s contraction principle by introducing partial metric spaces. And then, a lot of authors extended fixed point results in metric spaces to partial metric spaces.

Some authors, for instance [2, 6, 9, 11] investigated Caristi-Kirk fixed point theorem in partial metric spaces.

In this paper, we obtain a generalization of Theorem 2.3 of [12] and Theorem 4.1 of [6], and then we give an extension of Theorem 4 of [7] to partial metric spaces.

Recall that some definitions and basic results in partial metric spaces. For more details, we refer to [1].

Let $X$ be a nonempty set.

A mapping $p : X \times X \to \mathbb{R}^+$ is called a partial metric [1] on $X$ if and only if for each $x, y, z \in X$ the following axioms are satisfied:

1. $p(x, x) = p(x, y) = p(y, y)$ if and only if $x = y$;
2. $p(x, x) \leq p(x, y)$;
3. $p(x, y) = p(y, x)$;
4. $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

The pair $(X, p)$ with nonempty set $X$ and partial metric $p$ on $X$, is called a partial metric space.

Let $(X, p)$ be a partial metric space, and let

$$B_p(x, \epsilon) = \{ y \in X : p(x, y) < p(x, x) + \epsilon \}$$

for all $x \in X$ and $\epsilon > 0$.

It is well known that each partial metric $p$ on $X$ generates a $T_0$-topology $\sigma_p$ on $X$ which has, as a base, the class of open $p$-balls:

$$\{ B_p(x, \epsilon) : x \in X, \epsilon > 0 \}.$$  

Also, it is known that the functions $p_s, p_w : X \times X \to \mathbb{R}^+$ defined by

$$p_s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

and

$$p_w(x, y) = p(x, y) - \min\{p(x, x), p(y, y)\}$$

are metrics on $X$.

From now on, let $X$ be a partial metric space endowed with a partial metric $p$ on $X$.

Let $\{x_n\} \subset X$ be a sequence and $x \in X$. Then we say that
(1) \( \{x_n\} \) is **convergent to** \( x \) in \((X, p)\) if and only if \( \lim_{n \to \infty} p(x, x_n) = p(x, x) \);

(2) \( \{x_n\} \) is **properly convergent to** \( x \) in \((X, p)\) if and only if \( \lim_{n \to \infty} p(x, x_n) = \lim_{n \to \infty} p(x_n, x_n) = p(x, x) \);

(3) \( \{x_n\} \) is called **Cauchy sequence** if and only if there exists \( \lim_{n,m \to \infty} p(x_n, x_m) \) such that it is finite;

(4) \( X \) is **complete** if and only if every Cauchy sequence in \( X \) is convergent to a point \( z \in X \) such that

\[
\lim_{n,m \to \infty} p(x_n, x_m) = p(z, z).
\]

**Remark 1.1.** \( X \) is complete if and only if for every Cauchy sequence \( \{x_n\} \) in \( X \), there exists \( z \in X \) such that

\[
\lim_{n,m \to \infty} p(x_n, x_m) = \lim_{n,m \to \infty} p(x_n, z) = p(z, z).
\]

Thus every Cauchy sequence in \( X \) is properly convergent whenever \( X \) is complete.

**Remark 1.2.** Let \( \{x_n\} \) be a sequence of points in \( X \), and let \( x \in X \). If the sequence \( \{x_n\} \) is convergent to \( x \) in \((X, p_s)\), then it is convergent to \( x \) in \((X, p)\), and the converse is not true (see \([1, 4]\)).

**Lemma 1.1.** A sequence \( \{x_n\} \) of points in \( X \) is properly convergent to \( x \) in \((X, p)\) if and only if it is convergent to \( x \) in \((X, p_s)\).

**Proof.** Obviously, \( \{x_n\} \) is convergent to \( x \) in \((X, p_s)\) whenever it is properly convergent to \( x \) in \((X, p)\).

Assume that \( \{x_n\} \) is convergent to \( x \) in \((X, p_s)\).

Then, from Remark 1.2 \( \lim_{n \to \infty} p(x_n, x) = p(x, x) \). Thus we have

\[
\lim_{n \to \infty} p_s(x_n, x) = 2p(x, x) - \lim_{n \to \infty} p(x_n, x_n) - p(x, x) = 0
\]

and so

\[
\lim_{n \to \infty} p(x_n, x_n) = p(x, x).
\]

Hence,

\[
\lim_{n \to \infty} p(x_n, x) = \lim_{n \to \infty} p(x_n, x_n) = p(x, x)
\]

and hence the sequence \( \{x_n\} \) is properly convergent to \( x \) in \((X, p)\) \hfill \Box
Example 1.1. Let $X = \mathbb{R}^+$, and let $p(x, y) = \frac{1}{2} \{ |x - y| + |x| + |y| \}$ for all $x, y \in X$.

Then, $(X, p)$ is a complete partial metric space.

Let $x_n = \frac{1}{n}$ for all $n \in \mathbb{N}$.

Then $\lim_{n \to \infty} p(x_n, 1) = 1 = p(1, 1)$, and so it is convergent to 1 in $(X, p)$.

However, $\lim_{n \to \infty} p(x_n, x_n) = 0 \neq p(1, 1)$, and hence it is not properly convergent to 1 in $(X, p)$.

Note that $\lim_{n \to \infty} p_s(x_n, 1) \neq 0$ and it is not convergent to 1 in $(X, p_s)$.

Let $\phi : X \to \mathbb{R}^+$ be a function. Then, we say that

(1) $\phi$ is lower semicontinuous for $(X, p)$ if and only if it satisfies

\[
\phi(x) \leq \lim_{n \to \infty} \inf \phi(x_n)
\]

for any sequence $\{x_n\}$ in $X$ and $x \in X$ such that it is convergent to $x$ in $(X, p)$.

(2) $\phi$ is lower semicontinuous for $(X, p_s)$ if and only if (1.1) holds for any sequence $\{x_n\}$ in $X$ and $x \in X$ such that it is convergent to $x$ in $(X, p_s)$.

(3) $\phi$ is properly lower semicontinuous for $(X, p)$ if and only if (1.1) holds for any sequence $\{x_n\}$ in $X$ and $x \in X$ such that it is properly convergent to $x$ in $(X, p)$.

From Lemma 1.1, $\phi : X \to \mathbb{R}^+$ is lower semicontinuous for $(X, p_s)$ if and only if it is properly lower semicontinuous for $(X, p)$.

From Remark 1.2 we have the following result.

**Lemma 1.2.** Let $\phi : X \to \mathbb{R}^+$ be a function. We consider the following statements.

(1) $\phi$ is lower semicontinuous for $(X, p)$;

(2) $\phi$ is properly lower semicontinuous for $(X, p)$.

Then the following implications holds:

(1) $\Rightarrow$ (2).

Note that the converse is not true.

**Lemma 1.3.** Let $Y$ be a closed subset of $X$. If $\phi : X \to \mathbb{R}^+$ is lower semicontinuous (resp. properly lower semicontinuous), then the restriction function with respect to $Y$, $\phi|_Y : Y \to \mathbb{R}^+$ is lower semicontinuous (resp. properly lower semicontinuous) for $(Y, p)$.
Proof. Let \( \{x_n\} \) be a sequence in \( Y \) such that \( \lim_{n \to \infty} p(x_n, x) = p(x, x) \).

Since \( Y \) is closed, \( x \in Y \) and so \( x \in X \). Since \( \{x_n\} \subset X \) and \( \phi \) is lower semicontinuous, we have \( \phi(x) \leq \lim_{n \to \infty} \inf \phi(x_n) \). Owing to the fact that \( \{x_n\} \subset Y \) and \( x \in Y \), \( (\phi|_Y)(x) \leq \lim_{n \to \infty} \inf (\phi|_Y)(x_n) \). Hence \( \phi|_Y \) is lower semicontinuous for \((Y,p)\).

Lemma 1.4. If a sequence \( \{x_n\} \) of points in \( X \) properly converges to some \( x \in X \), then \( \lim_{n \to \infty} p(x_n, y) = p(x, y) \), for every \( y \in X \).

Proof. We deduce that

\[
p(x_n, y) \leq p(x_n, x) + p(x, y) - p(x, x)
\]

and so

\[
\lim_{n \to \infty} p(x_n, y) \leq p(x, y).
\]

Since \( p(x, y) \leq p(x, x_n) + p(x_n, y) - p(x_n, x_n) \), we have

\[
p(x, y) \leq \lim_{n \to \infty} p(x_n, y).
\]

Hence \( \lim_{n \to \infty} p(x_n, y) = p(x, y) \).

Lemma 1.5. [12] For each \( x \in X \), the function \( p_x : X \to \mathbb{R}_+ \) defined by

\[
p_x(y) = p(x, y)
\]

is lower semicontinuous for \((X,p_s)\), and hence it is properly lower semicontinuous for \((X,p)\).

Remark 1.3. For each \( x \in X \), \( p_x - p(x, x) \) is lower semicontinuous for \((X,p_s)\), and hence it is properly lower semicontinuous for \((X,p)\).

Lemma 1.6. [11] A sequence \( \{x_n\} \) of points in \( X \) is Cauchy if and only if

\[
\lim_{m,n \to \infty} \{p(x_n, x_m) - p(x_m, x_m)\} = 0.
\]

The following Theorem is a slightly generalization of Theorem 2.3 of [12] and Theorem 4.1 of [6].

Theorem 1.1. Let \( T : X \to X \) be a map such that

\[
p(x, Tx) - p(x, x) \leq \phi(x) - \phi(Tx)
\]

for all \( x \in X \), where \( \phi : X \to \mathbb{R}_+ \) is properly lower semicontinuous for \((X,p)\). If \( X \) is complete, then \( \text{Fix}(T) \neq \emptyset \), where \( \text{Fix}(T) = \{x \in X : x = Tx\} \).
\textbf{Proof.} For each \( x \in X \), let \( S(x) = \{ y \in X : p(x, y) - p(x, x) \leq \phi(x) - \phi(y) \} \).

Then \( S(x) \neq \emptyset \) since \( Tx \in S(x) \).

Because \( p_x + \phi - p(x, x) : X \to \mathbb{R}_+ \) is properly lower semicontinuous, for each \( x \in X \) the set \( \{ y \in X : p(x, y) + \phi(y) - p(x, x) \leq \phi(x) \} \) is closed in \( X \), and so \( S(x) \) is closed in \( X \).

Let \( x_0 \in X \) be fixed, and let \( x_1 \in S(x_0) \) be such that

\[
\phi(x_1) < \inf_{y \in S(x_0)} \phi(y) + \frac{1}{2}.
\]

Then we have

\[
S(x_1) \subset S(x_0).
\]  
(1.3)

In fact, if \( x \in S(x_1) \), then \( p(x_1, x) - p(x_1, x_1) \leq \phi(x_1) - \phi(x) \). Since \( x_1 \in S(x_0) \),

\[
p(x_0, x_1) - p(x_0, x_0) \leq \phi(x_0) - \phi(x_1).
\]

Hence we obtain

\[
p(x_0, x) \leq p(x_0, x_1) + p(x_1, x) - p(x_1, x_1)
\]

\[
\leq p(x_0, x_0) + \phi(x_0) - \phi(x_1) + p(x_1, x_1) + \phi(x_1) - \phi(x) - p(x_1, x_1)
\]

which implies

\[
p(x_0, x) - p(x_0, x_0) \leq \phi(x_0) - \phi(x_1).
\]

Hence \( x \in S(x_0) \), and hence (1.3) holds.

Inductively, we have a sequence \( \{x_n\} \) of points in \( X \) and a sequence \( \{S(x_n)\} \) of closed subsets of \( X \) such that for all \( n \in \mathbb{N} \)

\[
S(x_{n+1}) \subset S(x_n) \text{ and } x_{n+1} \in S(x_n)
\]  
(1.4)

and

\[
p(x_n, x) - p(x_n, x) < \frac{1}{2^n} \text{ for all } x \in S(x_n).
\]  
(1.5)

We now show that \( \{x_n\} \) is a Cauchy sequence.

For \( m > n \), from (1.4) we have \( x_m \in S(x_n) \).

From (1.5) we obtain

\[
p(x_n, x_m) - p(x_m, x_m) < \frac{1}{2^n}.
\]

Hence

\[
\lim_{n,m \to \infty} \{p(x_n, x_m) - p(x_m, x_m)\} = 0
\]

and so \( \{x_n\} \) is a Cauchy sequence by Lemma 1.6.

It follows from the completeness of \( X \) that there exists \( z \in X \) with

\[
\lim_{n,m \to \infty} p(x_n, x_m) = \lim_{n \to \infty} p(x_n, z) = p(z, z).
\]
Since $\phi$ is properly lower semicontinuous,

$$\phi(z) \leq \lim_{n \to \infty} \inf \phi(x_n).$$

It follows from the fact $x_m \in S(x_n)$ that

$$\phi(x_m) \leq \phi(x_n) - p(x_n, x_m) + p(x_n, x_n)$$
$$\leq \phi(x_n) - p(x_n, z) + p(x_m, z) - p(x_m, x_m) + p(x_n, x_n),$$

because $p(x_n, z) \leq p(x_n, x_m) + p(x_m, z) - p(x_m, x_m)$. 

Thus, we have

$$\phi(z) \leq \lim_{m \to \infty} \inf \phi(x_m)$$
$$\leq \lim_{m \to \infty} \inf \{\phi(x_n) - p(x_n, z) + p(x_m, z) - p(x_m, x_m) + p(x_n, x_n)\}$$
$$= \phi(x_n) - (p(x_n, z) - p(x_n, x_n)).$$

Hence $p(x_n, z) - p(x_n, x_n) \leq \phi(x_n) - \phi(z)$, and hence $z \in S(x_n)$ for all $n \in \mathbb{N}$.

From (1.2)

$$p(z, Tz) - p(z, z) \leq \phi(z) - \phi(Tz) \quad (1.6)$$

From the fact $z \in S(x_n)$ and (1.6) we deduce that

$$p(x_n, Tz) - p(x_n, x_n)$$
$$\leq p(x_n, z) - p(x_n, x_n) + p(z, Tz) - p(z, z)$$
$$\leq \phi(x_n) - \phi(z) + \phi(z) - \phi(Tz)$$
$$= \phi(x_n) - \phi(Tz).$$

Thus $Tz \in S(x_n)$ for all $n \in \mathbb{N}$.

Hence,

$$p(x_n, Tz) - p(Tz, Tz) < \frac{1}{2^n}.$$ 

Letting $n \to \infty$ in above inequality and using Lemma 1.4,

$$p(z, Tz) = p(Tz, Tz).$$

We deduce that

$$p(x_n, z) \leq p(x_n, Tz) + p(z, Tz) - p(Tz, Tz) = p(x_n, Tz).$$
because \( p(z, Tz) = p(Tz, Tz) \). Hence

\[
0 \leq p(x_n, z) - p(Tz, Tz) \leq p(x_n, Tz) - p(Tz, Tz) < \frac{1}{2^n},
\]

because \( Tz \in S(x_n) \). By letting \( \rightarrow \infty \) in above, we obtain

\[
p(z, z) = p(Tz, Tz).
\]

Hence \( p(z, Tz) = p(Tz, Tz) = p(z, z) \), and hence \( z = Tz \).

From Lemma 1.2 we have the following results.

**Corollary 1.2.** Let \( T : X \to X \) be a map such that

\[
p(x, Tx) - p(x, x) \leq \phi(x) - \phi(Tx)
\]

for all \( x \in X \), where \( \phi : X \to \mathbb{R}_+ \) is lower semicontinuous for \((X, p)\). If \( X \) is complete, then \( \text{Fix}(T) \neq \emptyset \).

**Corollary 1.3.** Let \( T : X \to X \) be a map such that

\[
p_w(x, Tx) \leq \phi(x) - \phi(Tx)
\]

for all \( x \in X \), where \( \phi : X \to \mathbb{R}_+ \) is lower semicontinuous (resp. properly lower semicontinuous) for \((X, p)\). If \( X \) is complete, then \( \text{Fix}(T) \neq \emptyset \).

**Proof.** From (1.7) we have either

\[
p(x, Tx) - p(Tx, Tx) \leq p(x, Tx) - p(x, x) \leq \phi(x) - \phi(Tx)
\]

or

\[
p(x, Tx) - p(x, x) \leq p(x, Tx) - p(Tx, Tx) \leq \phi(x) - \phi(Tx).
\]

Hence

\[
p(x, Tx) - p(x, x) \leq \phi(x) - \phi(Tx).
\]

By Corollary 1.2 (resp. Theorem 1.1), \( \text{Fix}(T) \neq \emptyset \).

By Corollary 1.2, we obtain the following result.

**Corollary 1.4.** [6] Let \( T : X \to X \) be such that

\[
p(x, Tx) \leq \phi(x) - \phi(Tx)
\]

for all \( x \in X \), where \( \phi : X \to \mathbb{R}_+ \) is lower semicontinuous for \((X, p)\). If \( X \) is complete, then \( \text{Fix}(T) \neq \emptyset \).

By Lemma 1.1, we have the following corollary.
Corollary 1.5. [3] Let $T : X \to X$ be a map such that
\[ p(x, Tx) - p(x, x) \leq \phi(x) - \phi(Tx) \]
for all $x \in X$, where $\phi : X \to \mathbb{R}_+$ is lower semicontinuous for $(X, p_s)$. If $X$ is complete, then $\text{Fix}(T) \neq \emptyset$.

Corollary 1.6. [12] Let $T : X \to X$ be a map such that
\[ p(x, Tx) \leq \phi(x) - \phi(Tx) \]
for all $x \in X$, where $\phi : X \to \mathbb{R}_+$ is lower semicontinuous for $(X, p_s)$. If $X$ is complete, then $\text{Fix}(T) \neq \emptyset$.

Note that if $\phi : X \to \mathbb{R}_+$ is lower semicontinuous (resp. properly lower semicontinuous), then $\nu \phi$ is lower semicontinuous (resp. properly lower semicontinuous), where $\nu$ is a positive constant.

Thus from Theorem 1.1 we have the following corollary.

Corollary 1.7. Let $T : X \to X$ be a map such that
\[ p(x, Tx) - p(x, x) \leq \nu(\phi(x) - \phi(Tx)) \]
for all $x \in X$, where $\nu$ is a positive constant and $\phi : X \to \mathbb{R}_+$ is properly lower semicontinuous for $(X, p)$. If $X$ is complete, then $\text{Fix}(T) \neq \emptyset$.

One can prove the next Theorem as same as proof of Theorem 1.1.

Theorem 1.8. Let $T : X \to X$ be a map such that
\[ p(x, Tx) - p(Tx, Tx) \leq \phi(x) - \phi(Tx) \]
for all $x \in X$, where $\phi : X \to \mathbb{R}_+$ is properly lower semicontinuous for $(X, p)$. If $X$ is complete, then $\text{Fix}(T) \neq \emptyset$.

Corollary 1.9. Let $T : X \to X$ be a map such that
\[ p(x, Tx) - p(Tx, Tx) \leq \nu(\phi(x) - \phi(Tx)) \]
for all $x \in X$, where $\phi : X \to \mathbb{R}_+$ is properly lower semicontinuous for $(X, p)$. If $X$ is complete, then $\text{Fix}(T) \neq \emptyset$.

Corollary 1.10. [11] Let $T : X \to X$ be a map such that
\[ p(x, Tx) - p(Tx, Tx) \leq \phi(x) - \phi(Tx) \]
for all $x \in X$, where $\phi : X \to \mathbb{R}_+$ is lower semicontinuous for $(X, p)$. If $X$ is complete, then $\text{Fix}(T) \neq \emptyset$. 
A function \( c : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is right upper semicontinuous if and only if, for any sequence \( \{ t_n \} \) in \( \mathbb{R}_+ \) and \( \alpha \geq 0 \) with \( t_n \downarrow \alpha \), we have
\[
c(\alpha) \geq \lim_{n \to \infty} \sup c(t_n).
\]

A function \( c : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is right locally bounded from above if and only if, for each \( \alpha \geq 0 \), there exists \( \lambda = \lambda(\alpha) > 0 \) such that
\[
\sup c([\alpha, \alpha + \lambda]) < \infty.
\]

**Lemma 1.7.** [7] If \( c : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is right upper semicontinuous, then it is right locally bounded from above.

Throughout this paper, let \( c : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) be right locally bounded from above, and let \( \phi : X \rightarrow \mathbb{R}_+ \) be a properly lower semicontinuous function for \((X,p)\).

### 2 Fixed point theorems

**Theorem 2.1.** Let \( T : X \rightarrow X \) be a map such that
\[
p(x,Tx) - p(x,x) \leq H(c(\phi(x)), c(\phi(Tx)))(\phi(x) - \phi(Tx))
\] (2.1)
for all \( x \in X \), where \( H : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is a locally bounded function. If \( X \) is complete, then \( \text{Fix}(T) \neq \emptyset \).

**Proof.** Let \( \alpha = \inf_{x \in X} \phi(x) \). Since \( c \) is right locally bounded from above, there exists \( \lambda > 0 \) such that \( \mu := \sup c([\alpha, \alpha + \lambda]) < \infty \). Hence there exists \( \nu > 0 \) such that \( H(s,t) \leq \nu \) for all \( s,t \in [0,\mu] \). For some \( x_0 \) such that \( \alpha \leq \phi(x_0) \leq \alpha + \lambda \), let \( X_0 = \{ x \in X : \phi(x) \leq \phi(x_0) \} \).

Then \( X_0 \) is a nonempty closed subset of \( X \). Hence \( (X_0,p) \) is complete.

For \( x \in X_0 \), we have
\[
\phi(Tx) \leq \phi(x) \leq \phi(x_0)
\]
and so
\[
T(X_0) \subset X_0.
\]

Since \( \phi(x), \phi(Tx) \in [\alpha, \alpha + \lambda] \), we have
\[
H(c(\phi(x)), c(\phi(Tx))) \leq \nu.
\] (2.2)

From (2.1) and (2.2) we have
\[
p(x,Tx) - p(x,x) \leq \nu(\phi(x) - \phi(Tx)) = \nu(\phi|_{X_0}(x) - \phi|_{X_0}(Tx))
\]
for all \( x \in X_0 \).

By lemma 1.3 and Corollary 1.7, we have \( \text{Fix}(T) \neq \emptyset \). \( \square \)
If we have $H(s, t) = \max\{s, t\}$, then we obtain the following Corollary.

**Corollary 2.2.** Let $T : X \to X$ be a map such that

$$p(x, Tx) - p(x, x) \leq \max\{c(\phi(x)), c(\phi(Tx))\}(\phi(x) - \phi(Tx))$$

for all $x \in X$. If $X$ is complete, then $\text{Fix}(T) \neq \emptyset$.

If we have either $H(s, t) = s$ or $H(s, t) = t$, then we obtain the following

**Corollary 2.3.** Let $T : X \to X$ be a map such that either

$$p(x, Tx) - p(x, x) \leq c(\phi(Tx))(\phi(x) - \phi(Tx))$$

for all $x \in X$ or

$$p(x, Tx) - p(x, x) \leq c(\phi(Tx))(\phi(x) - \phi(Tx))$$

for all $x \in X$. If $X$ is complete, then $\text{Fix}(T) \neq \emptyset$.

**Corollary 2.4.** Let $T : X \to X$ be a map such that

$$p(x, Tx) - p(x, x) \leq \psi(p(x, Tx))(\phi(x) - \phi(Tx))$$

for all $x \in X$, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is right upper semicontinuous.

If $X$ is complete and $p(x, Tx) \leq \phi(x)$ for all $x \in X$, then $\text{Fix}(T) \neq \emptyset$.

**Proof.** Define $c : \mathbb{R}_+ \to \mathbb{R}_+$ by $c(t) = \sup\{\psi(s) : 0 \leq s \leq t\}$.

Then $c$ is well defined and nondecreasing, and hence right locally bounded from above.

Since $\psi(p(x, Tx)) \leq c(p(x, Tx)) \leq c(\phi(x))$, (2.4) implies (2.3). Thus by Corollary 2.3, $\text{Fix}(T) \neq \emptyset$. \qed

**Corollary 2.5.** Let $T : X \to X$ be a map such that

$$p^w(x, Tx) \leq H(c(\phi(x)), c(\phi(Tx)))(\phi(x) - \phi(Tx))$$

for all $x \in X$, where $H : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a locally bounded function. If $X$ is complete, then $\text{Fix}(T) \neq \emptyset$.

**Corollary 2.6.** Let $T : X \to X$ be a map such that

$$p(x, Tx) \leq H(c(\phi(x)), c(\phi(Tx)))(\phi(x) - \phi(Tx))$$

for all $x \in X$, where $H : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a locally bounded function. If $X$ is complete, then $\text{Fix}(T) \neq \emptyset$. 


One can prove the following theorem as similar with the proof of Theorem 2.1.

**Theorem 2.7.** Let $T : X \rightarrow X$ be a map such that

$$p(x, Tx) - p(Tx, Ty) \leq H(c(\phi(x)), c(\phi(Tx)))(\phi(x) - \phi(Tx))$$

for all $x \in X$, where $H : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a locally bounded function. If $X$ is complete, then $\text{Fix}(T) \neq \emptyset$.

3 Applications

**Lemma 3.1.** Let $T : X \rightarrow X$ be a map such that

$$p(Tx, Ty) - p(Ty, Ty) \leq k(p(x, y) - p(y, y)) \quad (2.5)$$

for all $x, y \in X$, where $k \in (0, 1)$ is constant.

If a sequence $\{x_n\}$ of points in $X$ converges to some $x$ in $(X, p)$, then

$$\lim_{n \to \infty} p(Tx_n, Tx) = p(Tx, Tx).$$

**Proof.** From (2.5) we have

$$p(Tx_n, Tx) - p(Tx, Tx) \leq k(p(x_n, x) - p(x, x)).$$

Hence,

$$0 \leq \lim_{n \to \infty} p(Tx_n, Tx) - p(Tx, Tx) \leq k \lim_{n \to \infty} (p(x_n, x) - p(x, x)) = 0.$$

Thus $\lim_{n \to \infty} p(Tx_n, Tx) = p(Tx, Tx)$.

**Lemma 3.2.** Let $T : X \rightarrow X$ be a map such that (2.5) is satisfied. If a sequence $\{x_n\}$ of points in $X$ properly converges to some $x$ in $(X, p)$, then $\phi : X \rightarrow \mathbb{R}_+$ defined by $\phi(x) = p(x, Tx) - p(Tx, Tx)$ is properly lower semi-continuous in $(X, p)$.

**Proof.** Because $p(x_n, Tx) \leq p(Tx, Tx_n) + p(Tx_n, x_n) - p(Tx_n, Tx_n)$, we have

$$p(x_n, Tx) - p(Tx, Tx_n) \leq p(Tx_n, x_n) - p(Tx_n, Tx_n).$$

Hence $\phi(x_n) = p(x_n, Tx_n) - p(Tx_n, Tx_n) \geq p(x_n, Tx) - p(Tx, Tx_n)$.

By Lemma 1.4 and 3.1, we obtain

$$\lim_{n \to \infty} \inf \phi(x_n) \geq p(x, Tx) - p(Tx, Tx) = \phi(x).$$

Hence $\phi$ is a properly lower semicontinuous function.
Theorem 3.1. Let \( T : X \to X \) be a map such that
\[
p(Tx, Ty) - p(Ty, Ty) \leq k(p(x, y) - p(y, y)) + Lp_s(y, Tx) \tag{2.6}
\]
for all \( x, y \in X \), where \( k \in (0, 1) \) and \( L \geq 0 \). If \( X \) is complete, then \( \text{Fix}(T) \neq \emptyset \).

Proof. Let \( r \in (0, 1) \) be such that \( k = 1 - r \).

From (2.6) we have
\[
r(p(x, y) - p(y, y)) \leq p(x, y) - p(y, y) - \{p(Tx, Ty) - p(Ty, Ty)\} + Lp_s(y, Tx)
\]
and so
\[
p(x, y) - p(y, y) \leq \frac{1}{r}\{p(x, y) - p(y, y) - p(Tx, Ty) - p(Ty, Ty)\} + \frac{L}{r}p_s(y, Tx).
\]
Let \( y = Tx \) and \( \phi(x) = \frac{1}{r}\{p(x, Tx) - p(Tx, Tx)\} \).

Then \( \phi \) is properly lower semicontinuous, and
\[
p(x, Tx) - p(Tx, Tx) \leq \phi(x) - \phi(Tx)
\]
for all \( x \in X \).

Hence, by Theorem 1.2, \( \text{Fix}(T) \neq \emptyset \). \( \square \)

Corollary 3.2. [1] Let \( T : X \to X \) be a map such that
\[
p(Tx, Ty) - p(Ty, Ty) \leq k(p(x, y) - p(y, y))
\]
for all \( x, y \in X \), where \( k \in (0, 1) \). If \( X \) is complete, then \( \text{Fix}(T) \) is singleton.

Proof. By Theorem 3.1 with \( L = 0 \), \( \text{Fix}(T) \neq \emptyset \).

For the uniqueness of fixed point, let \( u, v \in \text{Fix}(T) \).

Then we obtain
\[
p(u, v) - p(v, v) = p(Tu, Tv) - p(Tv, Tv) \leq k(p(u, v) - p(v, v))
\]
which implies
\[
p(u, v) = p(v, v).
\]

Similarly, \( p(v, u) = p(u, u) \).

Thus we have \( p(u, v) = p(v, v) = p(u, u) \). Hence \( u = v \), and hence the set \( \text{Fix}(T) \) is singleton. \( \square \)

Similarly with the proof of Theorem 3.1, one can prove the following Theorem 3.2 and 3.3.
Theorem 3.3. Let $T : X \to X$ be a map such that
\[ p(Tx, Ty) - p(x, x) \leq k\{p(x, y) - p(x, x)\} + Lp_s(y, Tx) \]
for all $x, y \in X$, where $k \in (0, 1)$ and $L \geq 0$. If $X$ is complete, then $\text{Fix}(T) \neq \emptyset$. Moreover if $L = 0$, then $\text{Fix}(T)$ is singleton.

Theorem 3.4. Let $T : X \to X$ be a map such that
\[ p(Tx, Ty) - p(y, y) \leq k\{p(x, y) - p(y, y)\} + Lp_s(y, Tx) \]
for all $x, y \in X$, where $k \in (0, 1)$ and $L \geq 0$.
If $X$ is complete, then $\text{Fix}(T) \neq \emptyset$. Moreover if $L = 0$, then $\text{Fix}(T)$ is singleton.

Theorem 3.5. Let $T : X \to X$ be a map such that
\[ p(Tx, Ty) - p(x, y) \leq kp(x, y) + Lp_s(y, Tx). \quad (2.7) \]
for all $x, y \in X$, where $k \in (0, 1)$ and $L \geq 0$. If $X$ is complete, then $\text{Fix}(T) \neq \emptyset$.

Proof. Let $r \in (0, 1)$ be such that $k = 1 - r$.
From (2.7) we have
\[ rp(x, y) \leq p(x, y) - p(Tx, Ty) + Lp_s(y, Tx) \]
and so
\[ p(x, y) \leq \frac{1}{r}\{p(x, y) - p(Tx, Ty)\} + \frac{L}{r}p_s(y, Tx). \quad (2.8) \]

Let $y = Tx$ and $\phi(x) = p(x, Tx)$. Then $\phi$ is properly lower semicontinuous for $(X, p)$ by Lemma 1.5, and from (2.8) we have
\[ p(x, Tx) \leq \frac{1}{r}\{\phi(x) - \phi(Tx)\} \]
for all $x \in X$.
From Corollary 1.6 we have $\text{Fix}(T) \neq \emptyset$, because $\frac{1}{r}\phi$ is properly lower semicontinuous for $(X, p)$.

Corollary 3.6. [1] Let $T : X \to X$ be a map such that
\[ p(Tx, Ty) \leq kp(x, y). \]
for all $x, y \in X$, where $k \in (0, 1)$ is constant. If $X$ is complete, then $\text{Fix}(T)$ is singleton.
Proof. If \( u, v \in Fix(T) \), then we obtain
\[
p(u, v) = p(Tu, Tv) \leq kp(u, v)
\]
which implies
\[
p(u, v) = 0.
\]
Similarly,
\[
p(v, v) = p(u, u) = 0.
\]
Thus we have \( p(u, v) = p(v, v) = p(u, u) \). Hence \( u = v \), and hence the set \( Fix(T) \) is singleton.

References


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